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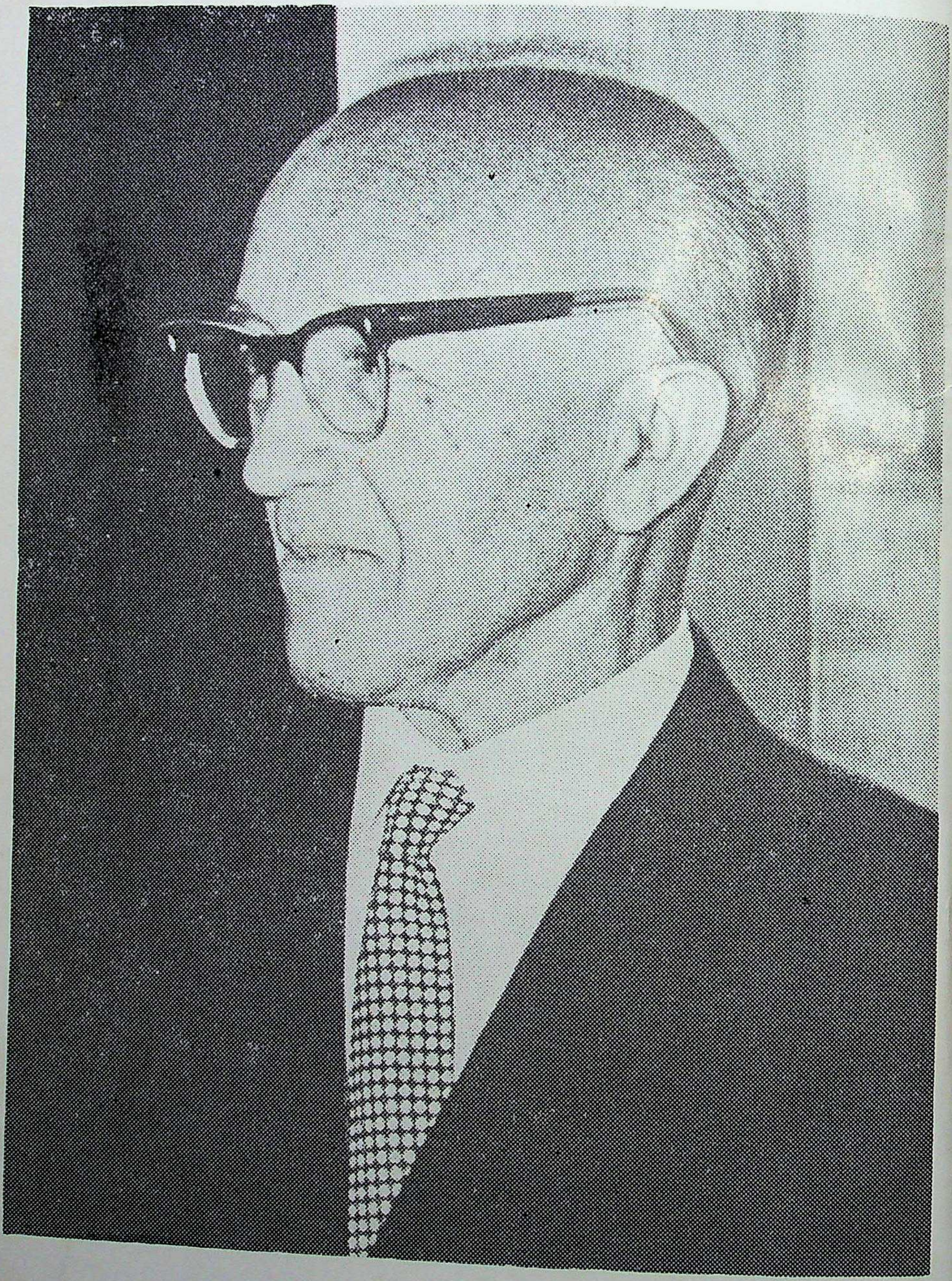

Edward Ott

DEMONSTRATIO MATHEMATICA
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Naresh Kumar, G.S. Srivastava

ON THE PROXIMATE ORDER OF AN ENTIRE FUNCTION WITH INDEX PAIR (p,q)

1. Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$, where $z = re^{i\theta}$ and $\{a_n\}$ is any sequence of complex numbers, be a non-constant entire function. We set

$$M(r) = \max_{|z|=r} |f(z)|, \quad \mu(r) = \max_{n\geq 0} \{ |a_n| r^n \}$$

and

$$\nu(r) = \max \{ n : \mu(r) = |a_n| r^n \}.$$

Then $M(r)$, $\mu(r)$ and $\nu(r)$ are called respectively the maximum modulus, the maximum term and the rank of the maximum term of $f(z)$ for $|z|=r$. It is well known that $\log M(r)$ and $\log \mu(r)$ are indefinitely increasing convex function of $\log r$ and $\nu(r)$ is a nondecreasing integer valued function of r. Further ([8], p. 31) $\nu(r)$ and $\mu(r)$ are related by the equation

$$\log \mu(r) = O(1) + \int_{r_0}^{r} \frac{\nu(x)}{x} dx, \quad 0 < r_0 < r. \tag{1.1}$$

An entire function $f(z)$ is said to be of order $\rho(1)$ if

(1.2) $$\limsup_{r\to\infty} \frac{\log\log M(r)}{\log r} = \rho, \quad 0 \le \rho \le \infty .$$

For entire functions of finite order $(\rho < \infty)$, we also have

(1.3) $$\log M(r) \approx \log \mu(r) \quad \text{as } r \to \infty .$$

If $\rho = 0$ or $\rho = \infty$, then further classification of $f(z)$ is required to study its growth precisely. For this, the notion of index pair is used.

Let us define for $n \ge 0$, $\log^{[0]} x = x$, $\log^{[n]} x = \log(\log^{[n-1]} x)$ for $0 < \log^{[n-1]} x < \infty$ and $F_{[n]}(x) = \log^{[n]} x \circ \log^{[n-1]} x \circ \dots \circ x$. Then we set

(1.4) $$\limsup_{r\to\infty} \frac{\log^{[p]} M(r)}{\log^{[q]} r} = \rho(p,q) ,$$

where p and q are integers such that $p \ge q \ge 0$.

An entire function $f(z)$ is said to have index pair (p,q), $p \ge q \ge 1$ if $b < \rho(p,q) < \infty$ and $\rho(p-1,q-1)$ is either 0 or ∞, where $b = 1$ if $p = q$ and $b=0$ if $p > q$. If $f(z)$ has index pair (p,q) then $\rho = \rho(p,q)$ is called its (p,q)-order [2].

Let $f(z)$ be of index pair (p,q) and of (p,q)-order $\rho (b < \rho < \infty)$. Then the (p,q)-type $T(p,q) = T$ and lower (p,q)-type $t(p,q) = t$ of $f(z)$ are defined as [3]

(1.5) $$\lim_{r\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log^{[p-1]} M(r)}{(\log^{[q-1]} r)^{\rho}} = \begin{matrix}T\\ t\end{matrix} , \quad 0 \le t \le T \le \infty .$$

A function $w(r)$ defined on $(0, \infty)$ is said to be a proximate order [4] of an entire function $f(z)$ having index pair (p,q) if it satisfies the following conditions

$$\lim_{r\to\infty} w(r) = \rho \;, \quad b < \rho < \infty \;, \tag{1.6}$$

$$\lim_{r\to\infty} w'(r)\, F_{[q]}(r) = 0 \,. \tag{1.7}$$

In this paper, we shall find some estimates for the growth of maximum modulus and maximum term etc. in relation to proximate order. For this, we consider a general growth function $\Phi(r)$ which is real valued, finite on every finite interval and increases to + ∞ as $r \to \infty$.

We set

$$\Psi(r) = \Psi(r_0) + \int_{r_0}^{r} \frac{\Phi(x)}{F_{[q-1]}(x)}\, dx \;, \quad r_0 > 0 \tag{1.8}$$

and define the following constans

$$\left\{\begin{aligned} &\lim_{r\to\infty}\begin{matrix}\sup\\ \inf\end{matrix}\ \frac{\Psi(r)}{(\log^{[q-1]} r)^{w(r)}} = \begin{matrix}\alpha\\ \beta\end{matrix} \;, \\ &\lim_{r\to\infty}\begin{matrix}\sup\\ \inf\end{matrix}\ \frac{\Phi(r)}{(\log^{[q-1]} r)^{w(r)}} = \begin{matrix}\gamma\\ \delta\end{matrix} \;, \\ &\lim_{r\to\infty}\begin{matrix}\sup\\ \inf\end{matrix}\ \frac{\Psi(r)}{\Phi(r)} = \begin{matrix}c\\ d\end{matrix} \;. \end{aligned}\right. \tag{1.9}$$

In the next section we prove two theorems involving the constans defined above. The first theorem and its corollary suppliment some results already obtained in this direction ([7], Theorems 1 and [2]) while the second theorem not only generalizes but improves upon the result of Shah [6]. Throughout this paper, it is assumed that the constans defined in (1.9) are non zero and finite. To avoid trivial cases, we shall consider only transcendental entire functions, that is, entire functions of index pair (p,q), $p \geq 2$, $q \geq 1$.

2. We now prove

Theorem 1. We have

$$(2.1)\qquad \limsup_{r\to\infty} \frac{\Phi(r)}{(\log^{[q-1]} r)^{w(r)}} \exp\left[\rho \frac{\Psi(r)}{\Phi(r)}\right] = e\rho\alpha \quad ,$$

where α is as defined in (1.9).

Proof. Since for any real x, $\exp(x) \ge ex$, we have

$$\exp\left[\rho \frac{\Psi(r)}{\Phi(r)}\right] \ge e\rho \frac{\Psi(r)}{\Phi(r)} .$$

Hence

$$\frac{\Phi(r)}{(\log^{[q-1]} r)^{w(r)}} \exp\left[\rho \frac{\Psi(r)}{\Phi(r)}\right] \ge e\rho \frac{\Psi(r)}{(\log^{[q-1]} r)^{w(r)}} ,$$

or, proceeding limits, we have

$$(2.2)\qquad \limsup_{r\to\infty} \frac{\Phi(r)}{(\log^{[q-1]} r)^{w(r)}} \exp\left[\rho \frac{\Psi(r)}{\Phi(r)}\right] \ge e\rho\alpha \quad .$$

To prove the reverse inequality, we take any increasing sequence of positive numbers $\{x_n\}$, $x_n \longrightarrow \infty$ as $n \longrightarrow \infty$. Then from (1.8),

$$\Psi(r) = \Psi(n_n) + \int_{x_n}^{r} \frac{\Phi(x)}{F_{[q-1]}(x)}\, dx \ge$$

$$\ge \Psi(x_n) + \Phi(x_n)[\log^{[q]} r - \log^{[q]} x_n] \quad .$$

Hence

$$(2.3)\qquad \frac{\Psi(r)}{(\log^{[q-1]} r)^{w(r}} \ge \frac{\Psi(x_n) + \Phi(x_n)[\log^{[q]} r - \log^{[q]} x_n]}{(\log^{[q-1]} r)^{w(r)}} .$$

We consider another sequence of real numbers $\{r_n\}$ defined by

$$\log^{[q]} r_n = \frac{\Phi(x_n) + \rho[\ \Phi(x_n)\log^{[q]} x_n - \Psi(x_n)\ ]}{\rho\Phi(x_n)} .$$

Then we have

$$\log^{[q-1]} r_n = (\ \log^{[q-1]} x_n\ )\ \exp\ [\ \frac{1}{\rho} - \frac{\Psi(x_n)}{\Phi(x_n)}\ ] .$$

From [7,Th.1] we have $0 < \frac{\delta}{\rho\gamma} \le d \le c \le \frac{\gamma}{\rho\delta} < \infty$. Hence

$$\exp\ (\ \frac{1}{\rho} - \frac{\gamma}{\rho\delta}\ )\ \log^{[q-1]} x_n < \log^{[q-1]} r_n <$$

$$< \exp\ (\ \frac{1}{\rho} - \frac{\gamma}{\rho\delta}\ )\ \log^{[q-1]} x_n ,$$

or, there exist finite constans A and B such that

$$(2.4) \qquad A \log^{[q-1]} x_n < \log^{[q-1]} r_n < B \log^{[q-1]} x_n .$$

Hence $r_n \longrightarrow \infty$ as $x_n \longrightarrow \infty$. Now consider the quantity

$$P = (\ \log^{[q-1]} x_n\ )^{w(r_n)-w(x_n)} .$$

Then

$$\log P = [\ w(r_n) - w(x_n)\ ]\ \log^{[q]} x_n =$$

$$= w'(t_n)\ (r_n - x_n)\ \log^{[q]} x_n \quad , \quad x_n < t_n < r_n$$

by Lagrange's Mean Value Theorem. Since $t_n \longrightarrow \infty$ as $n \longrightarrow \infty$, we have by (1.7),

$$\log P < \frac{\varepsilon\ (\ r_n - x_n\ )\ \log^{[q]} x_n}{F_{[q]}(t_n)} < \frac{\varepsilon r_n \log^{[q]} x_n}{F_{[q]}(x_n)}$$

where $\varepsilon > 0$ is arbitrarily small and n is sufficiently large. Now proceeding to limit, we have for $q \ge 1$,

$$\lim_{n\to\infty} \log P = 0 \quad , \text{ i.e.}$$

(2.5) $$\lim_{n\longrightarrow\infty} (\log^{[q-1]} x_n)^{w(r_n) - w(x_n)} = 1.$$

Since (2.3) holds for all $r \ge x_n$, we obtain on substituting $r = r_n$,

$$\frac{\Psi(r_n)}{\log^{[q-1]} r_n)^{w(r_n)}} \ge \frac{\Psi(x_n) + \Phi(x_n)[\log^{[q]} r_n - \log^{[q]} x_n]}{(\log^{[q-1]} r_n)^{w(r)}} =$$

$$= \frac{1}{\rho} \frac{\Phi(x_n)}{(\log^{[q-1]} x_n)^{w(r_n)}} \exp[\frac{w(r_n)\Psi(x_n)}{\Phi(x_n)} - \frac{w(r_n)}{\rho}] .$$

Since $w(r_n) \longrightarrow \rho$ as $n \longrightarrow \infty$ and $0 < d \le c < \infty$, therefore

$$\frac{w(r_n)\Psi(x_n)}{\Phi(x_n)} = \frac{[w(r_n) - \rho] \Psi(x_n)}{\Phi(x_n)} + \frac{\rho\Psi(x_n)}{\Phi(x_n)} = o(1) + \frac{\rho\Psi(x_n)}{\Phi(x_n)} .$$

Finally, using (2.5), we get

$$\limsup_{n\longrightarrow\infty} \frac{\Psi(r_n)}{\log^{[q-1]} r_n)^{w(r_n)}} \ge$$

$$\ge \frac{1}{e\rho} \limsup_{n\longrightarrow\infty} \{ \frac{\Phi(x_n)}{(\log^{[q-1]} x_n)^{w(r_n)}} \} \exp [\frac{\rho\Psi(x_n)}{\Phi(x_n)}] .$$

Since

$$\limsup_{r\longrightarrow\infty} \frac{\Psi(r)}{(\log^{[q-1]} r)^{w(r)}} \ge \limsup_{r\longrightarrow\infty} \frac{\Psi(r_n)}{(\log^{[q-1]} r_n)^{w(r_n)}}$$

and $\{x_n\}$ was any arbitrary sequence of real numbers, we have

$$e\rho\alpha \ge \limsup_{r\longrightarrow\infty} \frac{\Phi(r)}{(\log^{[q-1]} r)^{w(r)}} \exp(\frac{\rho\Psi(r)}{\Phi\ (r)}) .$$

Combining the above ineqality with (2.2), we get (2.1). This proves Theorem 1.

Corollary. For the constants defined by (1.9), we have

$$(2.6)\qquad \begin{cases} \dfrac{\beta}{\delta} \le c \le \dfrac{1}{\rho}\{\, 1 + \log(\, \rho\alpha/\delta\,)\,\} \le \dfrac{\alpha}{\delta}\,, \\[2ex] \dfrac{\beta}{\gamma} \le d \le \dfrac{1}{\rho}\{\, 1 + \log(\, \rho\alpha/\gamma\,)\,\} \le \dfrac{\alpha}{\gamma}\,. \end{cases}$$

Proof. From (2.1) we obtain on breaking up the left hand side,

$$(2.7)\qquad \delta \exp(\,\rho c\,) \le e\rho\alpha \text{ and } \gamma \exp(\,\rho d\,) \le e\rho\alpha\,.$$

Also, since

$$\frac{\Psi(r)}{(\log^{[q-1]} r)^{w(r)}} = \frac{\Psi(r)}{\Phi(r)} \cdot \frac{\Phi(r)}{(\log^{[q-1]} r)^{w(r)}}\,,$$

we obtain

$$(2.8)\qquad \beta \le \delta c\,, \text{ and } \beta \le \gamma d\,.$$

Hence (2.8) in view of (2.7) gives (2.6).

Theorem 2. We have

$$\frac{1}{\rho k} \le \lim_{r\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\Psi(r)}{\Phi(r)} \le \frac{k}{\rho}\,,$$

where k is that root of the equation $\alpha \log x = \beta x - \alpha$, which lies in the interval $[1,\infty)$.

Proof. We consider the expression

$$H(x) = \rho x\beta - \delta x \log x\,, \quad x > 0\,.$$

Then we can easly see that the maximum value of $H(x)$ is obtained for

$$x = x_0 = \exp\Big(\,\frac{\rho\beta - \delta}{\delta}\,\Big) \text{ and } H(x_0) = \delta\exp\Big(\,\frac{\rho\beta - \delta}{\delta}\,\Big)\,.$$

Hence we have for all $x > 0$,

$$\rho x\beta - \delta x\log x \le \delta\exp\left(\frac{\rho\beta - \delta}{\delta} \right) \le \rho\alpha ,$$

on using (2.6). Let k be the root of the equation $\alpha\log x = \beta x - \alpha$. Then we get $\frac{\rho\alpha}{\delta} \le k$. Hence from (2.6) we get $c \le \frac{k}{\rho}$.

To prove the left hand inequality of (2.9) we consider the expression

$$G(x) = \frac{\gamma\beta x}{\alpha} - \rho\beta x\log x , \quad x \ge 1 .$$

Proceeding as before, we get $G(x) \le \rho\beta\exp\left(\frac{\gamma}{\rho\alpha} - 1 \right)$.

From (2.6) we have $\rho\beta \le \gamma\{ 1 + \log(\frac{\rho\alpha}{\gamma}) \}$. Further, it is easy to see that

$$1 + \log\left(\frac{\rho\alpha}{\gamma} \right) \le \exp\left(1 - \frac{\gamma}{\rho\alpha} \right) .$$

Hence we get $G(x) \le \gamma$ for any $x \ge 1$. Taking $x = k$ we get $\frac{\gamma}{\beta} \le \rho k$. Therefore from (2.6) we have $d \ge \frac{\beta}{\gamma} \ge \frac{1}{\rho k}$. This proves Theorem 2.

Remarks. We can choose for $\Phi(r)$, various growth function associated with an entire function to obtain interesting results from the theorems proved above. As an illustration, we state some of these

(i) Let $\Phi(r) = \frac{M'(r)F_{[q-1]}(r)}{F_{[p-2]}(M(r))}$, where $M'(r) = \frac{dM(r)}{dr}$. Then from (1.8) we get $\Psi(r) = \log^{[q-1]}M(r)$. If we take $w(r) \equiv \rho$ in (1.9) then $\alpha = T(p,q)$ and $\beta = t(p,q)$. Theorem 1 and its Corollary now give estimates of (p,q)-type T and lower (p,q)-type t in terms of the (p,q)-growth numbers.

(ii) Let $\mu(r)$ and $\nu(r)$ denote the maximum term and the rank of the maximum term of f(z), as defined before. We choose $\Phi(r) = \nu(r)$, $w(r) = \rho$ for entire functions of index pair

$(p,q) = (2.1)$. Then from (1.1) we have $\Psi(r) = \log \mu(r)$ and in view of (1.3) $\alpha = T$ and $\beta = t$. From (2.6) we have

$$\rho t \le c\rho\delta \le \delta \, [\, 1 + \log(\frac{\rho T}{\delta}) \,] \le \rho T \,,$$

$$\rho t \le d\rho\delta \le \gamma \, [\, 1 + \log(\frac{\rho T}{\gamma}) \,] \le \rho T \,.$$

Since $\rho T \le \gamma$ and $d \le \frac{\delta}{\rho\gamma}$ [7, Theorem 1], we see that the above inequalities obtained by Shah [5, Theorem 1].

(iii) Let us take $\Phi(r)$ and $w(r)$ as in (ii). Then for index pair (2,1), inequalitie (2.9) present considerable improvement over (2) of Shah [6] for $L(r) \equiv 1$. It can be easly seen that the root k of the equation $T\log x = tx - T$ lying in the interval $[1,\infty)$ considered in (2.9) is always smaller than the root of the equation $eT\log x = tx - eT$ lying in the interval (e,∞) considered by Shah.

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REFERENCES

[1] R.P. Boas : Entire functions. Academic Press, New York 1954.

[2] O.P. Juneja, G.P. Kapoor, S.K. Bajpai :On the (p,q)-order and lower (p,q)-order of an entire function, J. Reine Angew. Math. 282 (1976) 53–67.

[3] O.P. Juneja , G.P. Kapoor , S.K. Bajpai : On the (p,q)-type and lower (p,q)-type of an entire

fnction, J. Reine Angew. Math. 290 (1977) 180-190.

[4] K. Nandan, R.P. Dohrey, R.S. Srivastava: Proximate order of an entire function with index pair (p,q), Indian J. Pure Appl. Math. 11(1) (1980) 33-39.

[5] S.M. Shah: The maximum term of an entire series III, Quart. J. Math. Oxford Ser., 19 (1948) 220-223.

[6] S.M. Shah: A note on entire functions of perfectly regular growth, Math. Z. 56 (1952) 254-257.

[7] G.S. Srivastava, N. Kumar: Some inequalities related to proximate order of an entire function with index pair (p,q), Pure Appl. Math. Sci. 20 (1984) 93-101.

[8] G. Valiron: Lectures on the general theory of integral functions, Chelsea, New York 1949.

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M. A. Pathan and Yasmeen

ON GENERATING FUNCTIONS OF A GENERAL TRIPLE HYPERGEOMETRIC SERIES

Some generating functions for triple series $F^{(3)}$ of Srivastava [15] are obtained. Special cases appear to give new and known generating functions for Appell's functions F_1, F_2, F_3, Kampé de Fériet's function $F^{p:q;r}_{1:m;n}$, generalized Rice Polynomials $H_n^{(\alpha,\beta)}$ and Jacobi polynomials $P_n^{(\alpha,\beta)}$.

1. In a paper [6; p. 688 (2.2)], Manocha generalized a result of Srivastava [15] on Appell's function F_2 [1].

$$(1.1)\qquad \sum_{n=0}^{\infty} \frac{(1+\alpha)_n}{n!} F_2(\lambda; -n, -n; 1+\alpha; 1+\alpha; x, y)\, t^n =$$

$$= (1-t)^{\lambda-\alpha-1}[1-(1-x-y)]^{-\lambda} \cdot$$

$$\cdot\ {}_2F_1\left[\begin{matrix} \frac{\lambda}{2}, & \frac{\lambda+1}{2} & ; \\ & 1+\alpha & & \frac{4xyt}{\{1-(1-x-y)t\}^2} \end{matrix}\right];$$

in the form of a generating function

$$(1.2)\qquad \sum_{n=0}^{\infty} \frac{(1+\alpha)_{m+n}}{n!} F_2(\lambda; -n, -m-n; 1+\alpha; 1+\alpha; x, y)\, t^n =$$

$$= (1+\alpha)_m \, (1-y)^{-\lambda} (1-t)^{-1-\alpha-m} \, .$$

$$\cdot \, F_4 \left[\lambda, \; 1+\alpha+m; \; 1+\alpha; \; 1+\alpha; \; \frac{-xt}{(1-y)(1-t)} \, , \; \frac{-y}{(1-y)(1-t)} \right] ,$$

where F_4 is also an Appell's function [1; p.14 (14)].

An interesting result involving Appell's F_2 ([1]; p.14 (12)) and Horn's H_3 functions ([20], p.860) was given by Srivastava ([18], p.681 (2.2)).

$$(1.3) \qquad \sum_{n=0}^{\infty} \frac{(1+\alpha)_n}{n!} F_2 \, (a;-n,-n;1+\beta;1+\alpha;x,y) \, t^n =$$

$$= (1-t)^{a-\alpha-1} \, \theta^{-a} \, H_3 \left[a, \; \beta-\alpha; \; 1+\beta; \; \frac{xyz}{\theta^2}, \; \frac{xz}{\theta} \right] ,$$

where $\theta = 1 - (1-x-y) \, t$.

This work can be extended further, and it is the purpose of the present note to obtain the generating functions for the Srivastava's triple hypergeometric series $F^{(3)}$ which unify (1.1) to (1.3) and generalize a number of other results of Manocha [6] and Srivastava [16] and [19]. Many known and unknown generating relations of Appell's functions F_1, F_2, and F_3, Kampé de Fériet's function $F^{p:q;r}_{1:m;n}$, Jacobi polynomials $P_n^{(\alpha,\beta)}$ and generalized Rice polynomials $H_n^{(\alpha,\beta)}$, also follow as special cases of our results.

2. We first prove our main generating relation

$$(2.1) \qquad \sum_{n=0}^{\infty} \frac{(1+\alpha)_{m+n}}{n!} F^{(3)} \left[\begin{matrix} a::b;-;-:-n \; ;-m-n;c-b; \\ c::-;-;-:1+\beta; \; 1+\alpha; \; \; -; \end{matrix} \; x,y,z \right] t^n =$$

$$= (1-y)^{-a}\,(1+\alpha)_m\,(1-t)^{-1-\alpha-m}\,.$$

$$\cdot\, F^{(3)}\left[\begin{array}{l} a:\ :b;\ 1+\alpha+m;\ -;\ -:\ -\ ;\ \ -;\ c-b; \\ c:\ :\ \ -\ ;\ -;\ -:\ 1+\beta;\ 1+\alpha;\ \ -; \end{array}\quad \frac{-xt}{(1-y)(1-t)},\ \frac{-y}{(1-y)(1-t)},\ \frac{z-y}{1-y}\right],$$

where $F^{(3)}$ is a Srivastava's triple hypergeometric series defined by ([14], p. 428)

$$F^{3}\left[\begin{array}{l} (a):\ :(b);\ (b');\ (b''):\ (c);\ (c');\ (c''); \\ (d):\ :(g);\ (g');\ (g''):\ (h);\ (h');\ (h''); \end{array}\ x,y,z\right] =$$

$$= \sum_{m,n,p=0}^{\infty} \frac{((a))_{m+n+p}\,((b))_{m+n}\,((b'))_{n+p}\,((b''))_{p+m}}{((d))_{m+n+p}\,((g))_{m+n}\,((g'))_{n+p}\,((g''))_{p+m}}\,\cdot$$

$$\cdot\,\frac{((c))_m\,((c'))_n\,((c''))_p\,x^m y^n z^p}{((h))_m\,((h'))_n\,((h''))_p\,m!\,n!\,p!}\,.$$

Here (a) is a sequence of A parameters $a_1, \ldots, a_A$, and

$$((a_n)) = \prod_{j=0}^{A} \frac{\Gamma(a_j+n)}{\Gamma(a_j)}\,.$$

$F^{(3)}$ is a generalization of Lauricella's fourteen triple hypergeometric functions $F_1, \ldots, F_{14}$ of three variables ([5], p.114), extended F_k function of Sharma ([13] p.613), and the additional functions H_A, H_B, H_C ([16] pp.99-100).

To prove (2.1), we require the following formula which is a slight variation of a recent result of Khan and Pathan ([3], p. 105 (2.3)).

$$(2.2)\qquad F^{(3)}\left[\begin{array}{l} a:\ :b;\ -;\ -:\ d;\ f;\ c-b; \\ \\ c:\ :-;\ -;\ -:\ e;\ g;\quad -; \end{array} x,y,z\right] =$$

$$= (1-y)^{-a}F^{(3)}\left[\begin{array}{l} a:\ :\ b;\ -;\ -;\ d;\ g-f;\ c-b; \\ \\ c:\ :\ -;\ -;\ -:\ e;\ \ g\ ;\ \ -\ ; \end{array} \frac{x}{1-y},\ \frac{-y}{1-y},\ \frac{z-y}{1-y}\right].$$

On using (2.2) into (2.1) we have

$$\sum_{n=0}^{\infty} \frac{(1+\alpha)_{m+n}}{n!}F^{(3)}\left[\begin{array}{l} a:\ :\ b;\ -;\ -:\ -n\ ;-m-n;\ c-b; \\ c:\ :\ -;\ -;\ -:\ 1+\beta;\ 1+\alpha;\quad -; \end{array} x,y,z\right] t^n =$$

$$= \sum_{n=0}^{\infty} \frac{(1+\alpha)_{m+n}}{n!}(1-y)^{-a}\ .$$

$$\circ\ F^{(3)}\left[\begin{array}{l} a:\ :\ b;-;\ -;\ -n\ ;\ 1+\alpha+m+n;\ c-b; \\ \\ c:\ :\ -;-;\ -:\ 1+\beta;\quad 1+\alpha\ \ ;\ \ -; \end{array} \frac{x}{1-y},\ \frac{-y}{1-y},\ \frac{z-y}{1-y}\right] t^n =$$

$$= (1-y)^{-a}\ \sum_{n=0}^{\infty}\frac{(1+\alpha)_{m+n}}{n!}\ .$$

$$\cdot\sum_{k=0}^{\infty}\ \sum_{r,s=0}^{\infty} \frac{(a)_{k+r+s}\ (b)_{k+r}\,(-n)_k(1+\alpha+m+n)_r(c-b)_s}{(c)_{k+r+s}\ (1+\beta)_k\qquad (1+\alpha)_r\quad k!r!s!}\ .$$

$$\cdot\left(\frac{x}{1-y}\right)^k\left(\frac{-y}{1-y}\right)^r\left(\frac{z-y}{1-y}\right)^s t^n =$$

$$=(1-y)^{-a} \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \sum_{r,s=0}^{\infty} \frac{(1+\alpha)_{m+n+r} (a)_{k+r+s} (b)_{k+r} (c-b)_s}{(c)_{k+r+s} \, (1+\beta)_k \, (1+\alpha)_r k!r!s!(n-k)!} \, \cdot$$

$$\cdot \left(\frac{-x}{1-y} \right)^k \left(\frac{-y}{1-y} \right)^r \left(\frac{z-y}{1-y} \right)^s =$$

$$= (1-y)^{-a} \sum_{n,k=0}^{\infty} \sum_{r,s=0}^{\infty} \frac{(1+\alpha)_{m+n+r+k} (a)_{k+r+s} (b)_{k+r} \; (c-b)_s}{(c)_{k+r+a} \, (1+\beta)_k \, (1+\alpha)_r \; n!k!r!s!} \, \cdot$$

$$\cdot \left(\frac{-x}{1-y} \right)^k \left(\frac{-y}{1-y} \right)^r \left(\frac{z-y}{1-y} \right)^s t^{n+k} =$$

$$= (1-y)^{-a} (1+\alpha)_m \sum_{k,r,s=0}^{\infty} \frac{(a)_{k+r+s} (b)_{k+r} (1+\alpha+m)_{k+r} (c-b)_s}{(c)_{k+r+s} \, (1+\beta)_k \, (1+\alpha)_r k!r!s!} \, \cdot$$

$$\cdot \left(\frac{-x}{1-y} \right)^k \left(\frac{-y}{1-y} \right)^r \left(\frac{z-y}{1-y} \right)^s t^{n+k} = \sum_{n=0}^{\infty} \frac{(1+\alpha+m+k+r)}{n!} \, n \, t^n =$$

$$= (1-y)^{-a} \, (1+\alpha)_m \, (1-t)^{-1-\alpha-m} \, .$$

$$\cdot F^{(3)} \left[\begin{array}{l} a: \, :b, \; 1+\alpha+m; -; -: \; - \; ; \quad - \; ; \; c-b; \\ \qquad\qquad\qquad\qquad \dfrac{-xt}{(1-y)(1-t)}, \; \dfrac{-y}{(1-y)(1-t)}, \; \dfrac{z-y}{1-y} \\ c: \; : \quad -----; -; -: \; 1+\beta; \; 1+\alpha; \; - \; ; \end{array} \right]$$

This completes the proof of (2.1).

3. For $a = b = c$ and $\beta = \alpha$, (2.1) reduces to (1.2), which is a known results of Manocha [6; p. 687 (2.1)].

Setting $m = 0$, $b = c$ and using a transformation ([20], p. 860)

(3.1) $$F_4 \, (\alpha, \; \beta; \; \gamma, \; \beta; \; x, \; y) = (1-x-y)^{-a} \, .$$

$$\cdot\, H_3\left[\,\alpha,\ \gamma-\beta;\ \gamma;\ \frac{xy}{(x+y-1)^2},\ \frac{x}{x+y-1}\,\right],$$

(2.1) yields (1.3).

For b = c, (2.1) reduces to

$$(3.2)\quad \sum_{n=0}^{\infty} \frac{(1+\alpha)_{m+n}}{n!}\, F_2(\,a;\ -n,\ -m-n;\ 1+\beta,\ 1+\alpha;\ x,\ y\,)\, t^n =$$

$$= (1+\alpha)_m (1-y)^{-a}\, (1-t)^{-1-\alpha-m}.$$

$$\cdot F_4\left(a, 1+\alpha+m;\ 1+\beta, 1+\alpha;\ \frac{-xt}{(1-y)(1-t)},\ \frac{-y}{(1-y)(1-t)}\right).$$

We put m = 0, a = c, b = 1+α+β in (2.1) and make use of a transformation formula ([7], p.155 (2.2)).

$$F_4\,(1+\alpha+\beta\,, 1+\alpha;\ 1+\beta,\ 1+\alpha;\ x,\ y) = (1+x-y)^{-1-\alpha-\beta}\ .$$

$$\cdot\ {}_2F_1\left[\begin{matrix}\dfrac{1+\alpha+\beta}{2},\ \dfrac{2+\alpha+\beta}{2}\ ; \\ 1+\beta\ ;\end{matrix}\quad \frac{4x}{(1+x-y)^2}\right]$$

to get

$$(3.3)\quad \sum_{n=0}^{\infty} \frac{(1+\alpha)_n}{n!}\, F_2(\,1+\alpha+\beta;\ -n,\ -n;\ 1+\beta,\ 1+\alpha;\ x,\ y\,)\, t^n =$$

$$= (1-t)^{\beta}\left[\,1-t(1+x-y)\,\right]^{-1-\alpha-\beta}\ .$$

$$\cdot\ {}_2F_1\left[\begin{matrix}\dfrac{1+\alpha+\beta}{2},\ \dfrac{2+\alpha+\beta}{2}\ ; \\ 1+\beta\ ;\end{matrix}\quad \frac{4xt(1-y)(1-t)}{\{1-(\,1+x-y\,)t\}^2}\right],$$

which is a special case of a result of Manocha ([6], p.688 (2.2)).

Special cases of (2.1) also give similar generating functions for Appell's functions F_1 and F_3 ([1], p.14 (11) and (13)).

For the Appell's function F_1 we have

$$\sum_{n=0}^{\infty} \frac{(1+\alpha)_{m+n}}{n!} F_1(a;-m-n,c-\alpha-1;c;y,z)\, t^n =$$

$$= (1-y)^{-a}\, (1+\alpha)_m (1-t)^{-1-\alpha-m}.$$

$$\cdot F_1\left(a;1+\alpha+m,\ c-\alpha-1;\ c;\frac{-y}{(1-y)(1-t)},\frac{z-y}{1-y}\right)$$

which on letting $m = 0$ and using a result of Bailey ([2] p.79 (2)) gives

$$\sum_{n=0}^{\infty} \frac{(1+\alpha)_n}{n!} F_1\ (a;-n,c-\alpha-1;c;y,z)\, t^n =$$

$$= (1-t)^{-1-\alpha}\, (1-z)^{-a}\ {}_2F_1\left[\begin{matrix} a,\ 1+\alpha; \\ c\ \ ; \end{matrix}\ \frac{z+(y-z)t}{z-1}\right].$$

For $y \longrightarrow 0$ and $b = 1+\beta$, (2.1) reduces to

$$(3.4) \qquad \sum_{n=0}^{\infty} \frac{(1+\alpha)_{m+n}}{n!} F_1(a;-m-n,c-\beta-1;c;x,z)\, t^n =$$

$$= (1+\alpha)_m\ (1-t)^{-1-\alpha-m}\ F_1\left(a;1+\alpha+m,c-\beta-1;c;\frac{-xt}{1-t},z\right) .$$

If we use a transformation ([8]; p.372 91.2):

$$F^{(3)}\left[\begin{matrix} a:\ :b;\ -;\ -:\ g-f;\ d;\ c-b; \\ c:\ :-;\ -;\ -:\ g\ ;\ e;\ -\ ; \end{matrix}\ x,y,z\right] = (1-z)^{b-c}.$$

$$\cdot\, F^{(3)}\left[\begin{matrix} -::a,b;-;-:g-f;d;c-a,c-b; \\ c::-;-;-:g;e;-; \end{matrix}\; x,y,\frac{z}{1-z}\right],$$

(2.1) would yield

$$(3.5)\qquad \sum_{n=0}\frac{(1+\alpha)_{m+n}}{n!}\;\cdot$$

$$\cdot\, F^{(3)}\left[\begin{matrix} -::a,\ b;\ -;\ -:\ -n;-m-n;\ c-a,\ c-b; \\ c::\ -;\ -;\ -:\ 1+\beta;\ 1+\alpha;\ -; \end{matrix}\; x,y,\frac{z}{1-z}\right] t^{n} =$$

$$= (1+\alpha)_{m}\,(1-y)^{-a}\,(1-t)^{-1-\alpha-m}\,(1-z)^{c-b}\;\cdot$$

$$\cdot F^{(3)}\left[\begin{matrix} a::b,\ 1+\alpha+m;-;-:\ -;\ -;\ c-b; \\ c::\ -;-;-:\ 1+\beta;\ 1+\alpha;\ -; \end{matrix}\; \frac{-xt}{(1-y)(1-t)},\ \frac{-y}{(1-y)(1-t)},\ \frac{z-y}{1-y}\right]$$

For the Appell's polynomial F_3, we have

$$(3.6)\qquad \sum_{n=0}^{\infty}\frac{(1+\alpha)_{m+n}}{n!}\,F_3(a,c-a;-m-n,c-\alpha-1;c;y,\frac{z}{z-1})\; t^{n} =$$

$$= (1+\alpha)_{m}\,(1-y)^{-a}\,(1-t)^{-1-\alpha-m}(1-z)^{c-\alpha-1}\;.$$

$$\cdot\, F_1(a;1+\alpha+m,c-\alpha-1;c;\frac{-y}{(1-y)(1-t)},\ \frac{z-y}{1-y})$$

which is a special case of (3.5) for $x \longrightarrow 0$ and $b = \alpha+1$.

When $y \longrightarrow 0$ and $b = \beta+1$, (93.5) gives

$$(3.7)\qquad \sum_{n=0}^{\infty}\frac{(1+\alpha)_{m+n}}{n!}\,F_3(a,c-a;-n,c-\beta-1;c;x,\frac{z}{z-1})t^{n} =$$

$$= (1+\alpha)_m \, (1-z)^{c-\beta-1} \, (1-t)^{-1-\alpha-m} \, .$$

$$\cdot \, F_1(\, a; \; 1: \; \alpha+m, \; c-\beta-1; \; c; \; \frac{-xt}{1-t}, \; z) \, .$$

For z ⟶ 0, (2.1) reduces to

$$(3.8) \qquad \sum_{n=0}^{\infty} \frac{(1+\alpha)_{m+n}}{n!} \, F^{2:1;1}_{1:1;1} \left[\begin{matrix} a,b: \; -n; -m-n; \\ c \; ; \; 1+\beta; \; 1+\alpha; \end{matrix} \; x,y \right] t^n =$$

$$= (1-y)^{-a} (1+\alpha)_m (1-t)^{-1-\alpha-m} \, .$$

$$\cdot F^{(3)} \left[\begin{matrix} a: \; :b, \; 1+\alpha+m; \; -; \; -: \; -; \; \; - \; ; \; c-b; \\ \qquad \qquad \qquad \qquad \frac{-xt}{(1-y)(1-t)} , \frac{-y}{(1-y)(1-t)}, \frac{z-y}{1-y} \\ c: \; : \qquad - \; ; \; -; \; -; \; -: \; 1+\beta; \; \; -; \end{matrix} \right] ,$$

where $F^{2:1;1}_{1:1;1}$ is Kampé de Fériet's function ([1], p.150 (29)).

Similarly putting z = y in (3.5) gives a generating function of $F^{(3)}$ in terms of Kampé de Fériet's function $F^{3:0;0}_{1:1;1}$.

For y ⟶ 0, (3.8) would give us the following generating function.

$$(3.9) \qquad \sum_{n=0}^{\infty} \frac{(1+\beta+m)_n}{(\alpha+1)_n} \, H_n^{(\alpha,\beta-n)} \, [\; b,c;x \;] \; t^n =$$

$$= F^{1:2;0}_{0:2;0} \left[\begin{matrix} 1+\beta+m: \alpha+\beta+1 \, , b; -; \\ - \; : \; \alpha+1 \; \; ,c; -; \end{matrix} \; -xt, t \right]$$

for the generalized Rice Polynomial defined by

([4]; p.158 (2.3)).

$$(3.10)\qquad H_n^{(\alpha,\beta)}[\, b,c;x\,] = \frac{(\alpha+1)_n}{n!}\; {}_3F_2\left[\begin{matrix} -n,\alpha+\beta+n+1 \;,\; b; \\ \alpha+\beta \;,\; c; \end{matrix}\; x\right],$$

$n = 0, 1, \ldots$

which, when $\alpha = \beta = 0$, reduces to the orginal form $H_n[\ b,c,x]$ given in ([11], p.108).

The generating function (3.9) is equivalent to a known relation of Srivastava ([19], p.77 (3.3)) for $m = 0$, since

$$H_n^{(\alpha,\beta)}[b,b,x] = P_n^{(\alpha,\beta)}(\,1-2x\,)\ ,$$

where $P_n^{(\alpha,\beta)}(\,x\,)$ denotes the Jacobi polynomial (3.9) can be further specialized to obtain generating functions for the special Jacobi Polynomials $P_n^{(\alpha-n,\beta)}(x)$ or $P_n^{(\alpha,\beta-n)}(x)$, since it is well known that

$$P_n^{(\alpha,\beta)}(\,x\,) = (-1)^n\, P_n^{(\beta,\alpha)}(-x\,)\ .$$

The generating function

$$(3.11)\qquad \sum_{n=0}^{\infty} \frac{(1+\beta+m)_n}{(\alpha+1)_n}\, P_n^{(\alpha,\beta-n)}(1-2x)\, t^n =$$

$$= F^{1:1;0}_{0:1;0}\left[\begin{matrix} 1+\beta+m: \alpha+\beta+1\,;-; \\ -\ :\ \alpha+\beta\ ;-; \end{matrix}\; -xt,\ -t\right]$$

follows from (3.9) by setting $b = c$. (3.11) and its generalization in a different direction was obtained by Srivastava [17].

4. In this section we shall show that (2.1) and (3.3)

enable us to obtain some new formulae of Srivastava's triple series $F^{(3)}$.

Pathan ([9], p.785) gave a summation formula

$$(4.1) \qquad \sum_{k=0}^{n} \frac{(1+\alpha)_k}{k!} F^{(3)}\left[\begin{matrix} a: \ :b; \ -: \ -k; \ -k; \ c-b; \\ c: \ :-; \ -:1+\alpha;1+\alpha; \ - \ ; \end{matrix} \ x,y,z \right] =$$

$$= \frac{(1+\alpha)_{n+1}(c-1)}{n!(a-1)(b-1)} (x-y)^{-1} .$$

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$$\circ \left(F^{(3)} \left[\begin{matrix} a-1: \ :b-1; \ -; \ -: \ -n;-n-1;c-b; \\ c-1: \ : \ - \ ; \ -; \ -:1+\alpha; \ 1+\alpha; \ - \ ; \end{matrix} \ x,y,z \right] + \text{''}x \Leftrightarrow y\text{''} \right).$$

where Re (α) > -1 and the abbreviation " $x \Leftrightarrow y$ " is used to indicate the presense of a second term that originates from the first by interchanging x and y.

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By (4.1) we have

$$\frac{(c-1)(x-1)^{-1}}{(a-1)(b-1)} \sum_{n=0}^{\infty} \frac{(1+\alpha)_{n+1}}{n!} .$$

$$\circ \left(F^{(3)} \left[\begin{matrix} a-1: \ :b-1; \ -; \ -: \ -n;-n-1;c-b; \\ c-1: \ : \ - \ ; \ -; \ -:1+\alpha; \ 1+\alpha; \ - \ ; \end{matrix} \ x,y,z \right] + x \Leftrightarrow y \right) t^n =$$

$$= \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{(\alpha+1)_k}{k!} F^{(3)} \left[\begin{matrix} a: \ :b; \ -; \ -; \ -k; \ -k; \ c-b; \\ c: \ :-; \ -; \ -:1+\alpha;1+\alpha; \ - \ ; \end{matrix} \ x,y,z \right] t^n =$$

$$= \sum_{n=0}^{\infty} t^n \left(\sum_{k=0}^{\infty} \frac{(1+\alpha)_k}{k!} F^{(3)} \left[\begin{matrix} a: \ :b; \ -; \ -; \ -k; \ -k; \ c-b; \\ c: \ :-; \ -; \ -:1+\alpha;1+\alpha; \ - \ ; \end{matrix} \ x,y,z \right] t^k \right)$$

Using (2.1) for $m = 0$ and $\beta = \alpha$, we arrive at

(4.2)
$$\sum_{n=0}^{\infty} \frac{(1+\alpha)_{n+1}}{n!} \cdot$$

$$\cdot \left(F^{(3)}\left[\begin{matrix} a-1: \ :b-1; \ -; \ -: \ -n;-n-1;c-b; \\ c-1: \ : \ - \ ; \ -; \ -:1+\alpha; \ 1+\alpha; \ - \ ; \end{matrix} \ x,y,z \right] + x \Leftrightarrow y \right) t^{n} =$$

$$= [(a-1)(b-1)(x-y)(1-t)^{-2-\alpha}] \,/\, (c-1)(1-y)^{a} \cdot$$

$$\cdot \, F^{(3)}\left[\begin{matrix} a: \ :b,1+\alpha; \ -; \ -: \ -; \ - \ ; \ c-b; \\ c: \ : \quad - \ ; \ -;-:1+\alpha;1+\alpha; \quad - \ ; \end{matrix} \ \frac{-xt}{(1-y)(1-t)}, \ \frac{-y}{(1-y)(1-t)}, \ \frac{z-y}{1-y}\right].$$

For $y = z$ and $c = 1+\alpha$, (4.2) gives

(4.3)
$$\sum_{n=0}^{\infty} \frac{(1+\alpha)_{n+1}}{n!} \cdot$$

$$\cdot \left(F^{(3)}\left[\begin{matrix} a-1: \ :b-1; \ -; \ -: \ -n;-n-1;1+\alpha-b; \\ \alpha \ : \ : \ - \ ; \ -; \ -:1+\alpha; \ 1+\alpha; \quad - \ ; \end{matrix} \ x,y,z \right] + x \Leftrightarrow y \right) t^{n} =$$

$$= \frac{(a-1)(b-1)}{\alpha} \, \frac{(x-y)(1-t)^{-2-\alpha}}{(1-y)^{a}} \cdot$$

$$\cdot \, F_{4}\left[a, \ b; \ 1+\alpha, \ 1+\alpha; \ \frac{-xt}{(1-y)(1-t)}, \ \frac{-y}{(1-y)(1-t)} \right].$$

For $b = c$, (4.2) reduces to result of Manocha ([6], p.688 (3.1)) with the help of a transformation formula ([12], p.432)

(4.4)
$$\sum_{n=0}^{\infty} \frac{(1+\alpha)_{n+1}}{n!} \cdot$$

$$\cdot [\, F_2(\lambda-1;\ -n,-n-1;\ 1+\alpha,\ 1+\alpha;\ x,\ y\,) - x\ y\ ]\ t^n =$$

$$= (\lambda-1)\ (x-y)\ (1-t)^{\lambda-\alpha-2}\ [1-(1-x-y)t]^{-\lambda}\ .$$

$$\circ\ {}_2F_1\left[\begin{matrix}\frac{\lambda}{2},\ \frac{\lambda+1}{2}\ ; \\ 1+\alpha\ ;\end{matrix}\ \frac{4\,xyt}{\{1-(1-x-y)\,t\}^2}\right].$$

It may be pointed out that in the paper of Manocha [6] some inaccuracies crept in and he ended up with λ instead of $\lambda-1$ in the right hand side of (4.4). For correction to results of [6], we refer a paper of Pathan, Khan and Qureshi [10].

Turning to (2.1) again, we put m = 1 and then combine it with (4.2) to get

(4.5)

$$F^{(3)}\left[\begin{matrix}a-1:\ :b-1,2+\alpha;-;-:\ -;\ -;\quad c-b; \\ \\ c-1:\qquad -\quad ;-;-:1+\alpha;\ 1+\alpha;\ -\ ;\end{matrix}\ \frac{-xt}{(1-y)(1-t)},\ \frac{-y}{(1-y)(1-t)},\ \frac{z-y}{1-y}\right] +$$

$+ x$ [illegible] $y = (a-1)\ (b-1)\ (x-y)\ /\ (c-1)\ (1+\alpha)\ (1-t)\ (1-y)\ \cdot$

$$\cdot\ F^{(3)}\left[\begin{matrix}a:\ :b,1+\alpha;\ -;\ -:\ -;\ -\ ;\ c-b; \\ \\ c:\ :\quad -\ ;\ -;-:1+\alpha;1+\alpha;\ \ -\ ;\end{matrix}\ \frac{-xt}{(1-y)(1-t)},\ \frac{-y}{(1-y)(1-t)},\ \frac{z-y}{1-y}\right].$$

(4.5) is generalization of corrected version of a result of Manocha ([6], p. 689 (3.4)).

REFERENCES

[1] P.Appell, J. Kampe dé Fériet : Fonctions Hypergéometriques et Hypersphèriques : Polynomes d'Hermite, Gauthier-Villars, Paris, 1926.

[2] W.N.Bailey : Generalized hypergeometric series, Cambridge University Press, London, 1935.

[3] B. Khan, M.A.Pathan : On some transformations of hypergeometric functions of three variables $F^{(3)}$, Bull. Inst. Math. Acad. Sinica 2 (2), (1984) 103-111.

[4] P.R.Khandekar : On a generalization of Rice's Polynomial I, Proc. Nat. Acad. Sci. India, Sect. A 34, (1964) 157-162.

[5] G.Lauricalla : Sulle funzioni ipergeometriche a più variabili, Rend. Circ. Math. Palermo 7, (1893) 111 - 158.

[6] H.L.Manocha : On Appell's function F_2, Proc. Cambridge Philos. Soc. 65, (1969) 687-689.

[7] H.L.Manocha : Some formulae involving Appell function F_4, Publ. Inst. Match. (Beograd) (N.S.), 9 (1969) 153-156.

[8] M.A.Pathan : On some transformations of triple hypergeometric series $F^{(3)}$, Indian J. Pure Appl. Math. 9 (4), (1978) 371-376.

[9] M.A.Pathan : A note on the generalization of the

summation formulas for Appell's and Kampé de Fériets hypergeometric function, . Math. Anal. Appl. 72 (2), (1979) 783–787.

[10] F.U.Khan, M.A.Pathan, M.I.Qureshi : On Appell's polynomial F_2, Tamkang J. math. 15 (2), (1984) 163–169.

[11] S.O.Rice : Some properties of ${}_3F_2(-n,n+1,\xi;1,p;\nu)$, Duke Math. j. 6, (1940) 108–119.

[12] B.L.Sharma : Integrals involving hypergeometric functions of two variables, Proc. Nat. Acad. Sci. India Sect. A 36, (1966) 713–718.

[13] B.L.Sharma : Some formulae for Appell's functions, Proc. Cambridge Philos. Soc. 67, (1970) 613–618.

[14] H.M.Srivastava : Generalized Neumann expansions involving hypergeometric functions, Proc. Cambridge Philos. Soc. 63, (1967) 425–429.

[15] H.M.Srivastava : On a summation formula for the Appell function F_2, Proc. Cambridge Philos. Soc.63, (1967) 1087–1089.

[16] H.M.Srivastava : Some integrals representing triple hypergeometric functions, Rend. Circ. Mat. Palermo Ser. II. 16, (1967) 99–115.

[17] H.M.Srivastava : On a generating function for the Jacobi Polynomial, J. Math. Sci. 4, (1969) 61–68.

[18] H.M.Srivastava : An infinite summation formula associated with Appell's function F_2, Proc. Cambridge Philos. Soc. 65, (1969) 679–682.

[19] H.M.Srivastava : Certain formulas associated with

generalized Rice Polynomials, II. Ann. Polon Math. 27, (1972) 73-83.

[20] H.M.Srivastava, R.Panda : Some hypergeometric transformations involving Horn's function H_3, Rand. Mat. 6 (6), (1973) 859-864.

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ON M-REPRESENTATION OF M-COMPLETE PSEUDO BOOLEAN ALGEBRA

This paper deals with the m-representation of an m-complete pseudo Boolean algebra. In this paper we define the notion of a Heyting ring. We also prove the theorem about the m-representation of an m-complete pseudo Boolean algebra, and the theorem about the neccesary and sufficient condition for an m-complete pseudo Boolean algebra to be isomorfic to an m-Heyting ring.

The m-complete pseudo Boolean algebra

Definition 1.1. Let A be a pseudo Boolean algebra, and let m be an infinite cardinal number. A is said to be an m-complete pseudo Boolean algebra if it satisfies the following conditions:

a) for every m-indexed set $(a_t)_{t\in T}$ of elements of A, the join $\bigcup_{t\in T} a_t$ exists in A,

b) for every m-indexed set $(a_t)_{t\in T}$ of elements of A, the meet $\bigcap_{t\in T} a_t$ exists in A.

If A is an m-complete pseudo Boolean algebra for every cardinal number m, then A is said to be a complete pseudo Boolean algebra.

Boolean algebra.

Definition 1.2. An ideal I of an m-complete pseudo Boolean algebra A is said to be an m-complete idel or an m - ideal provided if $a_t \in I$ for every $t \in T$ and $\overline{\overline{T}} \leq m$, then $\bigcup_{t\in T} a_t \in I$: By duality a filter F of an m- complete pseudo Boolean algebra A is said to be an m-complete filter or an m-filter provided, if $a_t \in T$ for every $t \in T$ and $\overline{\overline{T}} \leq m$, then $\bigcap_{t\in T} a_t \in F$.

Definition 1.3. Let h be a homomorphism of a pseudo Boolean algebra A into a pseudo Boolean algebra B. The homomorphism h is said to be an m-homomorphism if for every m-indexed sets $(a_t)_{t\in T}$, $(b_t)_{t\in T}$ the join $\bigcup_{t\in T} a_t$ and the meet $\bigcap_{t\in T} b_t$ exists in A and the join $\bigcup_{t\in T} h(a_t)$ and the meet $\bigcap_{t\in T} h(b_t)$ exists in B, then

$$h\left[\bigcup_{t\in T} a_t\right] = \bigcup_{t\in T} h(a_t),$$

$$h\left[\bigcap_{t\in T} b_t\right] = \bigcap_{t\in T} h(b_t).$$

2. The m-Heyting ring

Definition 2.1. A Heyting ring on a set X is said to be every class $H \subseteq 2^X$ in which for every $A, B \in 2^X$, the following conditions are satisfied:

1) $X \in H$,

2) if $A, B \in H$ then $A \cap B \in H$ and $A \cup B \in H$,

3) if $A, B \in H$ then $A \to B \in H$, where $A \to B = \bigcup \{ Y \in H : A \cap B \subseteq B \}$

$A \longrightarrow B$ is said to be a pseudo-complement of A relative to B.

If in H the condition

4) $\bigcap H \in H$ satisfied then H is said to be a Heyting ring with zero.

It is easy to note that every Heyting ring with zero is a pseudo Boolean algebra.

3. The representation theorem for a pseudo Boolean algebra

Let A be a given pseudo Boolean algebra. We assume the following notations. the symbol $P(A)$ denotes the class of all prime filters in A and

(*) $\qquad s(a) = \{ F \in P(A) : a \in F \}$,

(**) $\qquad S(A) = \{ s(a) \quad : a \in A \}$.

Theorem 1. Every pseudo Boolean algebra is isomorphic to a Heyting ring with zero on a some set X.

Proof. By the representation theorem for distributive lattices [1], we know that $S(A)$ is a set lattice isomorphic with a distributive lattice A. The isomorphism is a mapping $s : A \longrightarrow S(A)$ defined by (*), it suffices to prove that

(***) $\qquad s(a \rightarrow b) = \bigcup \{ Y \in S(A) : s(a) \cap Y \subseteq s(b) \}$.

Let $F \in s(a \rightarrow b)$, then $a \rightarrow b \in F$ and $F \in P(A)$, since in every pseudo Boolean algebra we have $a \cap (a \rightarrow b) \leq b$ such that $s(a) \cap s(a \rightarrow b) \subseteq s(b)$ for $Y = s(a \rightarrow b)$ we have $s(a) \cap Y \subseteq s(b)$ and $F \in Y$.

Let now $F \in \bigcup \{ Y \in S(A) : s(a) \cap Y \subseteq s(b) \}$. There exists a set $Y \in S(A)$ such that $s(a) \cap Y \subseteq s(b)$, $F \in Y$ for some $u \in A$, we have $Y = s(u)$. That is $s(a) \cap s(u) \subseteq s(b)$ i.e. $a \cap u \leq b$

consequently $u < a \to b$, since $F \in s(u)$ thus $a \to b \in F$. This proves (***).

Definition 3.1. A Heyting ring H is said to be an m-complete Heyting ring, if for every m-indexed family $(X_t)_{t\in T} \subseteq H$, $\bigcap_{t\in T} X_t$, $\bigcup_{t\in T} X_t$ belongs to H.

If H is an m-complete Heyting ring for every cardinal number m, then H is said to be a complete Heyting ring.

Definition 3.2. Let A be an m-complete pseudo Boolean algebra. Then A is said to be m-representable, when A is the image of an m-homomorphism of an m-Heyting ring with zero.

Let H be a Heyting ring and $F \subseteq H$ a filter in ring H. For any elements $a, b \in H$, we shall write

$$a \equiv_F b \Leftrightarrow a \to b,\ \ b \to a \in F.$$

The equivalence relation $\equiv_F$ is a congruence in H.

Theorem 2. Let H be an m-Heyting ring with zero and $F \subseteq H$ an m-filter in H. Then the quotient algebra H/F is an m-complete pseudo Boolean algebra.

Proof. For arbitrary m-indexed sets $(a_t)_{t\in T}$, $(b_t)_{t\in T}$ we must prove:

(i) $$\forall_{t\in T}\ a_t \equiv_F b_t \Leftrightarrow \bigcap_{t\in T} a_t \equiv_F \bigcap_{t\in T} b_t ,$$

(ii) $$\forall_{t\in T}\ a_t \equiv_F b_t \Leftrightarrow \bigcup_{t\in T} a_t \equiv_F \bigcup_{t\in T} a_t .$$

We have

$$\forall_{t\in T}\ (\, a_t \to b_t \in F \wedge b_t \to a_t \in F\,),$$

$$\bigcap_{t\in T} (a_t \to b_t) \in F \wedge \bigcap_{t\in T} (b_t \to a_t) \in F ,$$

$$\bigcap_{t\in T} (a_t \to b_t) \leq \bigcap_{t\in T} a_t \longrightarrow \bigcap_{t\in T} b_t ,$$

$$\bigcap_{t\in T} (b_t \to a_t) \leq \bigcap_{t\in T} b_t \longrightarrow \bigcap_{t\in T} a_t .$$

We know also that

$$\bigcap_{t\in T} (a_t \to b_t) \leq \bigcup_{t\in T} a_t \longrightarrow \bigcup_{t\in T} b_t ,$$

$$\bigcap_{t\in T} (b_t \to a_t) \leq \bigcup_{t\in T} b_t \longrightarrow \bigcup_{t\in T} a_t ,$$

$$\bigcap_{t\in T} a_t \longrightarrow \bigcap_{t\in T} b_t , \bigcap_{t\in T} b_t \longrightarrow \bigcap_{t\in T} a_t \in F ,$$

$$\bigcup_{t\in T} a_t \longrightarrow \bigcup_{t\in T} b_t , \bigcup_{t\in T} b_t \longrightarrow \bigcup_{t\in T} a_t \in F ,$$

$$\bigcap_{t\in T} a_t \equiv_F \bigcap_{t\in T} b_t , \bigcup_{t\in T} a_t \equiv_F \bigcup_{t\in T} b_t .$$

Hence we have proved that H/F is an m-complete pseudo Boolean algebra.

Theorem 3. For every m-pseudo Boolean algebra A the following two conditions are equivalent:

(i) A is m-isomorphic with m-Heyting ring with zero.

(ii) For any a, $b \in A$, if $a \leq b$, then there exists m-prime filter $F \subseteq A$ such that $a \in F$, $b \notin F$ and $A-F$ is m-ideal in A.

Proof. Let H be an m-Heyting ring with zero, and let $B: A \longrightarrow H$ be an isomorphism for A into H. Suppose that a, $b \in A$ such that $a \nleq b$. Then $B(a) \nleq B(b)$. For an element x_0 we have $x_0 \in B(a)$ and $x_0 \notin B(b)$.

We will define a family $F_0 \subseteq H$ as follows $F_0 = \{Y \in H: x_0 \in Y\}$.

By the definition of F_0 and from the fact that H is m-Heyting ring we get:

(1) if $(X_t)_{t\in T} \in F_0$, $\overline{\overline{T}} \le m$ then $\bigcap_{t\in T} X_t \in F_0$,

(2) if $X \subseteq Y$, $X \in F_0$ then $Y \in F_0$,

(3) $X \cup Y \in F_0$ then $X \in F_0$ or $Y \in F_0$ for any $X, Y \in H$.

In addition to this, we have $B(a) \in F_0$ and $B(b) \notin F_0$.

We will prove that

(4) $H-F_0$ is an m-ideal in H.

Let $(X_t)_{t\in T} \subseteq H-F$. For every $t \in T$ we have $x_t \notin F_0$, $\overline{\overline{T}} \le m$, $\bigcup_{t\in T} X_t \in H$ since $x_0 \in \bigcup_{t\in T} X_t$ then $\bigcup_{t\in T} X_t \in H-F_0$.

Let now $X \subseteq Y$, $Y \in H-F_0$. Then $x_0 \notin Y$, from this and that $X \subseteq Y$, we have $x_0 \notin X$, thus $X \in H-F_0$.

From (1), (2), (3) we get that F_0 is an m-prime filter in H, and the inverse image $B^{-1}(F_0)$ is an m-prime filter in A which satisfies the conditions

$a \in B^{-1}(F_0)$, $b \in B^{-1}(F_0)$ and $A-B^{-1}(F_0)$ is an m-ideal in A.

Let A be an m-pseudo Boolean algebra satysfying (ii). We construct an m-Heyting ring H with zero which will be isomorphic with A.

Let $a \in A$, if $a \ne 0$ then $a \not\le 0$. As there exists an m-prime filter F in A, for all $a \in A$ we will define $\hat{a}$, $\hat{A}$ as follows

$$(*)' \qquad \hat{a} = \{ F \in \mathcal{T}_m(A) : a \in F \},$$

$$(**)' \qquad \hat{A} = \{ \hat{a} : a \in A \}.$$

It is easy to note that by (*)', we have $\hat{0} = 0$. We will show that $\hat{A}$ is an m-Heyting ring with zero isomorphic to A.

We will prove that the mapping $\hat{}: A \longrightarrow \hat{A}$ is one-to-one. Suppose $a \neq b$, then $a \nleq b$ or $b \nleq a$. If $a \nleq b$ then there exists an m-prime filter $F \subseteq A$ such that $a \in F$ and $b \notin F$, $F \in \hat{a}$ and $F \notin \hat{b}$ i.e. $\hat{a} \neq \hat{b}$.

Analogically we can prove that $a \neq b$ if $b \nleq a$. We have also for all $F \in \mathcal{F}_m(A)$:

(a) $F \in \hat{a} \cap \hat{b} \Leftrightarrow F \in \hat{a}$ and $F \in \hat{b} \Leftrightarrow a \in F$ and $b \in F \Leftrightarrow$

$\Leftrightarrow a \cap b \in F \Leftrightarrow F \in \widehat{a \cap b}$,

(b) $F \in \hat{a} \cup \hat{b} \Leftrightarrow F \in \hat{a}$ or $F \in \hat{b} \Leftrightarrow a \in F$ and $b \in F \Leftrightarrow$

$\Leftrightarrow a \cup b \in F \Leftrightarrow F \in \widehat{a \cup b}$,

(c) $\hat{0} = 0$.

From (a)-(c) we get that the mapping $\hat{}: A \longrightarrow \hat{A}$ is an isomorphism for the lattice $\langle A, \cup, \cap, 0 \rangle$ with the lattice $\langle \hat{A}, \cup, \cap, 0 \rangle$.

We must prove that

(i) $a \rightarrow b = a \Rightarrow b$ where $a \Rightarrow b = \bigcup\{ x \in A : a \cap x \leq b \}$

and

(ii) $\bigcap\{ \hat{a}_t : (a_t)_{t \in T} \subseteq A, \overline{\overline{T}} \leq m \} \in \hat{A}$; $\bigcap_{t \in T} \hat{a}_t = \widehat{\bigcap_{t \in T} a_t}$,

(iii) $\bigcup\{ \hat{a}_t : (a_t)_{t \in T} \subseteq A, \overline{\overline{T}} \leq m \} \in \hat{A}$, $\bigcup_{t \in T} \hat{a}_t = \widehat{\bigcup_{t \in T} a_t}$,

Proof of (i). Since $\hat{}: A \longrightarrow \hat{A}$ is an isomorphism of the lattice A with the lattice $\hat{A}$, we have

(***)' $a \cap x \leq b \Leftrightarrow \hat{a} \cap \hat{x} \leq \hat{b}$, for arbitrary a, b, $x \in A$.

For any $F \in \mathcal{J}_m(A)$ we have $F \in \widehat{a \to b} \Leftrightarrow a \to b \in F$. Thus $a \to b \in F$ and in every pseudo Boolean algebra it is true that $a \cap (a \longrightarrow b) \leq b$. Moreover

$$\exists_{x \in A} (x \in F \text{ and } a \cap x \leq b).$$

By (***)' we have

$$\exists_{x \in A} (x \in F \text{ and } \hat{a} \cap \hat{x} \leq \hat{b}) \text{ and}$$

$\exists_{x \in A} (F \in \hat{X}\colon \hat{a} \cap \hat{x} \leq \hat{b})$. Then $F \in \bigcup \{ \hat{x} \in \hat{A} : \hat{a} \cap \hat{x} \leq \hat{b} \}$ thus $F \in \hat{a} \Rightarrow \hat{b}$. Hence we have proved $\widehat{a \to b} \leq \hat{a} \Rightarrow \hat{b}$.

Let $F \in \hat{a} \Rightarrow \hat{b} \Leftrightarrow F \in \bigcup \{ \hat{x} \in \hat{A} : \hat{a} \cap \hat{x} \leq \hat{b} \} \Leftrightarrow$

$\Leftrightarrow \exists_{x \in A} (x \in F \text{ and } \hat{a} \cap \hat{x} \leq \hat{b}) \Leftrightarrow \exists_{x \in A} (x \in F \text{ and } a \cap x \leq b) \Rightarrow$

$\Rightarrow x_1 \in F (x_1 \in F \text{ and } a \cap x_1 \leq b)$ then $x_1 \in F$ and $x_1 \leq a \longrightarrow b$, thus $a \longrightarrow b \in F$, $F \in \widehat{a \longrightarrow b}$.

Proof of (ii). Let $(a_t)_{t \in T}$ be an m-indexed set of elements of A. We have $\bigcap_{t \in T} a_t \in A$ since A is an m-complete pseudo Boolean algebra. From this we have $\widehat{\bigcap_{t \in T} a_t} \in \hat{A}$. For any $F \in \mathcal{J}_m(A)$ we have also $F \in \bigcap \{ \hat{a}_t : t \in T \} \Leftrightarrow \forall_{t \in T}\, a_t \in F \Leftrightarrow \bigcap_{t \in T} a_t \in F \Leftrightarrow F \in \widehat{\bigcap_{t \in T} a_t} \in \hat{A}$ thus $\bigcap \{ \hat{a}_t : t \in T \} \in \hat{A}$.

Proof of (iii). Let $(a_t)_{t \in T}$ be an m-indexed set of elements of A. We have $\bigcup_{t \in T} a_t \in A$ since A is an m-complete pseudo Boolean algebra. We also have $\forall_{t \in T}\, \hat{a}_t \leq \widehat{\bigcup_{t \in T} a_t}$, thus

(a) $$\bigcup \{ \hat{a}_t : t \in T \} \leq \widehat{\bigcup_{t \in T} a_t} .$$

We must show that

(b) $$\widehat{\bigcup_{t\in T} a_t} \leq \bigcup \{ \hat{a}_t : t \in T \},$$

Let $F \widehat{\bigcup_{t\in T} a_t}$ and $F \notin \bigcup \{ \hat{a}_t : t \in T \}$. Then we have $\forall_{t\in T} F \notin \hat{a}_t$ thus $\forall_{t\in T} a_t \notin F$ i.e. $\forall_{t\in T} a_t \in A-F$. Hence $A-F$ is an m-ideal thus $\bigcup_{t\in T} a_t \in A-F$, $\bigcup_{t\in T} a_t \notin F$, which contradicts the hypothesis that $F \in \bigcup_{t\in T} a_t$. This ends the proof of the theorem.

BIBLIOGRAPHY

[1] H.Rasiowa, R.Sikorski: The mathematics of metamathematics. PWN, Warszawa 1968.

[2] R.Sikorski: On the representation of Boolean algebras as fields of sets, Fund. Math. 35 (1948) 247-256.

[3] R.Balbes, P.Dwinger: Distributive lattice. Missouri 1974.

[4] M.M.Stone: The theory of representation for Boolean algebras, Trans. Amer. Math. Soc. 40 (1936) 37-111.

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ON THE MAXIMUM ABSOLUTE PARTIAL SUMS OF BOUNDED P-WEAKLY MULTIPLICATIVE SYSTEMS

In this note is investigated the rate of convergence in the law of large numbers and the law of the iterated logarithm. We begin with the definition.

Definition. A sequence (X_i) of random variables is called a p − weakly multiplicative system (p − WMS) for $p \in [1,2]$, if $\sum |EX_{i_1}X_{i_2}\dots X_{i_k}|^p < \infty$, where the sum is taken over all combinations of indices.

Behaviour of such systems were investigated by many authors e.g. Alexits [1], Jakubowski [2], Móricz [3], Révész [4].

In this paper will be used the following notations: $S_n = \sum_{i=1}^{n} a_iX_i$, $A_n = \left(\sum_{i=1}^{n} |a_i|^2\right)^{\frac{1}{2}}$, $M_n = \sup_{k\le n} |S_n|$ for a sequence (a_i) of real or complex numbers, and (X_n) is p-WMS system, $1 \le p \le 2$. Moreover the constants $C_p = \left(1+\sum | EX_{i_1}X_{i_2}\dots X_{i_k} |^p\right)^{\frac{1}{p}}$ where the sum is taken over all combination of indices, are always constants connected with the p-WMS systems (X_n). In the sequel will be needed some estimations presented in [2] (cf. Theorem 7 and Corollary 4).

Theorem A [2]. Let (X_n) be a p-WMS system uniformly bounded by $K \geq 1$ and let (a_n) be a sequence of real or complex numbers. Then, for every $\lambda \geq 0$, the estimations

$$E \exp(\lambda M_n) \leq 4^{\frac{1}{p}} C_p \exp(\tfrac{1}{2} pl \lambda^2 K^2 A_n^2),$$

$$P(M_n > \lambda) \leq 4^{\frac{1}{p}} C_p \exp\left(- \frac{\lambda^2}{2plK^2A_n^2}\right)$$

hold, where $l = 1$ or $l = 2$ for real or commplex numbers (a_n), respectively.

Theorem 1. Let (X_n) be a uniformly bounded p-WMS system and let (a_n) be a sequence of complex numbers such that $A_n \xrightarrow[n\to\infty]{} \infty$. Then for every $\gamma > 0$,

$$\frac{M_n}{A_n(\ln A_n)^{\gamma}} \xrightarrow[n\to\infty]{} 0$$

holds almost surely.

Proof is similar to the proof of Theorem 10 in [2]. We ought to make only a slight changes.

In the similar way we can prove a law of the iterated logarithm.

Theorem 2. Let (X_n) be a p-WMS system uniformly bounded by $K \geq 1$. If (a_n) is a sequence of complex numbers such that $A_n \xrightarrow[n\to\infty]{} \infty$, then (X_n) satisfies

$$\limsup_{n\to\infty} \frac{M_n}{(2A_n^2 \ln\ln A_n)^{1/2}} \leq \sqrt{pl}\, K$$

almost surely, where $l = 1$ or $l = 2$ for real or complex numbers (a_n), respectively.

Proof. Fix arbitrary $\varepsilon > 0$. We can find, for every

$m > 1$, a sequence of indices (n_k) tending to infinity and such that $A^2_{n_{k-1}} \le m^k < A^2_{n_k}$.

Let $T_n = (2A_n^2 \ln\ln A_n)^{1/2}$, then by Theorem A we get

$$P\left[M_{n_k} > (1+\varepsilon) \sqrt{pl}\; KT_{n_k} \right] \le 4C_p \exp\left(- \frac{(1+\varepsilon)^2 plK^2T^2_{n_k}}{2plK^2A^2_{n_k}} \right) =$$

$$= 4C_p \exp\left[- (\ln\ln A_{n_k}) (1+\varepsilon)^2 \right] = 4C_p \left(\frac{1}{\ln A_{n_k}} \right)^{(1+\varepsilon)^2} \le$$

$$\le B_1 \left(\frac{1}{k} \right)^{(1+\varepsilon)^2}.$$

Since $\varepsilon > 0$ can be chosen arbitrary small so that, according to Borel-Cantelli lemma, we have

$$\lim_{k\to\infty} \sup \frac{M_{n_k}}{(2A^2_{n_k} \ \ln\ln A_{n_k})^{1/2}} \le \sqrt{pl}\; K$$

If we take arbitrary n different from the elements of the sequence (n_k), then there exists k such that $n_k < n < n_{k+1}$. Hence

$$\frac{M_n}{T_n} \le \frac{M_{n_k}}{T_{n_k}} \cdot \frac{T_{n_k}}{T_n} + \frac{\max\limits_{n_k<l\le n_{k+1}} | S_l - S_{n_k} |}{T'_k} \cdot \frac{T'_k}{T_n},$$

where $T'_k = \left[2 (A^2_{n_{k+1}-1} - A^2_{n_k}) \ln\ln A_{n_k} \right]^{1/2}$.

But

$$P\left(\max_{n_k<l\le n_{k+1}} | S_l - S_{n_k} | \ge \sqrt{pl}\,KT'_k \right) \le$$

$$\le 4C_p \exp\left(-2\ln\ln A_{n_k} \right) \le B_2\left(\frac{1}{k}\right)^2$$

so that

$$\limsup_{k\longrightarrow\infty} \left[\max_{n_k<l\le n_{k+1}} \frac{| S_l - S_{n_k} |}{T'_k} \right] \ge \sqrt{plK}$$

almost surely. If $n_{k+1} > n_k$, then

$$\left(\frac{T'_k}{T_n} \right)^2 \le \left(\frac{T'_k}{T_{n_k}} \right)^2 = \frac{A^2_{n_{k+1}-1} - A^2_{n_k}}{A^2_{n_k}} \le \frac{m^{k+1} - m^k}{m^k} = m-1.$$

Since $\frac{T_{n_k}}{T_n} \le 1$ for $n > n_k$ and m is an arbitrary number greater than one, we obtain the desired result taking $m\to 1$.

Modifying the method of [4,5] we can obtain the following theorems.

Theorem 3. Let (X_n) be a uniformly bounded p-WMS system and let (a_n) be a sequence of real or complex numbers such that $A_n\longrightarrow\infty$, $| a_n | = o(A_n)$. Then, for every $\alpha > 0$ and $\beta > 0$, we have

$$\sum_{n=1}^{\infty} \frac{|a_n|^2 (\ln A_n)^\beta}{A^2_n} P\left(\sup_{k\ge n} \frac{M_k}{A_k(\ln A_k)^\alpha} \ge 1 \right) < \infty.$$

Proof. Choose arbitrary $m > 1$ and let (n_k) be a sequence of indices such that

(1) $$m^{2k} \le A^2_{n_k} < m^{2k+1}$$

for k greater than or equal to some k_1. We can find such a sequence of incides because in opposite case there exist infinite many natural numbers k such that $A^2_n < m^{2k}$, A^2_{n+1}).

$\cdot\, m^{2k+1}$ for some $n = n(k)$, and then

$$\frac{|a_{n+1}|^2}{A_{n+1}^2} = 1 - \frac{A_n^2}{A_{n+1}^2} > 1 - \frac{1}{m}$$

contradicing the fact that $|a_n| = o(A_n)$. Moreover we may and do assume, for simplicity of calculations, that $A_{n_{k_1}} \ge e$. Let $K \ge 1$ denote a uniform bound for (X_n) and let $Y_n = \dfrac{M_n}{A_n(\ln A_n)^{\alpha}}$. For every $n \ge n_{k_1}$ there exists $k_0 = k_0(n)$ such that $n_{k_0} < n \le n_{k_0+1}$. Thus

$$P\left(\sup_{k\ge n} Y_k \ge 1\right) \le \sum_{k=k_0}^{\infty} P\left(\max_{n_k<l\le n_{k+1}} Y_l \ge 1\right).$$

By theorem A we have

$$q_k = P\left(\max_{n_k<l\le n_{k+1}} Y_l \ge 1\right) \le P\left(M_{n_{k+1}} \ge A_{n_k}(\ln A_{n_k})^{\alpha}\right) \le$$

$$\le 4C_p \exp\left(-\frac{A_{n_k}^2(\ln A_{n_k})^{2\alpha}}{8K^2 A_{n_{k+1}}^2}\right).$$

By (1) we get

(2) $$\frac{A_{n_k}^2(\ln A_{n_k})^{2\alpha}}{A_{n_{k+1}}^2} \ge \frac{(\ln m^k)^{2\alpha}}{m^3}$$

and

(3) $$\frac{1}{m^{2k_0+3}} < \frac{1}{A_{n_{k_0+1}}^2} < \frac{1}{A_n^2}.$$

On account of (2) and a fact that for $\alpha > 0$, $\beta > 0$, $b > 0$ there exists B_3 such that

$$\exp[-b(\ln x)^{2\alpha}] \le B_3 (\ln x)^{-2(1+\beta)} \text{ for } x \ge e$$

we have

$$q_k \le B_4(\ln m^k)^{-2(1+\beta)}, \tag{4}$$

Now by (3) and (4) we obtain

$$\sum_{k=k_0}^{\infty} q_k \le B_4 \sum_{k=k_0}^{\infty} \left(\frac{1}{\ln m^k}\right)^{2(1+\beta)} =$$

$$= B_4 \left(\frac{1}{\ln m}\right)^{2(1+\beta)} \sum_{k=k_0}^{\infty} \left(\frac{1}{k}\right)^{2(1+\beta)} \le$$

$$\le B_4 \left(\frac{1}{\ln m}\right)^{2(1+\beta)} \frac{1}{2\beta+1} \left(\frac{1}{k_0-1}\right)^{2\beta+1} \le$$

$$\le B_5 \left(\frac{1}{\ln m^{2k_0+3}}\right)^{2\beta+1} \le B_5 \left(\frac{1}{\ln A_n^2}\right)^{2\beta+1}$$

because

$$\left(\frac{1}{k_0-1}\right)^{2\beta+1} \le \frac{3^{2\beta+1}}{(2k_0+3)^{2\beta+1}} \text{ for } k_0 \ge 6 .$$

This implies assertion of theorem, since

$$\sum_{n=1}^{\infty} \frac{|a_n|^2}{\left[A_n^2\right]\left[\ln A_n\right]^{1+\beta}} < \infty \text{ for } \beta > 0 \text{ and } A_n \longrightarrow \infty . \tag{5}$$

Theorem 4. Let (X_n) be a p-WMS system uniformly bounded by $K \ge 1$. If (a_n) is a sequence of real or complex numbers such that $A_n \xrightarrow[n\to\infty]{} \infty$, $|a_n| = o(A_n)$ then, for every $\theta > 2K^2pl$, we have

$$\sum_{n=1}^{\infty} \frac{|a_n|^2}{A_n^2 \ln A_n} P\left[\sup_{k\ge n} \frac{M_k}{(\theta A_k^2 \ln\ln A_k)^{1/2}} \ge 1 \right] < \infty$$

where $l = 1$ or $l = 2$ for real or complex numbers (a_n), respectively.

Proof. Let m be an arbitrary number such that $1 < m < \left[\frac{\theta}{2K^2 pl} \right]^{1/3}$. As in the proof of Theorem 3 we can choose a sequence (n_k) such that $m^{2k} \le A_{n_k}^2 < m^{2k+1}$ for $k \ge k_1$. Let $Z_k = \frac{M_k}{(\theta A_k^2 \ln\ln A_k)^{1/2}}$, and $p_n = P\left[\sup_{k\ge n} Z_k \ge 1 \right]$. On account of (5) it is enough to prove that $p_n \le B_6 (\ln A_n)^{-\varepsilon}$, for some $\varepsilon > 0$. For every $n \ge n_{k_1}$ there exists $k_0 = k_0(n)$ such that $n_{k_0} < n \le n_{k_0+1}$. By estimation

$$p_n \le \sum_{k=k_0}^{\infty} P\left[\max_{n_k < l \le n_{k+1}} Z_l \ge 1 \right]$$

and by Theorem A we obtain

$$q_k = P\left[\max_{n_k < l \le n_{k+1}} Z_l \ge 1 \right] \le P\left[M_{n_{k+1}} \ge (\theta A_{n_k}^2 \ln\ln A_{n_k})^{1/2} \right] \le$$

$$\le 4C_p \exp\left[- \frac{\theta A_{n_k}^2 \ln\ln A_{n_k}}{2K^2 pl A_{n_{k+1}}^2} \right].$$

Since

$$\frac{A_{n_k}^2 \ln\ln A_{n_k}}{A_{n_{k+1}}^2} \ge \frac{\ln\ln m^k}{m^3},$$

it is clear that $q_k \le 4C_p \left(\frac{1}{\ln m^k}\right)^{\frac{\theta}{2K^2 p l m^3}}$. The number m was taken in such way that $d = \frac{\theta}{2K^2 p l m^3} > 1$, hence

$$p_n \le \sum_{k=k_0}^{\infty} 4C_p \left(\frac{1}{\ln m}\right)^{d} \left(\frac{1}{k}\right)^{d} \le B_7 \left(\frac{1}{\ln m}\right)^{d-1} \left(\frac{1}{k_0-1}\right)^{d-1} \le$$

$$\le B_7 \frac{3^{d-1}}{(2k+3)^{d-1}} \left(\frac{1}{\ln m}\right)^{d-1} = B_8 \left(\frac{1}{\ln m^{2k_0+3}}\right)^{d-1} \le$$

$$\le B_8 \left(\frac{1}{\ln A_n^2}\right)^{d-1}$$

because

$$A_n^2 \le A^2_{n_{k_0+1}} \le m^{2k_0+3}, \text{ and } 2k_0+3 \le 3(k_0-1) \quad \text{for } k_0 \ge 6.$$

This implies, that we can take $\varepsilon = d-1 > 0$, and the proof of theorem is complete.

REFERENCES

[1] G. Alexits: On the convergence of function series, Acta Sci. Math. /Szeged/ 34 (1973) 1-9.

[2] J. Jakubowski: Structural theorems for multiplicative systems of function, Acta Math. Sci. Hungar. 46 (1985) 111-127.

[3] F. Móricz: On the convergence properties of weakly multiplicative systems, Acta Sci. Math. /Szeged/ 38 (1976) 127-144.

[4] P. Révész: A law of the iterated logarithm for weakly

multiplicative systems and its applications, Acta Math. Acad. Sci. Hungar. 25 (1974) 425–438.

[5] R.J.Serfling : Convergence properties of S_n under moment restrictions, Ann. Math. Statist. 41 (1970) 1235–1248.

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QUALITATIVE INVESTIGATIONS OF SOLUTIONS OF SOME HYPERBOLIC EQUATIONS WITH PERIODIC COEFFICIENTS

1. In this paper, using the results in [3], [4] we shall consider behaviour of solutions of hyperbolic equations with periodic coefficients.

Consider in the domain $G = \{(x,y): 0 < x < 1, t > 0\}$ the equation

$$u_{tt} + a(t)u_t + b(t)u = u_{xx} + c(x,t), \tag{1}$$

where $a(t+\pi) = a(t)$, $b(t+\pi) = b(t)$, $a, b \in L^2(0,\pi)$, $c(x,t+\pi) = c(x,t)$, $c \in L^2[(0,1)\times(0,\pi)]$, with the initial conditions

$$u(x,0) = f(x), \qquad u_t(x,0) = F(x) \tag{IC}$$

and the boundary conditions

$$u(0,t) = u(1,t) = 0. \tag{BC}$$

In [2] this problem is considered when the involved equation has countinuous coefficients and the assumption $a(t) \geq \alpha > 0$ is imposed to assure the boundedness and stability of the solutions. In the present paper, by the periodicity of coefficients this assumption can be replaced by

$$\int_0^\pi a(t)dt > 0 .$$

Moreover the case $a(t) \equiv 0$ is also investigated. As in [2] being not interested in consideration of the uniqueness of solution we always assume this property of the problem.

In the sequel all derivatives are understood in the Sobolev sense and the boundary values of functions in Sobolev spaces are referred to as the traces (see [1]).

Denote $G_T = \{(x,y) : x \in (0,1) , t \in (0,T) \}$.

Definition 1. Let a sequence of functions u_k be given so that

(i) u_k is an almost everywhere solution of equation

$$u_{tt} + a(t)u_t + b(t)u = u_{xx} + c_k(x,t)$$

and satisfy (BC) (see [1]),

(ii)
$$u_k\Big|_{t=0} \longrightarrow f \quad \text{as} \quad k \longrightarrow \infty \quad \text{in } H^1(0,1) \quad ,$$

$$\frac{\delta u_k}{\delta t}\Big|_{t=0} \longrightarrow F \quad \text{as } k \longrightarrow \infty \quad \text{in } L^2(0,1) \quad ,$$

$$c_k \longrightarrow c \quad \text{as} \quad k \longrightarrow \infty \quad \text{in } L^2(G_{\Pi}) \quad .$$

We shall say that a function u is a generalized solution of the problem (1), (IC), (BC) in the domain G if it is the limit of u_k in $H^1(G_T)$ for every $t > 0$.

Definition 2. A generalized solution u of the problem (1), (IC), (BC) is said to be stable if for each $\varepsilon > 0$ there exists $\delta > 0$ such that for every $\bar{f}$ and for every $\bar{F}$ satisfying

$$\| f - \bar{f} \|_{H^1(0,1)} \leq \delta \quad , \quad \| F - \bar{F} \|_{L^2(0,1)} \leq \delta$$

and if $\bar{u}$ is a generalized solution of the problem (1), ($\overline{IC}$),

$(\overline{BC})$, where

$(\overline{IC})$ $$\bar{u}(x,0) = \bar{f}(x) \quad , \quad \bar{u}_t(x,0) = \bar{F}(x) \quad ,$$

we have for all $t \ge 0$

$$\| u - \bar{u} \|^2_{H^1(0,1)} + \| \frac{\delta}{\delta t}(u - \bar{u}) \|_{L^2(0,1)} \le \varepsilon \quad .$$

Definition 3. A generalized solution of the problem (1), (IC) and (BC) is said to be asymptotically stable if it is stable and

$$\lim_{t \to \infty} (\| u - \bar{u} \|^2_{H^1(0,1)} + \| \frac{\delta}{\delta t}(u - \bar{u}) \|^2_{L^2(0,1)}) = 0 \quad .$$

We shall say that a function φ satisfies $B(r)$, $r \ge 0$, if

(i) $$\varphi \in H^r(0,1) \quad ,$$

(ii) $$\varphi(0) = \varphi(1) = 0 \quad ,$$

(iii) $\varphi^{2k}(0) = \varphi^{2k}(1) = 0$ for $0 < 2k < r$, $k \in N$.

Lemma 1. If φ satisfies $B(r)$, then the series $\sum_{n=1}^{\infty} |p_n| n^{r-1}$, $\sum_{n=1}^{\infty} p_n^2 n^{2r}$ are convergent and

$$\sum_{n=1}^{\infty} p_n^2 n^{2r} \le \frac{2l^{2r-1}}{\Pi^{2r}} \| \varphi^{(r)}(x) \|^2_{L^2(0,1)} ,$$

where

$$p_n = \frac{2}{l} \int_0^1 \varphi(x) \sin \frac{n\Pi}{l} x dx \ .$$

Proof. Integrating r times the expression p_n by parts we get

$$p_n = R(r) \frac{l^r}{n^r \Pi^r} \frac{2}{l} \int_0^1 \varphi^r(x) S(x) dx \quad ,$$

where $R(r) = \pm 1$ and

$$S(x) = \begin{cases} \sin\frac{n\Pi}{l}x & \text{for } r = 2k \quad , \quad k \in N , \\ \cos\frac{n\Pi}{l}x & \text{for } r = 2k-1 \quad , \quad k \in N . \end{cases}$$

Therefore

$$p_n \frac{n^r \Pi^r}{l^r} R(r) = \frac{2}{l} \int_0^l \varphi^{(r)}(x) S(x) dx.$$

Regarding the right-hand side as coefficient of the Fourier expansion of the function $\varphi^{(r)}$ we get the desired result.

Lemma 2. Let c belong to $H^{r,0}(G_\Pi)$ satisfy r) (t is regarded as a parameter) and

$c(x,t+\Pi) = c(x,t)$. THen the series $\sum_{n=1}^{\infty} C_n n^{r-1}$, $\sum_{n=1}^{\infty} C_n^2 n^{2r}$ are convergent, where

$$C_n = \int_0^{\pi} |c_n(t)| dt \quad , \quad c_n(t) = \frac{2}{l} \int_0^l c(x,t) \sin\frac{n\Pi}{l}x dx \ .$$

Proof. By Lemma 1, for almost all $t \in (o,\Pi)$ the series $\sum_{n=1}^{\infty} c_n^2(t) n^{2r}$ is convergent and

$$\sum_{n=1}^{\infty} c_n^2(t) n^{2r} \le 2 \frac{l^{2r-1}}{\Pi^{2r}} \left\| \frac{\delta^r}{\delta x^r} c(x,t) \right\|^2_{L^2(0,l)} .$$

From this we have

$$\sum_{n=1}^{\infty} \int_0^{\Pi} c_n^2(t) n^{2r} \le 2 \frac{l^{2r-1}}{\Pi^{2r}} \left\| \frac{\delta^r}{\delta x^r} c(x,t) \right\|^2_{L^2(G_\Pi)} .$$

Applying Cauchy inequality we get

$$\sum_{n=1}^{\infty} C_n n^{r-1} \le \frac{\Pi^{\frac{3}{2}}}{\sqrt{6}} \left(\sum_{n=1}^{\infty} \int_0^{\Pi} c_n^2(t) n^{2r} dt \right)^{\frac{1}{2}} ,$$

$$\sum_{n=1}^{\infty} C_n^2 n^{2r} \le \Pi \sum_{n=1}^{\infty} \int_0^{\Pi} c_n^2(t) n^{2r} dt \quad .$$

2. Consider the following homogeneous equation

$$v_{tt} + a(t)v_t + b(t)v = v_{xx} \tag{2}$$

with the initial conditions

$$v(x,0) = f(x) \quad , \quad v_t(x,0) = F(x) \quad , \tag{IC}$$

and the boundary conditions

$$v(0,t) = v(1,t) = 0 \quad . \tag{BC}$$

Theorem 1. If the functions a, b satisfy the hypotheses of Theorem 1 in [4], f satisfies B(1) and F satisfies B(0), then for the problem (2), (IC), (BC) there exists a generalized solution, which is continous, bounded in $\overline{G}$ and asymptotically stable.

Proof. We shall seek a solution of (2) in the form

$$v(x,t) = \sum_{n=1}^{\infty} X_n(x)\, T_n(t) \tag{3}$$

so that each function $X_n T_n$ is an almost everywhere solution of (1), (BC). From this

$$X_n(x) = \sin \frac{n\Pi}{1} x \quad , \quad T_n(t) = p_{n1} T_{n1}(t) + p_{n2} T_{n2}(t) \quad , \tag{4}$$

where T_{n1}, T_{n2} are the solutions of the equation

$$T'' + a(t)T' + \left[\frac{n^2\Pi^2}{1^2} + b(t) \right] T = 0 \tag{5}$$

satisfying

$$T_{n1}(0) = 1 \ , \ T'_{n1}(0) = 0 \ , \ T_{n2}(0) = 0 \ , \ T'_{n2}(0) = 1 \quad . \tag{6}$$

By Theorem 1 in [4] there exists a constant $D > 0$ independent of n such that the solutions T_{n1}, T_{n2} can be estimated as follows

$$(7)\qquad \begin{cases} |T_{n1}(t)| \le De^{-\varepsilon t} , & |T'_{n1}(t)| \le Dne^{-\varepsilon t} , \\ |T_{n2}(t)| \le Dn^{-1}e^{-\varepsilon t} , & |T'_{n2}(t)| \le De^{-\varepsilon t} . \end{cases}$$

Choose P_{n1}, P_{n2} so that v satisfies (IC). By (3)-(6) we have

$$P_{n1} = \frac{2}{l}\int_0^l f(x)\sin\frac{n\Pi}{l}x\,dx \quad , \; P_{n2} = \frac{2}{l}\int_0^l F(x)\sin\frac{n\Pi}{l}x\,dx \; .$$

Put $v_k(x,t) = \sum_{n=1}^{k} X_n(x)T_n(t)$. It is easy to check that v_k is an almost everywhere solution of (2) satisfying (BC) and

$$v_k\Big|_{t=0} \longrightarrow f \;\text{ as }\; k \longrightarrow \infty \;\text{ in }\; H^1(0,l) \; ,$$

$$\frac{\delta v_k}{\delta t}\Big|_{t=0} \longrightarrow F \;\text{ as }\; k \longrightarrow \infty \;\text{ in }\; L^2(0,l) \; ,$$

By (3), (4), (7) we have

$$\| v_k - v_j \|^2_{H^1(0,l)} = \int_0^l \left\{ \Big(\sum_{n=j+1}^{k} X_n T_n \Big)^2 + \Big(\sum_{n=j+1}^{k} X'_n T_n \Big)^2 \right\} dx =$$

$$= \int_0^l \sum_{n=j+1}^{k} T_n^2 (X_n^2 + X_n'^2)\, dx \le$$

$$\le lD^2 \sum_{n=j+1}^{k} (P_{n1}^2 + \frac{1}{n^2} P_{n2}^2) (1 + \frac{n^2\Pi^2}{l^2}) \; ,$$

$$\Big\| \frac{\delta}{\delta t}(v_k - v_j) \Big\|^2_{L^2(0,l)} = \int_0^l \Big(\sum_{n=j+1}^{k} X_n T'_n \Big)^2 dx = \sum_{n=j+1}^{k} \int_0^l X_n^2 T_n'^2 \le$$

$$\le D^2 l \sum_{n=j+1}^{k} (p_{n1}^2 n^2 + p_{n2}^2) .$$

Hence for arbitrary positive T we have

$$\| v_k - v_j \|^2_{H^1(G_T)} \le T D^2 l \sum_{n=j+1}^{k} (n^2 p_{n1}^2 + p_{n2}^2) (1 + \frac{1}{n^2} + \frac{\Pi^2}{l^2}).$$

By Lemma 1 the sequence v_k is convergent in $H^1(G_T)$. On the other hand, for all $(x,t) \in \overline{G}_T$,

$$| X_n(x) \; T_n(t) | \le D (|p_{n1}| + \frac{1}{n} |p_{n2}|) .$$

So, the sequence v_k is uniformly convergent in $\overline{G}_T$ and therefore the function v is continuous and bounded by the constant $D \sum_{n=1}^{\infty} \left[|P_{n1}| + \frac{1}{n} |P_{n2}| \right]$. This constant is independent of T, so v is bounded in $\overline{G}$.

Let $\overline{v}$ be a generalized solution of (2), $(\overline{IC})$, (BC). Similarly we have

$$\| v - \overline{v} \|^2_{H^1(0,l)} + \left\| \frac{\delta}{\delta t}(v - \overline{v}) \right\|^2_{L^2(0,l)} \le$$

$$\le l D^2 (2 + \frac{\Pi^2}{l^2}) \left\{ \| (v - \overline{v})|_{t=0} \|^2_{H^1(0,l)} + \right.$$

$$\left. + \left\| \frac{\delta}{\delta t}(v - \overline{v})|_{t=0} \right\|^2_{L^2(0,l)} \right\} e^{-\varepsilon t} ,$$

which implies the asymptotically stability of the solution v .

3. Now we consider the following nonhomogenuous equation

(1) $$u_{tt} + a(t)u_t + b(t)u = u_{xx} + c(x,t) .$$

Theorem 2. If the functions a, b, f and F satisfy the hypotheses of Theorem 1 and c satisfies B(0) (t is regarded as a parameter), then for the problem (1), (IC), (BC) there exists a generalized solution, which is continuous, bounded in $\bar{G}$ and asymptotically stable .

Proof. We shall seek a solution of this problem in the form $u(x,t) = v(x,t) + U(x,t)$, where v is the solution of (2), (IC), (BC) and U is the solution of (1) satisfying

$$\text{(IC)}' \qquad U(x,0) = U_t(x,0) = 0 \, ,$$

$$\text{(BC)} \qquad U(0,t) = U(1,t) = 0 \, .$$

By Theorem 1 is suffices to prove that there exists a solution of (1), (IC)', (BC) which is continuous and bounded in $\bar{G}$. We seek a solution of (1) in the form

$$U(x,t) = \sum_{n=1}^{\infty} T_{n0}(t) \sin \frac{n\Pi}{1}x \, ,$$

where T_{n0} is the solution of the eqation

$$T'' + a(t)T' + \left[\frac{n^2\Pi^2}{1^2} + b(t) \right] T = c_n(t)$$

with $c_n(t) = \frac{2}{1} \int_0^1 c(x,t) \sin \frac{n\Pi}{1}x \, dx$, satisfying

$T_{n0}(0) = T'_{n0}(0) = 0$. It is an easy matter to see that $U_k(x,t) = \sum_{n=1}^{k} T_{n0}(t) \sin \frac{n\Pi}{1}x$ is an almost everywhere solution of the equation

$$u_{tt} + a(t)u_t + b(t)u = u_{xx} + \sum_{n=1}^{k} c_n(t) \sin \frac{n\Pi}{1}x$$

satisfying (IC)', (BC) and $\sum_{n=1}^{k} c_n(t) \sin \frac{n\Pi}{l}x \longrightarrow c(x,t)$ as $k \longrightarrow \infty$ in $L^2(G_\Pi)$.

Due to Theorem 2 in [4] we have

$$|T_{n0}(t)| \le KC_n n^{-1} \quad , \quad |T'_{n0}(t)| \le KC_n \quad ,$$

where K is a constant independent of n and

$$C_n = \int_0^\Pi |c_n(t)|\, dt.$$

By Lemma 2 the series

$$\sum_{n=1}^{\infty} C_n n^{-1} \quad , \quad \sum_{n=1}^{\infty} C_n^2$$

are convergent. The remainder is proved as in the proof of Theorem 1.

4. Consider the cases where $a \equiv 0$:

(8) $$u_{tt} + b(t)u = u_{xx} \ ,$$

(9) $$u_{tt} + b(t)u = u_{xx} + c(x,t) \ .$$

We shall say that a function b satisfies A(r), $r \ge 0$, if there exists a constant $K > 0$ independent of n such that the solutions y_{n1}, y_{n2}, satisfying

$$y_{n1}(0) = 1 \quad , \quad y'_{n1}(0) = 0 \quad ,$$

$$y_{n2}(0) = 0 \quad , \quad y'_{n2}(0) = 1$$

of the equations $y'' + \left[\frac{n^2\Pi^2}{l^2} + b(t) \right] y = 0$ can be estimated as follows

$$|y_{n1}(t)| \le Kn^{r} \quad , \quad |y'_{n1}(t)| \le Kn^{r+1} \quad ,$$

$$|y_{n2}(t)| \le Kn^{r-1} \quad , \quad |y'_{n2}(t)| \le Kn^{r} \quad .$$

We shall say that a function b satisfies $A(r,s)$, $r \ge 0$, $s \ge -1$, if it satisfies $A(r)$ and the solutions z_{n0} satisfying

$$z_{n0}(0) = z'_{n0}(0) = 0$$

of the equations $z'' + \left[\frac{n^2\Pi^2}{l^2} + b(t) \right] z = c_n(t)$ can be estimated as follows

$$|z_{n0}(t)| \le KC_n n^{r+s} \quad , \quad |z'_{n0}(t)| \le KC_n n^{r+s+1} \quad ,$$

where $C_n = \int_0^{\Pi} |c_n(t)|\, dt$.

It is added that in [3] sufficient conditions for b to satisfy $A(r)$ and $A(r,s)$ are established.

Theorem 3. Let b satisfy $A(r)$, f satisfy $B(r+1)$ and F satisfy $B(r)$. Then the problem (8), (IC), (BC) has a generalized solution which is continuous and bounded in $\overline{G}$. Moreover, if $r = 0$, it is stable.

Theorem 4. Let b satisfy $A(r,s)$, f satisfy $B(r+1)$ and F satisfy $B(r)$, c satisfy $B(r+s+1)$ (t is regarded as a parameter) and $c \in H^{r+s+1,0}(G_{\Pi})$. Then the problem (9), (IC), (BC) has a generalized solution which is continuous and bounded in $\overline{G}$. Moreover, if $r = 0$ it is stable.

The proofs of Theorems 3–4 are similar to the proofs of Theorems 1–2 .

In a similar way we can consider the problems with the boundary conditions

$$u(0,t) = u_x(1,t) = 0$$

or

$$u_x(0,t) = u_x(1,t) = 0 \, .$$

REFERENCES

[1] В.П.Михайлов : Дифференцялъные уравнения в частных проувводных. Моцква 1983 .

[2] J. Muszyński : Badanie jakościowe rozwiązań niektórych równań typu hiperbolicznego, Zeszyty Naukowe Politech. Warszaw., Matematyka 13 (1967).

[3] Pham ba Trung : Common behaviour of solutions of some collections of Hill's equations, Demonstratio Math. 19 (1986) 1095–1100.

[4] Pham ba Trung : Common behaviour of solutions of some collections of second order linear ordinary differential equations with periodic coefficients, Demonstratio Math. 20 (1987) (in print).

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ON THE EXISTENCE OF A SOLUTION OF THE MIXED PROBLEM FOR A CERTAIN HYPERBOLIC SYSTEM.

Introduction

In the present paper we give a theorem on the existence of a solution of the mixed problem for a certain system of 2n partial linear differential equations of the first order with two independent variables. This system will be written in the following matricial form

$$A(x,t)D_t\tilde{u} + D_x\tilde{u} + B(x,t)\tilde{u} = g(x,t) .$$

We assume that there exists a solution of the corresponding mixed problem for the canonical form of the given system.

We prove lemmas concerning the properties of the roots of the characteristic polynomial of the matrix $A^{-1}(x,t)$ at any point of the set on which the mixed problem is considered. By means of these lemmas we prove that the system of equations in question is of the hyperbolic type. Moreover, proceeding as in paper [1] we establish a theorem on the existence of a solution of the mixed problem for the considered system of equations.

We introduce the following notations: $\overline{R}_+ = [0, +\infty[$, $X = [0,\alpha]$, where α is a positive constant, $\Omega = X \times \overline{R}_+$ and we assume the following definitions on the set Ω :

$$u(x,t):= \mathrm{col}\,[u_1(x,t),\ u_2(x,t),\ldots u_n(x,t)] \tag{1}$$

$$i(x,t):= \mathrm{col}\,[i_1(x,t),\ i_2(x,t),\ldots i_n(x,t)] \tag{2}$$

$$\tilde{u}(x,t):= \mathrm{col}\,[\tilde{u}^{(1)}(x,t),\ \tilde{u}^{(2)}(x,t)]\ , \tag{3}$$

where

$$\begin{cases} \tilde{u}^{(1)}(x,t):= u(x,t)\ , \\ \tilde{u}^{(2)}(x,t):= i(x,t)\ ; \end{cases} \tag{4}$$

$$L(x,t):= [\ l_{kj}(x,t)\ ]_{(n,n)} \tag{5}$$

$$C(x,t):= [\ c_{kj}(x,t)\ ]_{(n,n)} \tag{6}$$

$$R(x,t):= [\ r_{kj}(x,t)\ ]_{(n,n)} \tag{7}$$

$$G(x,t):= [\ g_{kj}(x,t)\ ]_{(n,n)} \tag{8}$$

$$A(x,t):= \begin{bmatrix} O_n & L(x,t) \\ C(x,t) & O_n \end{bmatrix} \tag{9}$$

$$B(x,t):= \begin{bmatrix} O_n & R(x,t) \\ G(x,t) & O_n \end{bmatrix} \tag{10}$$

$$K(x,t):= \mathrm{Diag}\,[\ K_1(x,t),\ K_2(x,t)\ ]\ , \tag{11}$$

where

$$\begin{cases} K_1(x,t):= \mathrm{diag}\,[\ k_1(x,t),\ k_2(x,t),\ \ldots,\ k_n(x,t)\ ], \\ K_2(x,t):= \mathrm{diag}\,[\ k_{n+1}(x,t),\ k_{n+2}(x,t),\ \ldots,\ k_{2n}(x,t)\ ], \end{cases} \tag{12}$$

$$M(x,t):= [\ m_{kj}(x,t)\ ]_{(2n,2n)} \tag{13}$$

$$W(x,t):= \mathrm{col}\,[\ w^{(1)}(x,t),\ w^{(2)}(x,t)\ ]\ , \tag{14}$$

where

$$(15)\quad \begin{cases} w^{(1)}(x,t) := \text{col}\ [\ w_1(x,t),\ w_2(x,t),\ \dots,\ w_n(x,t)\ ] \\ w^{(2)}(x,t) := \text{col}\ [\ w_{n+1}(x,t),\ w_{n+2}(x,t),\ \dots,\ w_{2n}(x,t)\ ] \end{cases},$$

where $w(x,t)$ denotes anyone of the vector-functions $g(x,t)$, $f(x,t)$, $v(x,t)$, $\tilde{v}(x,t)$, $\tilde{\tilde{u}}(x,t)$.

We introduce, moreover, the following definitions:

$$(16)\qquad d(x) := \text{col}\ [\ d^{(1)}(x),\ d^{(2)}(x)\ ]\ ,$$

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where

$$(17)\quad \begin{cases} d^{(1)} := \text{col}\ [\ d_1(x),\ d_2(x),\ \dots,\ d_n(x)\ ]\ , \\ d^{(2)} := \text{col}\ [\ d_{n+1}(x),\ d_{n+2}(x),\ \dots,\ d_{2n}(x)\ ]\ , \end{cases} \quad x \in X$$

where $d(x)$ denotes anyone of the vector-functions $p(x)$, $a(x)$.

$$(18)\quad \begin{cases} E_{12}(t) := [\ e_{kj}(t)\ ]_{(n,n)}\ ,\ k = 1,\ 2,\ \dots,\ n\ , \\ E_{21}(t) := [\ e_{jk}(t)\ ]_{(n,n)}\ ,\ j = n+1,\ n+2,\ \dots,\ 2n\ ; \end{cases} \quad t \in \bar{R}_+ .$$

Assume next the following notations:

$\tilde{C}^{(2)}(\Omega)$ – the class of functions $s: \Omega \longrightarrow R$ continuous on Ω and satisfying the conditions:

1) $s \in C^1(\Omega)$, 2) $D_t D_x s,\ D_t^2 s \in C(\Omega)$;

$\tilde{C}^{(3)}(\Omega)$ – the class of functions $s: \Omega \longrightarrow R$ continuous on Ω and satisfying the conditions:

1) $s \in C^1(\Omega)$, 2) $D_x D_t s,\ D_t^2 s,\ D_t^3 s \in C(\Omega)$;

$\tilde{C}^{(4)}(\Omega)$ – the class of functions $s: \Omega \longrightarrow R$ continuous on Ω and satisfying the conditions:

1) $s \in C^3(\Omega)$, 2) $D_t^3 D_x s,\ D_t^4 s, \in C(\Omega)$;

Remark 1. The results established in paper [1] remain unaffected, if we replace the definitions of classes of functions $\tilde{C}^{(2)}(\Omega)$, $\tilde{C}^{(3)}(\Omega)$ introduced in [1] by the above stated definitions. Hence we shall make use in the sequel of the results established in paper [1].

Consider the system of 2n linear partial differential equations of the first order with the unknown functions u_k, i_k: $\Omega \longrightarrow R$, $k = 1, 2, \ldots, n$, represented in the matricial notation by the equation

$$\begin{bmatrix} O_n & L(x,t) \\ C(x,t) & O_n \end{bmatrix} D_t \begin{bmatrix} u \\ i \end{bmatrix} + D_x \begin{bmatrix} u \\ i \end{bmatrix} + \tag{19}$$

$$+ \begin{bmatrix} O_n & R(x,t) \\ G(x,t) & O_n \end{bmatrix} \begin{bmatrix} u \\ i \end{bmatrix} = \begin{bmatrix} g^{(1)}(x,t) \\ g^{(2)}(x,t) \end{bmatrix} .$$

In this equation $L(x,t)$, $C(x,t)$, $R(x,t)$, $G(x,t)$, $g^{(\gamma)}(x,t)$, $\gamma = 1,2$, denote given real functional matrices defined by the formulas (5)–(8), (15) such that $L, C \in \tilde{C}^{(4)}(\tilde{\Omega})$, $R, G, g^{(\gamma)} \in \tilde{C}^{(3)}(\tilde{\Omega})$, $\gamma = 1,2$, (see [1]).

According to (3), (4), (9), (10), (15) the equation (19) will take the form

$$A(x,t) D_t \tilde{u} + D_x \tilde{u} + B(x,t) \tilde{u} = g(x,t) . \tag{20}$$

Taking into account the definitions (1)–(4), (16), (17) we pose for the equation (20) the initial condition

$$\tilde{u}(x,0) = p(x),\ x \in X, \tag{21}$$

with the given vector-function $p \in C^3(X)$, and the homogeneous boundary conditions

$$(22)\qquad \begin{cases} \tilde{u}^{(1)}(0,t) = 0_{(n,1)}\,, \\ \tilde{u}^{(2)}(\alpha,t) = 0_{(n,1)}\,, \end{cases} \qquad t \in \bar{R}_+$$

under the assumption that the compatibility conditions

$$(23)\qquad \begin{cases} p^{(1)}(0) = 0_{(n,1)}\,, \\ p^{(2)}(\alpha) = 0_{(n,1)}\,, \end{cases}$$

are satisfied.

The problem $\tilde{U}(\Omega)$. Assuming that the compatibility conditions (23) are satisfied and that $A \in \tilde{C}^4(\tilde{\Omega})$, B, $g \in \tilde{C}^3(\tilde{\Omega})$, $p \in C^3(X)$ we have to find on the set Ω the classical solution of the problem (20)–(22).

Remark 2. If the matrices $L(x,t)$, $G(x,t)$ are symmetric and positive definite on the set Ω and if the matrices $R(x,t)$, $G(x,t)$ are diagonal on this set, then the equation (20) respresents a system of equations for n long lines.

Let us now consider the canonical form of a system of $2n$ linear partial differential equations of the first order, represented in matricial form by the equation

$$(24)\qquad D_t v + K(x,t)D_x v + M(x,t)v = f(x,t)\ ,$$

where $K(x,t)$, $M(x,t)$, $f(x,t)$ denote given real functional matrices of the class $\tilde{C}^3(\Omega)$ defined by the formulas (11)–(15) and the elements of the diagonal matrix $K(x,t)$ satisfy the condition

$$(25)\qquad \operatorname{sgn} k_i(x,t) = \begin{cases} 1 & \text{for } i = 1,2,\dots,n \\ -1 & \text{for } i = n+1,n+2,\dots,2n \end{cases} \qquad (x,t) \in \Omega.$$

Taking into account the definitions (14)-(18) we pose for the equation (24) the initial condition

$$(26)\qquad v(x,0) = a(x)\ ,\quad x \in X\ ,$$

with the given vector-function $a \in C^3(X)$, the boundary conditions

$$(27)\qquad \begin{cases} v^{(1)}(0,t) = E_{12}(t)v^{(2)}(0,t)\ , \\ v^{(2)}(\alpha,t) = E_{21}(t)v^{(1)}(\alpha,t)\ , \end{cases} \qquad t \in \bar{R}_+\ ,$$

with the given functional matrices E_{12}, $E_{21} \in C^3(\bar{R}_+)$, under the assumption that the compatibility conditions

$$(28)\qquad \begin{cases} a^{(1)}(0) = E_{12}(0)a^{(2)}(0)\ , \\ a^{(2)}(\alpha) = E_{21}(0)a^{(1)}(\alpha)\ , \end{cases}$$

are satisfied.

The problem $V(\Omega)$. Assuming that the compatibility conditions (28) are satisfied, (25) holds and K, M, $f \in C^3(\Omega)$, $a \in C^3(X)$, $E_{12}, E_{21} \in C^3(\bar{R}_+)$ we have to find on the set Ω the classical solution of the problem (24), (26), (27).

1. Some auxiliary lemmas

In this paragraph we give some lemmas which we shall use in the proofs of theorems stated in the next paragraph.

Let us make the following assumptions:

A.1. The matrices $L(x,t)$, $C(x,t)$ are symmetric and positive definite on the set Ω.

A.2. On the set Ω the multiplicities of all the roots of the characteristic polynomial of the matrix $L^{-1}(x,t)C^{-1}(x,t)$ are constant.

A.3. $L,C \in \tilde{C}^4(\tilde{\Omega})$ or A.3' $L,C \in C^1(\tilde{\Omega})$.

Lemma 1 (1'). Under the assumptions A.1 – A.3 (A.1, A.2, A.3') the roots $\lambda_j(x,t)$, $j = 1,2,\dots,2n$, of the characteristic polynomial of the matrix $A^{-1}(x,t)$ satisfy on the set Ω the following conditions:

(1a) The functions λ_j, $j = 1,2,\dots,2n$, are real functions.

(1b)
$$\operatorname{sgn} \lambda_j(x,t) = \begin{cases} 1 & \text{for } j = 1,2,\dots,n\ , \\ -1 & \text{for } j = n+1,n+2,\dots,2n\ , \end{cases}$$

where $\lambda_{n+j}(x,t) = -\lambda_j(x,t)$, $j = 1,2,\dots,n$.

(1c) The multiplicities of all the roots of the characteristic polynomial of the matrix $A^{-1}(x,t)$ are constant.

(1d) $\lambda_j \in \tilde{C}^4(\Omega)$ ($\lambda_j \in C^1(\Omega)$), $j = 1,2,\dots,2n$.

Proof. From the assumption A.1 it follows that on the set Ω the matrix $A(x,t)$ is non-singular and that

(1.1)
$$A^{-1}(x,t) = \begin{bmatrix} O_n & C^{-1}(x,t) \\ L^{-1}(x,t) & O_n \end{bmatrix} .$$

From the assumption A.1 and from the Lemma 2 of paper [2] it follows that the equation

(1.2)
$$\det [\ J_{2n} - \lambda A(x,t)\ ] = 0$$

has n real roots $\lambda_j(x,t)$, $j = 1,2,\ldots,n$, on the set Ω which assume on Ω positive values, and n real roots $\lambda_{n+1}(x,t) = -\lambda_j(x,t)$, $j = 1,2,\ldots,n$. Moreover, as follows from the proof of this lemma, for any $(x,t) \in \Omega$ and $\lambda \in R$ we have the equality

$$(1.3) \qquad \det [\, J_{2n} - \lambda A(x,t) \,] =$$

$$= \det [\, C(x,t)L(x,t) \,] \det [\, L^{-1}(x,t)C^{-1}(x,t) - \nu J_n \,] ,$$

where

$$(1.4) \qquad \nu := \lambda^2 .$$

The characteristic polynomial of the matrix $A^{-1}(x,t)$ has the form

$$(1.5) \qquad h(\lambda,x,t) := \det [\, A^{-1}(x,t) - \lambda J_{2n} \,] ,$$

whence follows the equality

$$(1.6) \qquad h(\lambda.x,t) = \lambda^{2n} - h_1(x,t)\lambda^{2n-1} + h_2(x,t)\lambda^{2n-2} +$$

$$- \ldots + h_{2n}(x,t) ,$$

where $h_j(x,t)$, $j = 1,2,\ldots,2n$, is the sum of the principal minors of degree j of the matrix $A^{-1}(x,t)$.

From the equalities (1.2), (1.5) and from assumption A.1 we get for any $(x,t) \in \Omega$ and $\lambda \in R$ the equivalence

$$h(\lambda,x,t) = 0 \Leftrightarrow \det [\, J_{2n} - \lambda A(x,t) \,] = 0 ,$$

whence it follows that on the set Ω the roots $\lambda_j(x,t)$, $j = 1,2,\ldots,2n$, of the polynomial $h(\lambda,x,t)$ are real functions satisfying the condition

$$\operatorname{sgn} \lambda_j(x,t) = \begin{cases} 1 & \text{for } j = 1,2,\ldots,n \ , \\ -1 & \text{for } j = n+1, n+2, \ldots, 2n \ , \end{cases}$$

where $\lambda_{n+j}(x,t) = -\lambda_j(x,t)$, $j = 1,2,\ldots,n$.

The equalities (1.1), (1.3), the definition (1.4) and Cauchy's theorem on the determinant of the product of matrices imply on the set Ω the equality

$$(1.7) \qquad \det [\, A^{-1}(x,t) - \lambda J_{2n} \,] =$$

$$= (-1)^n \det [\, L^{-1}(x,t) C^{-1}(x,t) - \nu J_n \,] .$$

By the equalities (1.5), (1.7) and assumption A.2, we conclude that on the set Ω the multiplicities of all the roots of the polynomial $h(\lambda,x,t)$ are constant.

From assumptions A.1, A.3 (A.3') and from equality (1.1) follows that there holds

$$(1.8) \qquad A^{-1}(x,t) \in \tilde{C}^4(\tilde{\Omega}) \qquad (A^{-1}(x,t) \in C^1(\tilde{\Omega})) ,$$

and, consequently, $h_j \in \tilde{C}^4(\tilde{\Omega})$, ($h_j \in C^1(\tilde{\Omega})$), $j = 1,2,\ldots 2n$. Hence it follows, by equality (1.6), that we have

$$h \in \tilde{C}^{(\infty,4)}(R \times \tilde{\Omega}) \qquad (h \in C^{(\infty,1)}(R \times \tilde{\Omega})) \quad \text{(see [1])}.$$

Making use of (1c) and proceeding similarly to the proof of Lemma 1 from paper [1] we get the condition

$$\lambda_j \in \tilde{C}^4(\Omega) \quad (\lambda_j \in C^1(\Omega)) , \ j = 1,2,\ldots,2n ,$$

which achieves the proof of Lemma 1 (1').

Let us now introduce the notation

$$(1.9) \qquad \nu_k(x,t) := \lambda_k^2(x,t), \quad (x,t) \in \Omega \ .$$

L e m m a 2. Under the assumptions A.1, A.2 we have, for any $(x,t) \in \Omega$ the equality

$$r\,[\, A^{-1}(x,t) - \lambda_k(x,t)J_{2n} \,] = 2n - n_k \ ,$$

where n_k denotes the multiplicity of the root $\lambda_k(x,t)$ of the characteristic polynomial of the matrix $A^{-1}(x,t)$ at any point $(x,t) \in \Omega$.

P r o o f. From Lemma 1 and from the definition of the number n_k it follows that for any $(x,t) \in \Omega$ the multiplicities of the roots $\lambda_k(x,t)$ and $-\lambda_k(x,t)$ of the polynomial $h(\lambda,x,t)$ are equal to n_k. Hence we infer from (1.7) and (1.9) that for any $(x,t) \in \Omega$ the multiplicity of the root $\nu_k(x,t)$ of the characteristic polynomial of the matrix $L^{-1}(x,t)C^{-1}(x,t)$ is equal to n_k. Thence it follows, by assumption A.1, that the matrices $L(x,t)$, $C^{-1}(x,t)$ and the function $\nu_k = \nu_k(x,t)$ satisfy on the set Ω the assumptions of Lemma 2 of paper [1]. By virtue of this lemma, there holds for any $(x,t) \in \Omega$ the condition

$$r[\, C^{-1}(x,t) - \nu_k(x,t)L(x,t) \,] = n - n_k \ ,$$

and, by A.1, the condition

$$r[\, L^{-1}(x,t)C^{-1}(x,t) - \nu_k(x,t)J_n \,] = n - n_k. \tag{1.10}$$

From Lemma 1 follows that for any $(x,t) \in \Omega$ we have the inequality

(1.11 $$\lambda_k(x,t) \neq 0 \ .$$

From (1.9)-(1.11) follows that for any $(x,t) \in \Omega$ we have

$$r\left[\, \frac{1}{\lambda_k(x,t)}\, L^{-1}(x,t)C^{-1}(x,t) - \lambda_k(x,t)J_n \,\right] = n - n_k. \tag{1.12}$$

On the other hand, on the set Ω the following equality is true

$$\begin{bmatrix} J_n & O_n \\ \frac{1}{\lambda_k(x,t)}L^{-1}(x,t) & J_n \end{bmatrix} \begin{bmatrix} -\lambda_k(x,t)J_n & C^{-1}(x,t) \\ L^{-1}(x,t) & -\lambda_k(x,t)J_n \end{bmatrix} =$$

$$= \begin{bmatrix} -\lambda_k(x,t)J_n & C^{-1}(x,t) \\ O_n & \frac{1}{\lambda_k(x,t)}L^{-1}(x,t)C^{-1}(x,t) - \lambda_k(x,t)J_n \end{bmatrix} .$$

Thence it follows, on account of formula (1.1), that for any $(x,t) \in \Omega$ we have the condition:

$$r[\ A^{-1}(x,t) - \lambda_k(x,t)J_{2n}\ ] = \tag{1.13}$$

$$= r\begin{bmatrix} -\lambda_k(x,t)J_n & C^{-1}(x,t) \\ O_n & \frac{1}{\lambda_k(x,t)}L^{-1}(x,t)C^{-1}(x,t) - \lambda_k(x,t)J_n \end{bmatrix} .$$

The equalities (1.12), (1.13) imply that for any $(x,t) \in \Omega$ there holds the equality

$$r[\ A^{-1}(x,t) - \lambda_k(x,t)J_{2n}\ ] = 2n - n_k ,$$

and this achieves the proof of Lemma 2.

Proceeding as in the proof of Lemma 3 of paper [1] and making use of Lemmas 1.2 (1'.2) we can establish the following lemmas:

Lemma 3 (3'). Under the assumptions A.1–A.3 (A.1, A.2, A.3') there exists a matrix

$$W(x,t) := \left[\ {}^{k}w_j(x,t)\ \right]_{(2n,2n)}$$

defined on the set Ω and satisfying on this set the following conditions:

(3a) The matrix $W(x,t)$ is non-singular ,

(3b) $W \in \tilde{C}^4(\Omega)$ ($W \in C^1(\Omega)$) .

(3c) $W^{-1}(x,t)\ A^{-1}(x,t)\ W(x,t) =$

$$= \operatorname{diag}[\ \lambda_1(x,t),\ \ldots,\ \lambda_n(x,t),\ \lambda_{n+1}(x,t),\ \ldots,\ \lambda_{2n}(x,t)\ ],$$

where $\lambda_j(x,t)$ $j = 1,\ldots,n,n+1,\ldots 2n$, denote the roots of the characteristic polynomial of the matrix $A^{-1}(x,t)$ satisfying the condition

$$\operatorname{sgn} \lambda_j(x,t) = \begin{cases} 1 & \text{for } j = 1,2,\ldots,n\ , \\ -1 & \text{for } j = n+1,n+2,\ldots,2n\ , \end{cases}$$

From Lemma 3 (3') follows that the matrix $W(x,t)$ is the modal matrix of the matrix $A^{-1}(x,t)$. We represent it in the form of the following block matrix

$$W(x,t) := \begin{bmatrix} W_{11}(x,t) & W_{12}(x,t) \\ W_{21}(x,t) & W_{22}(x,t) \end{bmatrix},$$

where

$$W_{lm}(x,t) := \Big[\ W_j^k(x,t)\ \Big]_{(n,n)},$$

where

$$\begin{cases} j,k = 1,2,\ldots n & \text{for } l = m = 1\ , \\ j\ = 1,2,\ldots n;\ \ k = n+1,\ldots,2n & \text{for } l = 1,\ m = 2\ , \\ j\ = n+1,\ldots,2n;\ \ k = 1,2,\ldots n & \text{for } l = 2,\ m = 1\ , \\ j,k = n+1,\ldots,2n & \text{for } l = m = 2\ . \end{cases}$$

Lemma 4 (see Lemma 4 in [1]). If for any $t \in \bar{R}_+$ and for a fixed $x_0 \in X$ the following conditions are satisfied:

C.1 $\det W(x_0,t) \neq 0$,

C.2 $\det W_{ll}(x_0,t) \neq 0$, $l = 1,2$,

then for any $t \in \overline{R}_+$ the matrix $W^{-1}(x_0,t)$ has the form

$$W^{-1}(x_0,t) = \begin{bmatrix} \tilde{W}_{11}^{-1}(x_0,t) & -W_{11}^{-1}(x_0,t)W_{12}(x_0,t)\tilde{W}_{22}^{-1}(x_0,t) \\ -W_{22}^{-1}(x_0,t)W_{21}(x_0,t)\tilde{W}_{11}^{-1}(x_0,t) & \tilde{W}_{22}^{-1}(x_0,t) \end{bmatrix},$$

where for any $t \in \overline{R}_+$ there is

$$\tilde{W}_{11}(x_0,t) := W_{11}(x_0,t) - W_{12}(x_0,t)W_{22}^{-1}(x_0,t)W_{21}(x_0,t),$$

$$\tilde{W}_{22}(x_0,t) := W_{22}(x_0,t) - W_{21}(x_0,t)W_{11}^{-1}(x_0,t)W_{12}(x_0,t).$$

2. Theorem on the existence of a solution of the problem $U(\tilde{\Omega})$

We make the following assumptions:

A.4 For any $t \in \overline{R}_+$ the matrices $W_{11}(0,t)$, $W_{11}(\alpha,t)$, $W_{22}(0,t)$, $W_{22}(\alpha,t)$ are non-singular.

A.5 $B, g \in \tilde{C}^3(\Omega)$, $p \in C^3(X)$.

A.6 The problem $V(\Omega)$ has a solution of the class $\tilde{C}^2(\Omega)$.

Theorem 1. Under the assumptions A.1–A.6, there exists a solution of the class $\tilde{C}^2(\Omega)$ of the problem $\tilde{U}(\Omega)$.

Proof. Multiplying the equation (20) on the left by the matrix $A^{-1}(x,t)$ we get the equivalent equation

(2.1) $$D_t\tilde{u} + A^{-1}(x,t)D_x\tilde{u} + A^{-1}(x,t)B(x,t)\tilde{u} = A^{-1}(x,t)g(x,t) .$$

which can be written, by means of the substitution

$$\tilde{u} = W(x,t)v, \quad (x,t) \in \Omega \tag{2.2}$$

in the form

$$W(x,t)D_t v + A^{-1}(x,t)W(x,t)D_x v + [\, D_t W(x,t) + \tag{2.3}$$

$$A^{-1}(x,t)D_x W(x,t) + A^{-1}(x,t)B(x,t)W(x,t) \,]\, v = A^{-1}(x,t)g(x,t) \; .$$

Multiplying equation (2.3) on the left by the matrix $W^{-1}(x,t)$ and applying Lemma 3 we obtain the equation

$$D_t v + \Lambda(x,t)D_x v + \tilde{M}(x,t)v = \tilde{g}(x,t) \; , \tag{2.4}$$

where the matrices $\tilde{g}(x,t)$, $\tilde{M}(x,t)$, $\Lambda(x,t)$ are defined on the set Ω by the following formulas:

$$\tilde{g}(x,t) := W^{-1}(x,t)A^{-1}(x,t)g(x,t) \; , \tag{2.5}$$

$$\tilde{M}(x,t) := W^{-1}(x,t) \,[\, D_t W(x,t) + A^{-1}(x,t)D_x W(x,t) + \tag{2.6}$$

$$+ A^{-1}(x,t)B(x,t)W(x,t) \,] \; .$$

(2.7)

$$\Lambda(x,t) := \operatorname{diag}[\, \lambda_1(x,t),\dots,\lambda_n(x,t),\lambda_{n+1}(x,t),\dots,\lambda_{2n}(x,t) \,],$$

where the functions λ_j satisfy for any $(x,t) \in \Omega$ the condition

$$\operatorname{sgn} \lambda_j(x,t) = \begin{cases} 1 & \text{for } j = 1,2,\dots,n \; , \\ -1 & \text{for } j = n+1,n+2,\dots,2n \; , \end{cases} \tag{2.8}$$

By virtue of equality (2.2), definitions (3), (14)–(16) and Lemma 4, for $x_0 = 0$ ($x_0 = \alpha$), the limit conditions (21), (22) take the form

$$v(x,0) = q(x) \; , \quad x \in X \; , \tag{2.9}$$

(2.10)
$$\begin{cases} v^{(1)}(0,t) = F_{12}(t)v^{(2)}(0,t) , \\ v^{(2)}(\alpha,t) = F_{21}(t)v^{(1)}(\alpha,t) , \end{cases} \quad t \in \overline{R}_+,$$

where

(2.11)
$$q(x) := W^{-1}(0,t)p(x) \quad \text{for } x \in X ,$$

(2.12)
$$\begin{cases} F_{12}(t) := -W_{11}^{-1}(0,t)W_{12}(0,t) , \\ F_{21}(t) := -W_{22}^{-1}(\alpha,t)W_{21}(\alpha,t) , \end{cases} \quad t \in \overline{R}_+$$

By the definition (2.11), we have

$$\begin{cases} q(0) := W^{-1}(0,0)p(0) , \\ q(\alpha) := W^{-1}(\alpha,0)p(\alpha) . \end{cases}$$

Thence by the definitions (1.7), (2.12), Lemma 4 for $x_0 = 0$ ($x_0 = \alpha$) and by the compatibility conditions (23), we get the equalities

(2.13)
$$\begin{cases} q^{(1)}(0) = F_{12}(0)q^{(2)}(0) , \\ q^{(2)}(\alpha) = F_{21}(0)q^{(1)}(\alpha) , \end{cases}$$

which constitute the compatibility conditions for the problem (2.4), (2.9), (2.10).

From Lemmas 1, 3, the assumptions A.3, A.5, the equality (1.1) and the definitions (2.5)–(2.7), (2.11), (2.12) we obtain the conditions

$$(2.14)\qquad \begin{cases} \Lambda \in \tilde{C}^4(\Omega),\ \tilde{M},\tilde{g} \in C^3(\Omega)\ , \\ q \in C^3(X),\ F_{12},F_{21} \in C^4(\overline{R}_+)\ . \end{cases}$$

Hence it follows from the equalities (2.13) and the conditions (2.8), (2.14) that the mixed problem (2.4), (2.9), (2.10) constitutes a particular case of the problem $V(\Omega)$ and has, by assumption A.6, the solution $\tilde{v} \in \tilde{C}^2(\Omega)$.

Making use of Lemmas 3 and 4 for $x_0 = 0$ ($x_0 = \alpha$) and proceeding as in the proof of the theorem (see [1]) on the existence of a solution of problem $U(\Omega)$, we can show that the vector-function $\tilde{\tilde{u}} : \Omega \longrightarrow R^{2n}$, defined by the formula

$$\tilde{\tilde{u}}(x,t) := W(x,t)\tilde{v}(x,t)\ ,$$

is a solution of class $\tilde{C}^2(\Omega)$ of the problem $\tilde{U}(\Omega)$, which completes the proof of Theorem 1.

Theorem 2. Under the assumptions A.1, A.2, A.3′ the equation (20) on the set Ω of the hyperbolic type.

Proof. By assumption A.1, the equations (20) and (2.1) are equivalent on the set Ω. By means of the substitution (2.2) we reduce the equation (2.1) to the form (2.3). Multiplying on the left the equation (2.3) by the matrix $W^{-1}(x,t)$ and making use of Lemma 3′, we get the equation (2.4) equivalent on the set Ω to the equation (2.3). The form of the equation (2.4), the definition (2.7) and the condition (2.8) imply that this equation is of the hyperbolic type on the set Ω. Consequently, the equation (20) is also of the hyperbolic type on the set Ω.

Bibliography

[1] J. Radzikowski : The existence of a solution of the mixed problem for a certain symmetric t-hyperbolic system, Demonstratio Math. 17 (1984) 913-938.

[2] J. Radzikowski : Qualitative study of a linear homogeneous hyperbolic system, Demonstratio Math. 18 (1985) 945-962.

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ON A GENERALIZED ITÔ TYPE STOCHASTIC HEREDITARY EQUATION

1. Introduction

A great number of real problems in various branches of control theory, physics, biology and other areas of applied mathematics lead to the study of stochastic hereditary dynamical systems (see [14]–[16]). In recent years significant emphasis has been given to the study of various types of stochastic differential and integral equations because of the increasing use of such equations in the formulation of problems of the physical, engineering and other areas of sciences (see [1], [3]–[5], [7], [13]). However, a little attention has been given to the study of Itô type stochastic hereditary systems even though for many physical phenomena such equations work as good models because of their accuracy and reliability. In this paper, we are interested in the study of a generalized Itô type stochastic hereditary equation of the form

$$dx(t) = F(\cdot\, t,\ x^t,\ Ax^t,\ Bx^t\,)dt + \tag{1.1}$$

$$+\ H(\,t,\ x^t,\ Px^t,\ Qx^t\,)d\beta(t),\ t \ge t_0\,,$$

$$x^{t_0} = \Psi^{t_0}, \tag{1.2}$$

where

$$Ax^t = \int_{t_0}^{t} f_1(t,s,x^s)ds ,$$

$$Bx^t = \int_{t_0}^{t} f_2(t,s,x^s)d\beta(s) ,$$

$$Px^t = \int_{t_0}^{t} h_1(t,s,x^s)ds ,$$

$$Qx^t = \int_{t_0}^{t} h_2(t,s,x^s)d\beta(s) .$$

Here x^t denotes the history up to t of the measurable real valued stochastic process x, $\beta(t)$ is a real-valued Wiener process based on some probability space $(\Omega, \mathcal{P}, P)$, f_i, h_i $(i = 1,2)$ are functionals defined on $R^2 \times X$, X being a past history space, and F, H are functionals defined on $R \times X \times R^2$ and f_i, h_i $(i = 1,2)$ and F, H are Borel measurable on $R^2 \times X$ and $R \times X \times R^2$ respectively.

In a recent paper [8] Mizel and Trutzer have studied the problems of existence and uniqueness of the solutions of (1.1) when F and H are independent of Ax^t, Bx^t, Px^t, Qx^t and

$$F(t, x^t, Ax^t, Bx^t) = F_0(t,x^t) + Ax^t + Bx^t ,$$

$$H(t, x^t, Px^t, Qx^t) = H_0(t,x^t) + Px^t + Qx^t .$$

The object of the present paper is to obtain sufficient conditions for the existence and uniqueness of the solutions of (1.1)–(1.2). The method of successive approximations is used to establish our results. In fact our results in this paper are motivated by the recent work of Mizel and Trutzer [8] and also influenced by the general form of Itô type stochastic equations studied by Berger and Mizel [2], Murge and Pachpatte [9]-[10], Pachpatte [12] and others.

2. Statements of results

Before proceeding to the statements of our main results, we shall setforth some preliminaries from Mizel and Trutzer [8] and hypotheses on the functionals involved in (1.1)-(1.2) which are to be used throughout the paper.

Let X denote a past history space i.e. a product space $R \times L_p^{\rho}(R^+:R)$, ($R^+ = [0, \infty)$, $R = (-\infty, \infty)$), where L_p^{ρ}, $1 \le p < \infty$, is the space of all measurable functions $\Psi : R^+ \longrightarrow R$ such that

$$\int_0^\infty |\Psi(s)|^p \rho(s)ds < \infty .$$

On the space X we define the norm as

$$\|\Psi\|_X = \left(|\Psi(0)|^p + \int_0^\infty |\Psi(s)|^p \rho(s)ds \right)^{\frac{1}{p}} = \tag{2.1}$$

$$= \left(|\Psi(0)|^p + \|\Psi\|_r^p \right)^{\frac{1}{p}} .$$

The function $\rho : R^+ \longrightarrow R^+$ is called an influence function with relaxation property and satisfies the following conditions :

(i) ρ is summable on R^+ .

(ii) For every $u \ge 0$ one has

(2.2)
$$\overline{C}(u) = \operatorname*{ess\,sup}_{s\in R^+} \frac{\rho(s+u)}{\rho(s)} \le \overline{C} < \infty ,$$

$$\underline{C}(u) = \operatorname*{ess\,sup}_{s\in R^+} \frac{\rho(s)}{\rho(s+u)} < \infty ,$$

(iii) ρ is essentially bounded on R^+.

(iv) ρ is essentially strictly positive on $(0,\infty)$.

(v) $s\rho(s) \longrightarrow 0$ as $s \to \infty$.

We define the mappings $\{T^u\}$, $\{T_u\}$ and $\{T^{-u}\}$ as follows : For $u \ge 0$:

$$(T^u\Psi)(s) = \begin{cases} \Psi(0), & 0 \le s \le u, \\ \Psi(s-u) & s \ge u,\ s \in R^+,\ \Psi \in X \end{cases}$$

$$(T_u\Psi)(s) = \Psi(s+u),\ s \in R^+,\ \Psi \in X,$$

$$(T^{-u}\Psi)(s) = \begin{cases} 0, & 0 \le s \le u, \\ \Psi(s-u), & s \ge u. \end{cases}$$

These mappings are closely related with the properties of the spaces stated above. We refer [8] for the details about the properties of these mappings.

If $x : R \longrightarrow R$ (or $x : (-\infty,a) \longrightarrow R$) is measurable we call its history up to t, x^t, the function on R^+ defined by

$$x^t(s) = x(t-s),\quad s \in R^+.$$

A function $x : (-\infty,a\,] \longrightarrow R$ is said to be admissible with respect to X if $x^t \in X$ for each $t \le a$.

We first list the following basic assumption used in our subsequent discussion.

(A_1) Let $\beta(t)$, $t \geq t_0$ be a one-dimensional motion on a probability space (Ω, $\mathscr{P}$, P). Further let $\mathscr{P}_t$ be an increasing family of σ-fields in the probability space Ω generated such that $\beta(t)$ is $\mathscr{P}_t$-measurable for any $t \geq 0$. Suppose $\mathscr{A}_{t_0}$, $t_0 \geq 0$ be the space of measurable stochastic processes $x(t)$, $t \leq t_0$ such that $x^{t_0} \in X$ for a.e. ω and such that for every such t, $x(t)$ is independent of $\{ \beta_u - \beta_{t_0}, u \geq t_0 \}$. The first condition implies, by the structure of the space X, that $x^t \in X$ for all $t \leq t_0$ a.s. The initial condition Ψ^{t_0} is assumed to belong to $\mathscr{A}_{t_0}$.

A stochastic process $x(t)$, $t \in (-\infty, T]$ is a (strong) solution of the equation (1.1)-(1.2) for $t \in [t_0, T]$ if :

(i) $x(t)$ is non-anticipating for $t \leq T$,

(ii) for a.e. ω, x^t is in X for $t \in [t_0, T]$,

(iii) functionals $f_i(t) = f_i(t, s, x^s)$, $h_i(t) = h_i(t, s, x^s)$, $(i = 1,2)$, $F(t) = F(t, x^t, Ax^t, Bx^t)$, $H(t) = H(t, x^t, Px^t, Qx^t)$ are such that

$$\int_{t_0}^{T}\int_{t_0}^{t} |f_1(s)|^2 ds dt < \infty , \qquad \int_{t_0}^{T}\int_{t_0}^{t} |h_1(s)|^2 ds dt < \infty ,$$

$$\int_{t_0}^{T}\int_{t_0}^{t} |f_2(s)|^2 ds dt < \infty , \qquad \int_{t_0}^{T}\int_{t_0}^{t} |h_2(s)|^2 ds dt < \infty ,$$

$$\int_{t_0}^{T} |F(t)|^2 dt < \infty , \qquad \int_{t_0}^{T} |H(t)|^2 dt < \infty , \quad \text{a.s.},$$

(iv) $x^{t_0} = \Psi^{t_0}$, a.s.,

(v) the equation (1.1)-(1.2) holds for each $t \in [t_0, T]$.

It is easy to observe from the proof of Lemma 2.A in [8] that the process x^t is measurable.

In the process of proofs we are using the following notation,

$$|x(\cdot)|^{*}_{t_0}(t) = \sup_{t_0 \le s \le t} |x(s)|,$$

for a function x defined on R and

$$x_r^t = \begin{cases} x(t-s) & , 0 \le s \le t \\ (T^v\Psi)(s) & , s > t \end{cases} \tag{2.3}$$

$$= (T^v\Psi)(s) + (\delta x^t)(s),$$

where

$$(\delta x^t)(s) = \begin{cases} x(t-s) & , 0 \le s \le t \\ 0 & , s > t \end{cases}$$

We suppress the subscript t_0 in case no confusion arises.

The following consequence of the Burkholder-Davis inequality ([6], p.110), (see, also [8]) will be used repeatedly. For any $l > 0$, there exists a constant $C_l > 0$ such that

$$E\left|\int_{t_0}^{t} \phi(u)\,d\beta(u)\right|^{*l}_{t_0}(t) \le C_l E\left(\int_{t_0}^{t} |\phi(u)|^2 du\right)^{\frac{l}{2}} \tag{2.4}$$

for any measurable $\mathcal{F}_t$-adapted process $\phi(t)$ such that.

$$\int_0^T \phi^2(t,\omega)dt < \infty , \quad \text{a.s.}$$

Also we shall use the following elementary inequalities on constants

$$(2.5)\quad \begin{cases} |a+b|^r \le K_r(|a|^r + |b|^r), \; r \ge 0, \; K_r = 2^{r-1} \vee 1. \\ |a+b+c|^r \le \tilde{K}_r(|a|^r + |b|^r + |c|^r), \\ \qquad\qquad r \ge 0, \; \tilde{K}_r = 3^{r-1} \vee 1. \\ |a+b+c+d|^r \le \tilde{\tilde{K}}_r(|a|^r + |b|^r + |c|^r + |d|^r), \\ \qquad\qquad r \ge 0, \; \tilde{\tilde{K}}_r = 4^{r-1} \vee 1 . \end{cases}$$

For convenience we list the following assumptions :

(A_2) (a) (Uniform Lipschitz condition). For all $x, y, x_j \in X, y_j, z_j \in R, (j = 1,2), t,s \in [t_0,T]$ there exist constans L_1 and L_2 such that

$$|f_i(t, s, x) - f_i(t, s, y)|^2 + |h_i(t, s, x) - h_i(t, s, y)|^2 \le$$

$$\le L_1\|x - y\|^2, \quad i = 1,2,$$

$$|F(t, x_1, y_1, z_1) - F(t, x_2, y_2, z_2)|^2 +$$

$$+ |H(t, x_1, y_1, z_1) - H(t, x_2, y_2, z_2)|^2 \le$$

$$\le L_2 (\|x_1 - x_2\|^2 + \|y_1 - y_2\|^2 + \|z_1 - z_2\|^2).$$

(b) (growth condition). For all $x \in X, y,z \in R$ $s,t \in [t_0,T]$ there exist constants L_3 and L_4 such that

$$|f_i(t, s, x)|^2 + |h_i(t, s, x)|^2 \leq L_3(1 + \|x\|^2), \ i = 1,2,$$

$$|F(t,x,y, z)|^2 + |H(t,x,y,z)|^2 \leq L_4(1 + \|x\|^2 + |y|^2 + |z|^2).$$

Next we state the following propositions concerning the estimates and properties of the history processes which are needed in the proofs of our results.

Proposition 1. Suppose assumption (A_1) and condition (a) of the assumption (A_2) hold. Let $x(t)$, $y(t)$ be nonanticipative measurable stochastic processes which are continuous for $t \geq t_0$ and x^{t_0}, $y^{t_0} \in X$ a.s. and define

$$R(t,x) = \int_{t_0}^{t} F(s, x^s, Ax^s, Bx^s)ds + \int_{t_0}^{t} H(s, x^s, Px^s, Qx^s)d\beta(s) =$$

$$= R_1(t,x) + Q_1(t,x)$$

and

$$\delta(t) = x(t) - y(t).$$

Then for every $l > 2$ the following inequality holds

$$(2.6) \qquad E|R(.,x) - R(.,y)|_{t_0}^{*l}(t) \leq$$

$$\leq K_1 E \|\delta^{t_0}\|^l + K_2\Big[\int_{t_0}^{t} |\delta(s)|^l ds + \int_{t_0}^{t} \Big(\int_{t_0}^{s} |\delta(\tau)|^p \rho(s-\tau)d\tau\Big)^{\frac{1}{p}} ds\Big] +$$

$$+ K_3 E\Big[\int_{t_0}^{t} (s-t_0)^{l-1} \int_{t_0}^{s} |\delta(\tau)|^l d\tau ds +$$

$$+\int_{t_0}^{t}(s-t_0)^{l-1}\int_{t_0}^{s}\Big(\int_{t_0}^{\tau}|\delta(\theta)|^{p}\rho(\tau-\theta)d\theta\Big)^{\frac{1}{p}}d\tau ds\Big]+$$

$$+K_4E\Big[\int_{t_0}^{t}\Big(\int_{t_0}^{s}|\delta(\tau)|^{2}d\tau\Big)^{\frac{1}{2}}ds+$$

$$+\int_{t_0}^{t}\Big(\int_{t_0}^{s}\Big(\int_{t_0}^{s}|\delta(\theta)|^{p}\rho(\tau-\theta)d\theta\Big)^{\frac{2}{p}}d\tau\Big)^{\frac{1}{2}}ds\Big],$$

where K_1, K_2, K_3, K_4 depend only on l, p, L_1, L_2, $t-t_0$ and are bounded for $t_0 \le t \le T$. Also K_1 is $O(1)$ as $t \to t_0$.

Corollary 1. Let the conditions of Proposition 1 hold. Then

(2.7) $E|R(.,x) - R(.,y)|^{*l}_{t_0}(t) \le K_1E\|\delta^{t_0}\|^{l} + M^{t}_{t_0}\, E|\delta(.)|^{*l}_{t_0}(t)$

where K_1 is the same as in Proposition 1 and M depends on $t-t_0$, K_2, K_3, K_4 and

(2.8) $$M^{t}_{t_0} = O(1) \quad \text{as } t-t_0\to 0.$$

Proposition 2. Suppose $x(t)$, $t \ge t_0$ be the solution of the equation (1.1)-(1.2) with initial history $x^{t_0} = \Psi^{t_0}$ for some $\Psi \in \mathscr{A}_{t_0}$ and satysfying

$$E\|x^{t_0}\|^{l}_{X} < \infty ,$$

for some $l \ge 2$. Then

$$\lim_{t\to t_0} E\|x^{\circ} - x^{t_0}\|^{*l}(t) = 0.$$

We note that if $x(t)$ be the solution process of the equation (1.1)–(1.2) with initial history $x \in \mathscr{A}$ then

$$\lim_{t \to 0} E\|x^\circ - x\|^{*l}(t) = 0.$$

Proposition 3. Suppose $x(\circ)$, $y(\circ)$ be the solutions of the equation (1.1)–(1.2) for $t \geq t_0$ and $t \geq t_0'$ with initial conditions $x^{t_0} \in \mathscr{A}$, $x^{t_0'} \in \mathscr{A}$ respectively for a time $t_0' \geq t_0$. Then

$$\lim_{t_0 \downarrow t_0} E\|x^\circ - y^\circ\|^{*l}_{t_0}(T) = 0, \quad \|x^{t_0} - x^{t_0'}\| \longrightarrow 0.$$

Remark 1. We note that our results established in Propositions 1–3 are the further generalization of the results established by the authors in [8] which in turn are readily applicable to establish our main results on the existence and uniqueness of the solutions of equation (1.1)–(1.2). We also believe that our results established in Propositions 1–3 are of independent interest.

Now we are in a position to state our main results on the existence and uniqueness of solution of the equation (1.1)–(1.2).

Theorem 1. Let assumptions (A_1) and (A_2) hold. Suppose $l \geq 2$ and x_- be a process such that $x_- \in \mathscr{A}_{t_0}$ and

$$E\|x_-^{t_0}\|^{l}_{X} < \infty .$$

Then there exists a unique strong solution of (1.1)–(1.2), as continuous with

(2.9)
$$x^{t_0} = x_-^{t_0}$$

and

(2.10)
$$E|x(\circ)|^{*1}_{t_0}(T) < \infty .$$

Next we state the following localization theorem modeled after the classical theorem for Itô type stochastic differential equations without hereditary analysis (see, [2], [4]-[5], [9]-[10]).

We consider the equation (1.1) – (1.2) along with the equation

(2.11)
$$dy(t) = F^*(t,\ y^t,\ A^*y^t,\ B^*y^t)dt + $$
$$+ H^*(t,\ y^t,\ P^*y^t,\ Q^*y^t)d\beta(t),\ t \le t_0,$$

(2.12)
$$y^{t_0} = \Psi^{*t_0},$$

where

$$A^*y^t = \int_{t_0}^{t} f_1^*(t,\ s,\ y^s)ds,$$

$$B^*y^t = \int_{t_0}^{t} f_2^*(t,\ s,\ y^s)d\beta(s),$$

$$P^*y^t = \int_{t_0}^{t} h_1^*(t,\ s,\ y^s)ds,$$

$$Q^*y^t = \int_{t_0}^{t} h_2^*(t,\ s,\ y^s)d\beta(s),$$

and the functionals F^*, H^*, f_i^*, h_i^* $(i = 1,2)$ are as defined to the corresponding functionals F, H, f_i, h_i respectively.

Theorem 2. Suppose the assumptions (A_1) and (A_2) hold. Let the assumption (A_2) hold when F, H, f_i, h_i $(i = 1,2)$ are replaced by F^*, H^*, f_i^*, h_i^* respectively and the assumption (A_1) hold when x^{t_0} is replaced by y^{t_0}. Suppose $x(t)$ and $y(t)$ be solutions of equations (1.1)–(1.2) and (2.11)–(2.12) respectively. Let D be an open domain in X and suppose that for $x(t) \in D$, $t \geq t_0$

$$Ax(t) = A^*x(t), \quad Bx(t) = B^*x(t),$$

$$Px(t) = P^*x(t), \quad Qx(t) = Q^*x(t),$$

$$F(t,x(t),Ax(t),\ Bx(t)) = F^*(t,x(t),P^*x(t),\ B^*x(t)),$$

$$H(t,x(t),Px(t),\ Qx(t)) = H^*(t,x(t),P^*x(t),\ Q^*x(t)).$$

Further assume that for some $l \geq 2$,

$$E\|x^{t_0}\|^l < \infty, \quad E\|y^{t_0}\|^l < \infty$$

and

$$x^{t_0} = y^{t_0}$$

for almost all $\omega \in \left\{ x^{t_0} \in D \text{ or } y^{t_0} \in D \right\} = \Omega_D$. Denote by θ the first time (after $t = t_0$) that the process x^t intersects $X \setminus D$ if such time $t \in [t_0, T]$ exists, and $\theta = T$, otherwise. Also denote by θ^* the analogous stopping time for the process y^t. Then $\theta = \theta^*$ a.s. in Ω_D and

$$|x(\circ) - y(\circ)|^{*}_{t_0}(\theta) = 0, \quad \text{a.s. in } \Omega_D.$$

As an immediate consequence of this result we note that

$$|x^{\circ} - y^{\circ}|^{*}_{t_0}(\theta) = 0, \quad \text{a.s. in } \Omega_D.$$

Using Theorem 2 we can prove the following stronger version of our Theorem 1.

Theorem 3. Let the assumption (A_1) hold and F, H, f_i, h_i $(i = 1,2)$ in equation (1.1)–(1.2) satisfy the condition (b) of the assumption (A_2). Suppose further that for each $N > 0$ there exist constants $L_{N1} > 0$ and $L_{N2} > 0$ such that for all $x, y, x_j \in X$, $y_j, z_j \in R$, $\|x\| < N$, $\|y\| < N$, $\|x_j\| < N$ $(j = 1,2)$ the functionals F, H, f_i, h_i $(i = 1,2)$ in equation (1.1)–(1.2) satisfy the condition (a) of assumption (A_2) by replacing L_1 and L_2 by L_{N1} and L_{n2} respectively. Then for any $\Psi \in \mathscr{A}_{t_0}$, there exists a unique a–s. continuous solution $x(t)$ of (1.1)–(1.2) on $[t_0, T]$ satisfying

$$x^{t_0} = \Psi^{t_0} = x_{-}.$$

Remark 2. The problems of existence and uniqueness of the solutions of two different special forms of (1.1)–(1.2) are studied by the authors in [8]. Here we note that, our results in the general frame work of Theorems 1–3 are the extensions of our results in [10] and the generalizations of the recent results in [8] and are motivated in part by the recent studies of the present authors in [10] (see, also [2]) which deals with the study of more general form of equation (1.1)–(1.2) without hereditary arguments.

3. Proofs of Propositions 1–3

By using (2.4)–(2.5), Hölder's inequality repeatedly with indices l, $\frac{l}{l-1}$ and $\frac{l}{2}$, $\frac{l}{l-2}$ and Schwarz's inequality, for $l \geq 2$ we obtain

$$E\left|R(\circ,x) - R(\circ,y)\right|^{*l}_{t_0}(t) \leq \tag{3.1}$$

$$\leq K_1 E\Big(\left|R_1(\circ,x) - R_1(\circ,y)\right|^{*l}_{t_0}(t) + \left|Q_1(\circ,x) - Q_1(\circ,y)\right|^{*l}_{t_0}(t) \Big) \leq$$

$$\leq K_1 E\Big[(t-t_0)^{l-1} \int_{t_0}^{t} \left|F(s,x^s,Ax^s,Bx^s) - F(s,y^s,Ay^s,By^s)\right|^{l} ds \Big] +$$

$$+ K_1 C_1 E\Big[\int_{t_0}^{t} \left|H(s,x^s,Px^s,Qx^s) - H(s,y^s,Py^s,Qy^s)\right|^{2} ds \Big]^{\frac{l}{2}} \leq$$

$$\leq K_1 L_2^{\frac{l}{2}} \tilde{K}_{\frac{l}{2}}\, E\Big\{ (t-t_0)^{\frac{l}{2}} + C_1 \Big\} (t-t_0)^{\frac{l}{2}-1} \cdot \Big[\int_{t_0}^{t} \|\delta^s\|^{l} ds +$$

$$+ L_1^{\frac{l}{2}} \int_{t_0}^{t} (s-t_0)^{l-1} \int_{t_0}^{s} \|\delta^\tau\|^{l} d\tau ds + C_1 L_1^{\frac{l}{2}} \int_{t_0}^{t} \Big(\int_{t_0}^{s} \|\delta^\tau\|^{2} d\tau \Big)^{\frac{l}{2}} ds \Big].$$

We note from (2.1) (also, see [8])

$$\|\delta^s\|_X^p = |\delta(s)|^p + \|\delta^s\|_r^p$$

and

$$\|\delta^s\|_r^p = \int_{s-t_0}^{\infty} |\delta(s-\tau)|^p \rho(\tau) d\tau + \int_{0}^{s-t_0} |\delta(s-\tau)|^p \rho(\tau) d\tau = \tag{3.2}$$

$$= \int_0^\infty |\delta(t_0-\tau)|^p \frac{\rho(s+\tau+t_0)}{\rho(\tau)} \rho(\tau)d\tau + \int_{t_0}^s |\delta(\tau)|^p \rho(s-\tau)d\tau \le$$

$$\le \bar{C} \|\delta^{t_0}\|_r^p + \int_{t_0}^s |\delta(\tau)|^p \rho(s-\tau)d\tau ,$$

where $\bar{C}$ is as defined in (2.2). By using $\frac{l}{p} = \alpha$ and $\frac{2}{p} = \lambda$ and (3.2) we get

$$(3.3) \qquad \|\delta^s\|^l \le \hat{K}_\alpha \left[|\delta(s)|^l + \bar{C}^\alpha \|\delta^{t_0}\|_r^l + \left(\int_{t_0}^s |\delta(\tau)|^p \rho(s-\tau)d\tau \right)^\alpha \right],$$

and on integrating (3.3), we obtain

$$(3.4) \qquad \int_{t_0}^t \|\delta^s\|^l ds \le \hat{K}_\alpha \Big[\int_{t_0}^t |\delta(s)|^l ds + \bar{C}^\alpha (t - t_0) \|\delta^{t_0}\|_r^l +$$

$$+ \int_{t_0}^t \left(\int_{t_0}^s |\delta(\tau)|^p \rho(s-\tau)d\tau \right)^\alpha ds \Big].$$

Proceeding on similar lines as in (3.2)–(3.3), we have

$$(3.5) \qquad \int_{t_0}^t (s-t_0)^{l-1} \int_{t_0}^s \|\delta^\tau\|^l d\tau ds \le$$

$$\leq \tilde{K}_\alpha \Big[\int_{t_0}^{t} (s-t_0)^{l-1} \int_{t_0}^{s} |\delta^\tau|^{l} d\tau ds + \bar{C}^{\alpha} \|\delta^{t_0}\|_r^{l} \int_{t_0}^{t} (s-t_0)^{l-1} \int_{t_0}^{s} d\tau ds +$$

$$+ \int_{t_0}^{t} (s-t_0)^{l-1} \int_{t_0}^{s} \Big(\int_{t_0}^{\tau} |\delta(\theta)|^{p} \rho(\tau-\theta) d\theta \Big)^{\alpha} d\tau ds \Big]$$

and

$$(3.6) \qquad \int_{t_0}^{t} \Big(\int_{t_0}^{s} \|\delta^\tau\|^{2} d\tau \Big)^{\frac{1}{2}} ds \leq$$

$$\leq \tilde{K}_\lambda^{\frac{1}{2}} \tilde{K}_{\frac{1}{2}} \int_{t_0}^{t} \Big[\Big(\int_{t_0}^{s} |\delta^\tau|^{2} d\tau \Big)^{\frac{1}{2}} + \bar{C}^{\frac{1}{p}} (s-t_0)^{\frac{1}{2}} \|\delta^{t_0}\|_r^{l} +$$

$$+ \Big(\int_{t_0}^{s} \Big(\int_{t_0}^{\tau} |\delta(\theta)|^{p} \rho(\tau-\theta) d\theta \Big)^{\lambda} d\tau \Big)^{\frac{1}{2}} \Big] ds.$$

Substituting (3.4)-(3.6) in (3.1), we get (2.6). This proves Proposition 1.

By using $|\delta(u)| \leq |\delta|^{*}_{t_0}(t)$ for $t_0 \leq u \leq t$ in (2.6) we get

$$E|R(.,x) - R(.,y)|^{*l}_{t_0}(t) \leq$$

$$\leq K_1 E \|\delta^{t_0}\|^{l} + K_2 E|\delta(.)|^{*l}_{t_0}(t) \Big[\int_{t_0}^{t} ds + \int_{t_0}^{t} \Big(\int_{t_0}^{s} \rho(s-\tau) d\tau \Big)^{\frac{1}{p}} ds \Big] +$$

$$+ K_3 E |\delta(.)|^{*l}_{t_0}(t) \Big[\int_{t_0}^{t} (s-t_0)^{l-1} \int_{t_0}^{s} d\tau ds +$$

$$+\int_{t_0}^{t}(s-t_0)^{l-1}\int_{t_0}^{s}\Big[\int_{t_0}^{s}\rho(\tau-\theta)d\theta\Big]^{\frac{1}{p}}d\tau ds\Big] + K_4E\,|\delta(.)|_{t_0}^{*l}(t)\,\cdot$$

$$\cdot\Big[\int_{t_0}^{t}\Big(\int_{t_0}^{s}d\tau\Big)^{\frac{l}{2}}ds + \int_{t_0}^{t}\Big(\int_{t_0}^{s}\Big(\int_{t_0}^{\tau}\rho(\tau-\theta)d\theta\Big)^{\lambda}d\tau\Big)^{\frac{l}{2}}ds\Big] \le$$

$$K_1E\,\|\delta^{t_0}\|^{l} + K_2\,E|\delta(.)|_{t_0}^{*l}(t)\,\Big(1+\|\rho\|_{L_1}^{\frac{1}{p}}\Big)(t-t_0)+$$

$$+K_3E\,|\delta(.)|_{t_0}^{*l}(t)\,\Big(1+\|\rho\|_{L_1}^{\frac{1}{p}}\Big)\frac{(t-t_0)^{l+1}}{l+1}+$$

$$+K_4E\,|\delta(.)|_{t_0}^{*l}(t)\,\Big(1+\|\rho\|_{L_1}^{\frac{1}{p}}\Big)\frac{(t-t_0)^{\frac{l}{2}+1}}{\frac{l}{2}+1}=$$

$$=K_1E\,\|\delta^{t_0}\|^{l}+M_{t_0}^{t}\;E|\delta(.)|_{t_0}^{*l}(t)\;,$$

where

$$M_{t_0}^{t}=\Big[K_2+K_3\frac{(t-t_0)^{l}}{l+1}+K_4\frac{(t-t_0)^{\frac{l}{2}}}{\frac{l}{2}+1}\Big]\cdot$$

$$\cdot\Big[1+\|\rho\|_{L_1}^{\frac{1}{p}}\Big](t-t_0)=O(1)\text{ as }t-t_0\longrightarrow 0.$$

This completes the proof of Corollary 1.

We note that proofs of Propositions 2–3 can be worked out on similar lines as the proofs of Lemmas 2.C and 2.D in [8] with suitable modifications. We omit the details.

4. Proofs of Theorems 1–3

We define the sequence of random functions x_n as follows :

$$x_n(t) = x\_(t), \quad \text{for } t \le t_0 \text{ and for all } n \ge 0,$$

$$x_0(t) = x\_(t_0), \quad \text{for } t \ge t_0.$$

For $n \ge 1$

$$x_n(t) = x\_(t_0) + \int_{t_0}^{t} F(s, x_{n-1}^s, Ax_{n-1}^s, Bx_{n-1}^s)ds +$$

$$+ \int_{t_0}^{t} H(s, x_{n-1}^s, Px_{n-1}^s, Qx_{n-1}^s)d\beta(s), \quad t \ge t_0 ,$$

where x_n^s is as defined in (2.3). It is easy to observe from the above definition, assumptions on the functionals, Proposition 1 that the stochastic processes $x_n(t)$, $t \ge 0$ are nonanticipative, measurable, continuous and $x_n^t \in X$ for $t \ge t_0$.

First we shall prove by induction that

$$E\|x_n^{\circ}\|_{t_0}^{*1}(t) < \infty, \quad \text{for } t_0 \le t \le T.$$

For $n = 0$, $x_0^{t_0} \in X$ and for $t_0 \le t \le T$ we have (see, [8])

$$\|x_0^t\| = \|T^{t-t_0}x_0^{t_0}\| \le \text{const } \|x_0^{t_0}\| = \text{const. } \|x\_^{t_0}\|,$$

where the constant depends on $\bar{C}$ as defined in (2.2). From this ir follows that

$$E\|x_0^\circ\|_{t_0}^{*l}(T) \le \text{const } E\|x\_^{t_0}\|_X^l < \infty .$$

Assume

$$E\|x_{n-1}^\circ\|_{t_0}^{*l}(t) < \infty , \quad t_0 \le t \le T.$$

Now, using (2.4)–(2.5), condition (b) of the assumption (A_2) and Hölder's inequality with indices l, $\frac{l}{l-1}$ and $\frac{l}{2}$, $\frac{l}{l-2}$ we get, for $t_0 \le t \le T$

$$E\|x_n^\circ\|_{t_0}^{*l}(t) \le \tilde{K}_l \; E\Big[\; |x\_(t_0)|^l + \Big|\int_{t_0}^{t} F(s, x_{n-1}^s, Ax_{n-1}^s, Bx_{n-1}^s)ds\Big|^l +$$

$$+ \Big|\int_{t_0}^{t} H(s, x_{n-1}^s, Px_{n-1}^s, Qx_{n-1}^s)d\beta(s)\Big|^l \; \Big] \le$$

$$\le \tilde{K}_l \; E\Big[\; |x\_(t_0)|^l + (t-t_0)^{l-1} \int_{t}^{t} |F(s, x_{n-1}^s, Ax_{n-1}^s, Bx_{n-1}^s)|^l ds +$$

$$+ C_l \left\{ \int_{t_0}^{t} |H(s, x_{n-1}^s, Px_{n-1}^s, Qx_{n-1}^s)|^2 ds \right\}^{\frac{l}{2}} \Big] \le$$

$$\le \tilde{K}_l \; E\Big[\; |x\_(t_0)|^l + L_4^{\frac{l}{2}} \tilde{K}_{\frac{l}{2}} K_{\frac{l}{2}} \left\{ (t-t_0)^{\frac{l}{2}} + C_l \right\} (t-t_0)^{\frac{l}{2}} \; .$$

$$\cdot \left\{ 1 + \frac{L_3^{\frac{l}{2}} (t-t_0)^l}{l+1} + \frac{C_l \; L_3^{\frac{l}{2}} (t-t_0)^{\frac{l}{2}}}{\frac{l}{2}+1} \right\} (1 + \|x_{n-1}^\circ\|_{t_0}^{*l}(T)) \Big] < \infty .$$

By using $R(t,x)$ as defined in Proposition 1 and Corollary 1, we obtain

$$(4.1).\; E|x_n(\circ) - x_{n-1}(\circ)|_{t_0}^{*l}(t) = E|R(\circ, x_{n-1}) - R(\circ, x_{n-2})|_{t_0}^{*l}(t) \le$$

$$\leq M_t^{t_0}\ E|x_{n-1}(\circ) - x_{n-2}(\circ)|_{t_0}^{*1}(t)\ .$$

Noting (2.8) choose $t_1 > 0$ such that $M_t^{t_0} < \frac{1}{2}$ for $t_0 \leq t \leq t_0+t_1$. The choice of t_1 can be independent of the magnitude of t_0. By using (2.4)–(2.5), condition (b) of the assumption (A_2) and Hölder's inequality with indices 1, $\frac{1}{1-1}$ and $\frac{1}{2}$, $\frac{1}{1-2}$ we get

$$E|x_1(\circ) - x_0(\circ)|_{t_0}^{*1}(t) \leq K_1\ E\Big[\ |\int_{t_0}^{\circ} F(s,x_0^s,Ax_0^s,Bx_0^s)ds\ |_{t_0}^{*1}(t) +$$

$$+\ |\int_{t_0}^{\circ} H(s,x_0^s,Px_0^s,Qx_0^s)d\beta(s)\ |_{t_0}^{*1}(t)\ \Big] \leq$$

$$\leq K_1\ L_4^{\frac{1}{2}}\ \tilde{K}_{\frac{1}{2}}\ K_{\frac{1}{2}}\ E\Big\{\ (t-t_0)^{\frac{1}{2}} + C_1\ \Big\}\Big\{\ 1+ \frac{L_3^{\frac{1}{2}}\ (t-t_0)^1}{1+1} +$$

$$+ \frac{C_1\ L_3^{\frac{1}{2}}\ (t-t_0)^{\frac{1}{2}}}{\frac{1}{2}+1}\Big\}\ (t-t_0)^{\frac{1}{2}}\ (\ 1 + \|x_0\|_{t_0}^{*1}(T)\ ) < \infty$$

We define

$$E|x_1(\circ) - x_0(\circ)|_{t_0}^{*1}(T) = D < \infty\ .$$

By successive iterations on (4.1) we get

$$E|x_n(\circ) - x_{n-1}(\circ)|_{t_0}^{*1}(t) \leq \frac{1}{2^{n-1}}\ D,\ t_0 \leq t \leq t_0+t_1\ .$$

By using Chebyshev's inequality we observe that

$$\sum_{n=1}^{\infty} P\left\{ |x_n(\circ) - x_{n-1}(\circ)|_{t_0}^{*1}(t) > \frac{1}{n^2} \right\} \le$$

$$\le \sum_{n=1}^{\infty} n^{21}\, E|x_n(\circ) - x_{n-1}(\circ)|_{t_0}^{*1}(t) \le D \sum_{n=1}^{\infty} \frac{n^{21}}{2^{n-1}} < \infty$$

and by Borel- Cantelli Lemma it follows that the sums

$$x_n(t) = x\_(t_0) + \sum_{k=1}^{n-1} (x_k(t) - x_{k-1}(t))$$

are a.s. uniformly convergent on $[t_0, t_0+t_1]$. Hence

$$x(t) = \lim_{n \longrightarrow \infty} x_n(t)$$

exists. We observe that $x(t)$ is a.s. continuous on $[t_0, t_0+t_1]$ and non-anticipative.

Next we show that $x(t)$, $t \le t_0+t_1$ solves equation (1.1) -- (1.2).

By using (2.7) we observe that

$$E|x(\circ) - x(t_0) - \int_{t_0}^{\circ} F(s,x^s,Ax^s,Bx^s)ds -$$

$$- \int_{t_0}^{\circ} H(s,x^s,Px^s,Qx^s)d\beta(s)\ |_{t_0}^{*1}(t) =$$

$$= E|x(\circ) - x_n(\circ) - \int_{t_0}^{\circ} \Big\{ F(s,x^s,Ax^s,Bx^s) -$$

$$- F(s,x_{n-1}^s,Ax_{n-1}^s,Bx_{n-1}^s) \Big\}\, ds - \int_{t_0}^{\circ} \Big\{ H(s,x^s,Px^s,Qx^s) -$$

$$- H(s, x^s_{n-1}, Px^s_{n-1}, Qx^s_{n-1}) \Big\} \, d\beta(s) \Big|^{*l}_{t_0}(t) \le$$

$$\le K_1 E|x(\circ) - x_n(\circ)|^{*l}(t) + K_1 E|R(\circ, x) - R(\circ, x_{n-1})|^{*l}(t) \le$$

$$\le K_1 E|x(\circ) - x_n(\circ)|^{*l}(t) + K_1 \, M^t_{t_0} \, E|x(\circ) - x_{n-1}(\circ)|^{*l}(t).$$

Now, to prove the assertion we have to show that

$$\lim_{n \to \infty} E|x(\circ) - x_n(\circ)|^{*l}(t) = 0.$$

Let

$$b_i = i^{-\frac{2}{l'}}, \quad l' = \frac{l}{l-1}.$$

By using Hölder's inequality with indices l, $\frac{l}{l-1}$ for $l \ge 2$ we observe that (see, [8])

$$E|x_n(\circ) - x_{n+k}(\circ)|^{*l}(t) \le$$

$$\le E\Big(\sum_{i=n}^{n+k-1} |x_i(\circ) - x_{i+1}(\circ)|^{*}(t) b_i b_i^{-1} \Big)^l \le$$

$$\le \Big(\sum_{i=n}^{\infty} \frac{D b_i^{-l}}{2^i} \Big) \Big(\sum_{i=n}^{\infty} b_i^{l'} \Big)^{l-1},$$

and

$$|x(\circ) - x_n(\circ)|^{*l}(t) \le \lim_k \inf |x_n(\circ) - x_{n+k}(\circ)|^{*l}(t).$$

Finally by using Fatou's Lemma we obtain

$$E|x_n(\circ) - x(\circ)|^{*l}(t) \le D\Big(\sum_{i=n}^{\infty} \frac{b_i^{-1}}{2^i}\Big)\Big(\sum_{i=n}^{\infty} b_i^{l'}\Big)^{l-1} \longrightarrow 0 \text{ as } n\to\infty.$$

This proves the existence of a.s. continuous solution process $x(t)$ of the equation (1.1)–(1.2) for $t_0 \le t \le t_0+t_1$.

Next, we establish the existence the solution of the equation (1.1)–(1.2) beyond $t_0 + t_1$. First we obtain a solution of the equation (1.1)–(1.2) for $t_0 + t_1 \le t \le t_0 + 2t_1$ with initial history up to t_0+t_1 given by the process

$$\Psi_1(t) = \begin{cases} x\_(t), & t \le t_0 \\ x(t), & t_0 \le t \le t_0+t_1 \end{cases}.$$

We note that $\Psi_1 \in \mathscr{A}_{t_0+t_1}$. By repeating the above procedure for $t_0+t_1 \le t \le t_0+2t_1$ we get a solution $x(t)$ such that

$$x(t) = \Psi(t_0+t_1) + \int_{t_0+t_1}^{t} F(s,x^s,Ax^s,Bx^s)ds +$$

$$+ \int_{t_0+t_1}^{t} H(s,x^s,Px^s,Qx^s)d\beta(s) = x\_(t_0) + \int_{t_0}^{t_0+t_1} F(s,x^s,Ax^s,Bx^s)ds +$$

$$+ \int_{t_0}^{t_0+t_1} H(s,x^s,Px^s,Qx^s)d\beta(s) + \int_{t_0+t_1}^{t} F(s,x^s,Ax^s,Bx^s)ds +$$

$$+ \int_{t_0+t_1}^{t} H(s,x^s,Px^s,Qx^s)d\beta(s) = x\_(t_0) + \int_{t_0}^{t} F(s,x^s,Ax^s,Bx^s)ds +$$

$$+\int_{t_0}^{t} H(s,x^s,Px^s,Qx^s)\,d\beta(s).$$

This shows that $x(t)$ is a continuous solution of the equation (1.1)-(1.2) with initial condition (2.9).

By assuming a solution $x(t)$ of the equation (1.1)-(1.2) for $t_0 \le t \le t_0+kt_1$, $k \ge 1$, exists and proceeding as above successively we construct a solution for $t_0 \le t \le t_0+(k+1)t_1$ with initial history process $\Psi_k(t)$, up to t_0+kt_1, as

$$\Psi(t) = \begin{cases} x\_(t), & t \le t_0 \\ x(t), & t_0 \le t \le t_0+kt_1 \end{cases}.$$

and

$$\Psi_k \in \mathscr{A}_{t_0+kt_1}.$$

Extending this procedure until $t_0+kt_1 \ge T$ we get a solution of the equation (1.1)-(1.2) for $t_0 \le t \le T$ with initial condition (2.9). and satisfying (2.10). This completes the proof of existence.

The details of the proof of uniqueness follows by closely looking at the proof of uniqueness of Theorem 2.1 given in [8] with suitable modifications. We omit the details.

The proof of Theorem 2 follows by an argument similar to that of Theorems 2.3 and 2.3' given in [8] with suitable modifications. We leave the details to the reader.

To prove Theorem 3 we define $B_N = \left\{ x \in X : \|x\| < N \right\}$,

$$x_{N_-}(\omega) = \begin{cases} x_-(\omega), & \text{if } \|x_-(\omega)\| \le N, \\ N\,\dfrac{x_-(\omega)}{\|x_-(\omega)\|}, & \text{if } \|x_-(\omega)\| > N, \end{cases}$$

and for $y,\ z \in R$,

$$F_N(t,s,x,y,z) = \begin{cases} F(t,s,x,y,z), & \text{for } x_N \in B\ , \\ F(t,s,x,y,z)(N+1)-\|x\|), & N \le \|x\| \le N+1, \\ 0, & \|x\| \ge N+1, \end{cases}$$

$$f_{iN}(t,s,x) = \begin{cases} f_i(t,s,x), & \text{for } x \in B_N, \\ f_i(t,s,x)(N+1-\|x\|), & N \le \|x\| \le N+1, \\ 0\ , & \|x\| \ge N+1, \quad i=1,2. \end{cases}$$

Similarly $H_N(t,x,y,z)$ and $h_{iN}(t,s,x)$, $i=1,2$ can be defined. The rest of the proof follows by an argument similar to that given in ([8], Theorems 2.4 and 2.4') with suitable modifications and application of Theorem 2. We do not discuss the details of the proof.

Remark 3. We note that in [11] the present authors have obtained some interesting results on the explosion and asymptotic behavior of solutions of (1.1)–(1.2) without hereditary arguments. It should be noted here that the study of explosion and asymptotic behavior of solutions (1.1)–(1.2) in fact will be of great interest in view of their usefulness in various applications and we hope to succeed in the study of these problems in our forthcoming paper.

REFERENCES

[1] L. Arnold: Stochastic differential equations : Theory and applications , John Wiley and Sons, London, 1974.

[2] M.A.Berger , V.J.Mizel : Volterra equations with Itô integrals, J. Integral Equations 2,(1980), 188–245.

[3] A.T.Bharucha–Reid : Random integral equations, Academic Press, New York, 1972.

[4] A.Friedman : Stochastic differential equations and applications, Vol. 1, Academic Press, New York, 1975.

[5] I.I.Gihman , A.V.Skorohod : Stochastic differential equations, Springer–Verlag, New York, 1972.

[6] N.Ikeda , S.Watanabe : Stochastic differential equations and diffusion processes, (North Holland, 1981).

[7] D. Kannan , A. T. Bharuha–Reid : Random integral equation formulation of a generalized Langevin equation, J. Statistical Physics, 5 (1972) 209–233.

[8] V.J. Mizel , V. Trutzer : Stochastic hereditary equations: existence and asymptotic stability, J. Integral Equations 7 (1984) 1–72.

[9] M.G.Murge , B.G.Pachpatte : On second order Itô type stochastic integrodifferential equations, An.Sti. Univ. ,Al.I.Cuza, Iasi 30, (1984) 25–34.

[10] M.G.Murge , B.G.Pachpatte : On generalized Itô type stochastic integral equation, Yokohama Math.J. 34

(1986) (to appear).

[11] M.G. Murge, B.G. Pachpatte: Explosion and asymptotic behavior of nonlinear Itô type stochastic integrodifferential equations, Kodai Math. J., 9 (1986) 1-18.

[12] B.G.Pachpatte: On Itô type stochastic integro – – differential equations, Tamkang J.Math. 10 (1979) 1-18.

[13] C.P.Tsokos, W.J. Padgett: Random integral equations with applications to life sciences and engineering, Academic Press, New York, 1974.

[14] C.Tudor: Successive approximations for solutions of stochastic integral equations of Volterra type, J.Math. Anal. Appl. 10 (1984), 27-37.

[15] B.S.White: The effects of a rapidly - fluctuating random environment on systems of interesting species, SIAM J. Appl.Math.32 (1977) 666-693.

[16] B.S.White: Some limit theorems for stochastic delay - differencial equations, Comm. Pure. App. Math. 29 (1976) 113-141.

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ON SOME METHOD OF SOLVING LINEAR EQUATIONS, III

1. This paper is a continuation of [2], [3]. Its main purpose is the further study of the solution of the equation $A\varphi = \psi$ (ψ is fixed) in the case when A is the sum of the finite number of the linear operators. As in the papers [2], [3] we look for the solutions of such an equation in the class of infinite series defined in [3]. In the second part of this paper we show, without the assumption of ELI of H, the construction of the sequence H. The symbols and the definitions used in [1]–[3] are retained.

Let $\langle X, \|\cdot\| \rangle$ be a real Banach space and $H = (h_n)_{n=1,2,..}$ –ELI sequence of the elements of the space X (cf. [1]). We may easily to prove the following theorem.

Theorem 1.1. If the linear operators $A^{(i)}: X \longrightarrow X$, $i = 1,2,...n$, map H^L onto U_H, then the operator

$$A := A^{(1)} + A^{(2)} + \dots + A^{(n)}$$

maps H^L onto U_H and

$$\overline{A} := \overline{A}^{(1)} + \overline{A}^{(2)} + \dots + \overline{A}^{(n)}.$$

Moreover, if $A^{(i)}$, $i = 1,2,...n$, are H–continuous, then the operator A is H–continuous.

Further, as in [3], we shall restrict ourselves to such operators A, for which matrices $\overline{A}$, connected with A, are of the class $\tilde{s}_0^{\,2}$ and $\tilde{s}_2^{\,2}$.

In general, we can observe that, for $\overline{A}$, $\overline{B} \in \tilde{s}_2^{\,2}$ the matrix $\overline{A} + \overline{B}$ is not an element of $\tilde{s}_2^{\,2}$, however, the following theorem can be proved.

Theorem 1.2.

a) If $\overline{A}^{(i)} \in \tilde{s}_2^{\,2}(w_i^0, w_i^+)$ for $i = 1, 2, \ldots, n$ and $w_i^0 \le w_n^0$, $w_i < w_n$ for $i = 1, 2, \ldots, n-1$, then

$$\overline{A} = \overline{A}^{(1)} + \overline{A}^{(2)} + \ldots + \overline{A}^{(n)} \in \tilde{s}_2^{\,2}(w_n^0, w_n^+),$$

b) if $\overline{A}^{(i)} \in cs^2$ ($\overline{A}^{(i)} \in ls^2$), then $\overline{A} \in cs^2$ ($\overline{A} \in ls^2$),

c) if $w_1 = w_2 = \ldots = w_n$ and

$$\sum_{i=1}^{n} \mu_k(\overline{A}^{(i)}) \ne 0 \text{ for } k \ge 1 + \min_{i \le n} w_i^+ , \tag{1.1}$$

then

$$\overline{A} \in \tilde{s}_2^{\,2}(\max_{i \le n} w_i^0, \min_{i \le n} w_i^+).$$

Theorem 1.3. Let $\overline{A}$, $\overline{B} \in \tilde{s}_2^{\,2}$ and $r \in \mathcal{N}$.

a) If $w_A^0 \le w_B^0$ and $w_A \le w_B$, then $\rho^{A+B}(r) = \rho^B(r)$,

b) if $w_A = w_B$, $\mu_n(\overline{A}) + \mu_n(\overline{B}) \ne 0$, and $\mu_n(\overline{A})\,\mu_n(\overline{B}) > 0$ for $n \ge 1 + \min(w_A^+, w_B^+)$ and $\rho^A(r) < \infty$, $\rho^B(r) < \infty$, then $\rho^{A+B}(r) < \infty$.

Proof. We omit the easy proof of a). In the proof of b) assume that $w_A^0 = w_B^0 = 0$. Then we have $w_A^+ = w_B^+ := w$ and, applying Lemma 1.4 of [3], we get

$$\rho_k^{A+B}(r) = \max_{1 \le j \le r} \frac{(a_{k+j,k+j+w} + b_{k+j,k+j+w})^2}{(a_{k,k+w} + b_{k,k+w})^2} \le$$

$$\le \max_{1 \le j \le r} \left[\frac{a^2_{k+j,k+j+w}}{a^2_{k,k+w}} + \frac{b^2_{k+j,k+j+w}}{b^2_{k,k+w}} \right] \le \rho_k^A(r) + \rho_k^B(r).$$

Hence we have $\rho^{A+B}(r) \le \rho^A(r) + \rho^B(r)$. The proof is analogous if $w_A^0 = w_B^0 \ne 0$.

Assume now that $w_A^0 \le w_B^0$. Then $w_A^+ \le w_B^+$ and, by Theorem 1.2, $\bar{A} + \bar{B} \in \tilde{s}_2^{\,2}(w_B^0, w_B^+)$. Consider the matrix $\bar{B}' = [b_{ij}]_{\substack{i \ge 1-w_A^0 \\ j=1,2,\dots}}$. We see, that $\rho^{A+B}(r)$ differs from $\rho^{A^+ + B'}(r)$ at most by a finite number. To $\rho^{A^+ + B'}(r)$ can be applied the previous result. Hence we see that $\rho^{A^+ + B'}(r) < \infty$ and the proof of b) is finished.

In similar way the following theorem may be proved.

Theorem 1.4. If $\bar{A}, \bar{B} \in \tilde{s}_2^{\,2}$, $w_A = w_B$, $\mu_n(\bar{A}) + \mu_n(\bar{B}) \ne 0$, $\mu_n(\bar{A})\, \mu_n(\bar{B}) \ne 0$ for $n \ge 1 + \min(w_A^+, w_B^+)$ and $\Sigma_A < \infty$, $\Sigma_B < \infty$, then $\Sigma_{A+B} < \infty$.

Next we shall investigate the condition : $\Sigma_{A+B} < \infty$ in more general case. Let $\bar{A}, \bar{B} \in \tilde{s}_2^{\,2}$ and $w_A \ne w_B$. We denote, for $n \ge 1 + \max(w_A^+, w_B^+) =: n_0$

$$c_n^{A,B} := \begin{cases} \dfrac{\mu_n^2(\bar{A})}{\mu_n^2(\bar{B})} & \text{if } w_A < w_B, \\[2ex] \dfrac{\mu_n^2(\bar{B})}{\mu_n^2(\bar{A})} & \text{if } w_A > w_B, \end{cases}$$

$$c^{A,B} := \sum_{n=n_0}^{\infty} c_n^{A,B}, \quad \sigma^{A,B} := \sup_n c_n^{A,B}$$

Theorem 1.5. If $\bar{A}, \bar{B} \in \tilde{s}_2^2$, $w_A < w_B$, $w_A^0 \le w_B^0$, $\Sigma_A < \infty$, $\Sigma_B < \infty$ and $c^{A,B} < \infty$, then $\Sigma_{A+B} < \infty$.

Proof. First, suppose that $w_A^0 = w_B^0 = 0$ and let $w := w_A^+$, $w' := w_B^+$. Since $w_A < w_B$ we have $w < w'$. By Theorem 1.2, we infer that $\bar{A} + \bar{B} \in \tilde{s}_2^2(0, w)$. Assume that $w' - w \ge 2$. From the definition of the functional Σ_A for $\bar{A} \in \tilde{s}_2^2(0, w)$ we have

$$\Sigma_A = \sum_{j=1+w}^{\infty} \Bigg(\sum_{i=j-w+1}^{\infty} \frac{a_{ij}^2}{a_{j-w,j}^2} \Bigg).$$

Hence, having $\bar{A} + \bar{B} \in \tilde{s}_2^2(0, w')$, we get

$$\Sigma_{A+B} = \sum_{j=1+w'}^{\infty} \Bigg(\sum_{i=j-w'+1}^{j-w-1} \frac{b_{ij}^2}{b_{j-w',j}^2} + \sum_{i=j-w}^{\infty} \frac{(a_{ij} + b_{ij})^2}{b_{j-w',j}^2} \Bigg).$$

From Lemma 1.4 of [3] we have

$$\frac{(a_{ij} + b_{ij})^2}{b_{j-w',j}^2} = \frac{(a_{ij} + b_{ij})^2}{\Big(\frac{1}{2} b_{j-w',j} + \frac{1}{2} b_{j-w',j} \Big)^2} \le$$

$$\le 4 \left[\frac{a_{ij}^2}{b_{j-w',j}^2} + \frac{b_{ij}^2}{b_{j-w',j}^2} \right].$$

Thus

$$(1.3) \quad \Sigma_{A+B} \le 4 \sum_{j=1+w'}^{\infty} \left[\sum_{i=j-w'+1}^{\infty} \frac{b_{ij}^2}{b_{j-w',j}^2} + \sum_{i=j-w}^{\infty} \frac{a_{ij}^2}{b_{j-w',j}^2} \right] =$$

$$= 4 \left[\Sigma_B + \sum_{j=1+w'}^{\infty} \sum_{i=j-w}^{\infty} \frac{a_{ij}^2}{b_{j-w',j}^2} \right].$$

Denote by g the second element of the last sum. By definition of $c^{A,B}$, we have

$$g = \sum_{j=1+w'}^{\infty} \left[\frac{a_{j-w,j}^2}{b_{j-w',j}^2} + \sum_{i=j-w+1}^{\infty} \frac{a_{j-w,j}^2}{b_{j-w',j}^2} \, \frac{a_{ij}^2}{a_{j-w,j}^2} \right] \le$$

$$\le c^{A,B} + \sigma^{A,B} \sum_{j=1+w'}^{\infty} \left[\sum_{i=j-w+1}^{\infty} \frac{a_{ij}^2}{a_{j-w,j}^2} \right] =$$

$$= c^{A,B} + \sigma^{A,B} \left[\Sigma_A - \sum_{j=1+w}^{w'} \left[\sum_{i=j-w+1}^{\infty} \frac{a_{ij}^2}{a_{j-w,j}^2} \right] \right] \le$$

$$\le c^{A,B} + \sigma^{A,B} \, \Sigma_A.$$

Hence and from (1.3) we infer that $\Sigma_{A+B} < \infty$.

Now, suppose that $w' - w = 1$. Similarly as above, we get that

(1.4) $$\Sigma_{A+B} = \sum_{j=1+w'}^{\infty} \left(\sum_{i=j-w+1}^{\infty} \frac{(a_{ij} + b_{ij})^2}{b^2_{j-w,j}} \right) \le$$

$$\le 4 \left(\Sigma_B + \sum_{j=1+w'}^{\infty} \left(\sum_{i=j-w+1}^{\infty} \frac{a^2_{ij}}{b^2_{j-w',j}} \right) \right).$$

Denote by $\tilde{g}$ the second element of the last sum. Sinc $w' = 1 + w$, we have

$$\tilde{g} = \sum_{j=2+w}^{\infty} \left(\frac{a^2_{j-w,j}}{b^2_{j-w-1,j}} + \right.$$

$$\left. + \sum_{i=j-w+1}^{\infty} \frac{a^2_{j-w,j}}{b^2_{j-w,j}} \, \frac{a^2_{ij}}{a^2_{j-w,j}} \right) \le c^{A,B} + \sigma^{A,B} \, \Sigma_A.$$

Hence and from (1.4) we infer that $\Sigma_{A+B} < \infty$.

Now, in general case $w^0_A \neq 0$ or $w^0_B \neq 0$ the proof i immediate. Indeed, it is enough to observe, that $\Sigma_{A^+ + B'} <$ where B' is determined in the proof of Theorem 1.3.

Remark 1.1. The following example shows that th assumption $c^{A,B} < \infty$ is essential.

Let

$$\overline{A} = \begin{bmatrix} 1 & 0 & 0 & 0 & \ldots \\ 0 & 1 & 0 & 0 & \ldots \\ 0 & 0 & 1 & 0 & \ldots \\ \ldots & & & & \end{bmatrix} \in \tilde{s}^2_2(0,0), \quad \overline{B} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & \ldots \\ 0 & 0 & 1 & 0 & 0 & \ldots \\ 0 & 0 & 0 & 1 & 0 & \ldots \\ \ldots & & & & & \end{bmatrix} \in \tilde{s}^2_2(0,1)$$

We see that $\Sigma_A = \Sigma_B = 0 < \infty$, however $c^{A,B}_j = 1$ for $j \ge 2$, $c^{A,B} = \infty$. It is easy to prove that $\Sigma_{A+B} = \infty$.

We can generalize Theorems 1.4 and 1.5 as follows.

Theorem 1.6. If $\overline{A}^{(i)} \in \tilde{s}_2^2(w_i^0, w_i^+)$, $\Sigma_{A^{(i)}} < \infty$ for $i = 1, 2, \dots, n$ and if

a) if $w_1 = w_2 = \dots = w_n$, $\sum_{i=1}^{n} \mu_k(\overline{A}^{(i)}) \neq 0$,

$\mu_k(\overline{A}^{(i)})\, \mu_k(\overline{A}^{(j)}) \neq 0$ for $k \geq 1 + \min_{i \leq n} w_i^+$ and $i, j = 1, 2, \dots, n$;

or

b) $w_j^0 < w_n^0$, $w_j < w_n$ and $c^{A^{(j)}, A^{(j+1)}} < \infty$ for $j = 1, 2, \dots, n-1$,

then $\Sigma_A < \infty$, for $\overline{A} = \overline{A}^{(1)} + \dots + \overline{A}^{(n)}$.

The following example ilustrates certain applications of the above results.

Example 1.1. Let $\langle X, \|\cdot\| \rangle$ be the space of real functions bounded on $I_r := \mathcal{R} - (-r, r)$, $r > 1$, with the norm defined in the standard way. We put $h_n := x^{1-n}$ for $n = 1, 2, \dots$, $H = (h_n)$. Of course H is ELI and $\|h_n\| = r^{n-1}$, $n = 1, 2, \dots$. Since $r > 1$, we have $\sum_{n=1}^{\infty} \|h_n\| = \frac{r}{r-1}$.

Let A be the operator such that $Ay = y'$, where $y \in X$ is a function with bounded derivative on I_r.

Let Λ_k, $k = 1, 2, \dots$, be the linear operators from X to X, such that for an arbitrary $\varphi \in X$ we have $\Lambda_k \varphi = h_k \varphi$. Because for $\varphi \in X$ we have $\|\Lambda_k \varphi\| \leq \|h_k\|\, \|\varphi\|$, $k = 1, 2, \dots$, so Λ_k are continuous on X, hence Λ_k are H-continuous too.

Denote $h_{j_k}^k := \sum_{n=1}^{\wedge} c_n^k h_n$ for $k = 0, 1, 2, \dots$ and $j_k \in \mathcal{N}$, where $c_n^k \in \mathcal{R}$ for $n = 1, 2, \dots$.

We construct for an arbitrary fixed $m = 0, 1, 2, \dots$ the differential equation of the form

(1.5) $$\sum_{k=0}^{m} h_{j_k}^{k}\, y^{(k)} = \Psi$$

where $\Psi := \sum_{n=1}^{\infty} b_n h_n$ is an arbitrary fixed element of the space U_H^2.

Further we shall apply the following notations :

$$B_{ik} = \Lambda_i\, A^k,\quad B_k = \left[c_1^k\, \Lambda_1 + c_2^k\, \Lambda_2 + \dots + c_{j_k}^k\, \Lambda_{j_k} \right] A^k,$$

where $i = 1,2,\dots$; $k = 0, 1, 2,\dots$ and A^0 is the identical operator.

Now we can rewrite the equation (1.5) in the form

(1.6) $$By := (B_0 + B_1 + \dots + B_m)\, y = \Psi.$$

It is easy to verify that B_i, $i = 1,2,\dots m$, map H^L onto U_H and B_i are H-continuous. From Theorem 1.1 we infer that the operator B is H-continuous too, it maps H^L onto U_H and

$$\bar{B} = \bar{B}_0 + \bar{B}_1 + \dots + \bar{B}_m.$$

Moreover for $i = 1,2,\dots$ and $k = 0,1,2,\dots$ we have

$$\bar{B}_k = c_1^k \bar{B}_{1k} + c_2^k \bar{B}_{2k} + \dots + c_{j_k}^k \bar{B}_{j_k k}\ ,\quad \bar{B}_{ik} = \bar{\Lambda}_i \bar{A}^k$$

Since $A(h_k) = (1-k)h_{k+1}$ for $k = 1,2,\dots$, we state that $\bar{A}^0 \in \tilde{s}_2^2(0,0) \cap cs^2(1)$ and $\bar{A}^n \in \tilde{s}_2^2(-n-1,1) \cap cs^2(1)$, $n = 1,2,\dots$

From definition of Λ_n and H we also have

$$\bar{\Lambda}_n \in \tilde{s}_2^2(-n+1,0) \cap cs^2(1) \quad \text{for } n = 1,2,\dots$$

Finally by Theorem 1.6 of [3] we get

$$\bar{B}_{k0} = \bar{\Lambda}_k,\ \bar{B}_{kn} \in \tilde{s}_2^2(-n-k,1) \cap cs^2(1), \quad \text{for } k,n = 1,2,\dots$$

Since $\Sigma_A = \Sigma_{\Lambda_k} = 0 < \infty$ for $k = 1,2,\ldots$ $\bar{A} \in cs^2(1)$, $\rho^A(1) = 4 < \infty$ and $\rho^{\Lambda_k}(t) = 1$ for $t = 1,2,\ldots$ we get, by Theorem 1.8 and Corollary 1.4 of [3], the conditions $\Sigma_{A^n} < \infty$, $\Sigma_{B_{kn}} < \infty$ for $k = 1,2,\ldots$; $n = 0,1,2,\ldots$. Hence we see that the matrices $\bar{B}_{kn}$ are regular.

Let us denote, for $k = 0,1,\ldots m$,

$$m_k := \begin{cases} \min\{n:\ c_n^k \neq 0\} \\ 0 \text{ if } c_n^k = 0 \text{ for } n = 1,2,\ldots,j_k \end{cases}$$

We can easly see that

(1.7) $\bar{B}_0 \in \tilde{s}_2^{\,2}(-m_0+1,0)$, $\bar{B}_k \in \tilde{s}_2^{\,2}(-m_k-k,1)$ for $k = 1,2,\ldots,m$;

(1.8) $$\mu_n\left[c_1^k\bar{\Lambda}_1 + c_2^k\bar{\Lambda}_2 + \ldots + c_{j_k}^k\bar{\Lambda}_{j_k}\right] = c_{m_k}^k,\quad k = 0,1,\ldots,m,\ n = 1,2,\ldots .$$

Hence we have $\rho^{c_1^k\Lambda_1+c_2^k\Lambda_2+\ldots+c_{j_k}^k\Lambda_{j_k}}(1) = 1$ by $\bar{A}^k \in cs^2(1)$. By $\bar{A}_k \in cs^2(1)$ the definition of B_k and Theorem 1.8 of [3] we infer that

(1.9) $$\Sigma_{B_k} < \infty,\ k = 1,2,\ldots,m.$$

Further, we see from (1.7) that $w^+_{B_k} = 1$ for $k = 1,2,\ldots,m$. Thus, by Theorem 1.7 of [3], we get for $n \geq 2$ and $k \in \mathbb{N}$ that

(1.10) $$\mu_n(\bar{B}_k) = \mu_{n-w_{A^k}}\left[c_1^k\bar{\Lambda}_1 + c_2^k\bar{\Lambda}_2 + \ldots + c_{j_k}^k\bar{\Lambda}_{j_k}\right]\mu_n\bar{A}_k.$$

On the other hand $w_{A^k} = -k$, thus again applying Theorem 1.7 we have, for $n \geq w^+_{A^k} + 1 = 2$,

$$\mu_n(\overline{A}^k) = \mu_{n+k-1}(\overline{A}) \dots \mu_n(\overline{A}) = (-1)^k \, (n+k-2) \, (n+k-3) \, \dots \, (n-1).$$

Hence, (1.8) and (1.10) imply, for $k = 1,2,\dots,m$ and $n \geq 2$, the relations

$$\text{(1.11)} \quad \begin{cases} \mu_n(\overline{B}_k) = (-1)^k \, c^k_{m_k} (n+k-2)(n+k-3)\dots(n-1), \\ \\ \mu_n(\overline{B}_0) = \mu_n(c^{0\ k}_{m_0} \, \overline{\Lambda}_{m_0}) = c^{0\ k}_{m_0}. \end{cases}$$

Finally, for $k = 0,1,\dots,m$ and $n \geq 1 + \max(w^+_{B_0}, w^+_{B_1}, \dots, w^+_{B_m}) = 2$, we have

$$\frac{\mu_n(\overline{B}_k)}{\mu_n(\overline{B}_{k+1})} = \frac{c^k_{m_k}}{c^{k+1}_{m_{k+1}}} \, \frac{1}{n+k-1} \, ,$$

hence

$$\text{(1.12)} \qquad c^{B_k, B_{k+1}} = \left(\frac{c^k_{m_k}}{c^{k+1}_{m_{k+1}}} \right)^2 \sum_{n=2}^{\infty} \frac{1}{(n+k-1)^2} \, .$$

Now assume that $w^0_{B_i} \leq w^0_{B_m}$ and $w_{B_i} \leq w_{B_m}$ for $i = 0,1,\dots m-1$ thus

$$\text{(1.13)} \qquad m_i > m_m + m - 1 \quad \text{for } i = 0,1,\dots m-1.$$

Hence using (1.12), (1.13) and Theorem 1.6(a), we infer that $\overline{B} \in \tilde{s}^2_2(-m_m-m,1)$ and $\Sigma_B < \infty$, thus $\overline{B}$ is the regular matrix.

Basing on the results of [2] we can conclude that if $b_1 =$

$= b_2 = \dots = b_{m_m+1}$ and the inequalities (1.13) are fulfilled, then the equation (1.5) has a solution in U_H and

$$\dim U_H(\Psi, B) \geq 1.$$

If $m \geq 1$, then, in view of (1.11), we have for $n \geq 2$

$$\mu_n(\bar{B}) = \mu_n(\bar{B}_m) = (-1)^m c^m_{m_m} (n+m-2)(n+m-3) \dots (n-1).$$

Hence, for $n > 2$ $\|\bar{B}_n\|_2 \geq |c^m_{m_m}| > 0$ and $U_H(\Psi, B) \subset U^1_H$.

Let us suppose that $w_{B_0} = w_{B_1} = \dots = w_{B_m}$, $\sum_{n=1}^{m} \mu_k(\bar{B}_n) \neq 0$ and $\mu_k(\bar{B}_i)\, \mu_k(\bar{B}_j) > 0$ for $i, j = 0,1,\dots,m$ and $k \geq 2 = 1 + {}$ $+ \min_{0 \leq i \leq m} w^+_{B_i}$.

The above assumptions are fulfilled if and only if

$$(1.14)\quad \begin{cases} m_i + i = m_j + j,\ (-1)^{i+j} c^{j_0}_{m_j} c^{j}_{m_i} \geq 0 \\ \text{for } i,j = 0,1,\dots,m, \\ c^0_{m_0} + \sum_{k=1}^{m} (-1)^k c^k_{m_k} (n+k-2)(n+k-3)\dots(n-1) \neq 0 \\ \text{for } n \geq 2. \end{cases}$$

From Theorem 1.6 b) we have that $\bar{B} \in \tilde{s}^2_2(\max_{1 \leq k \leq m} (-m_0+1, -m_k-k), 0)$ and $\Sigma_B < \infty$. So, if $b_1 = b_2. \dots = b_{-w^0_B} = 0$, then the equation (1.5) has a solution in U_H. As previously we state that $U_H(\Psi, B) \subset U^1_H$.

Let us remark that we may choose the coefficients c^j_i and $w^0_{B_i}, w^+_{B_i}$ in the equation (1.5) in order that $\bar{B} \in \tilde{s}^2_2$ and $w^+_B = m$.

Then we have $U_H(\Psi,B) = m$ and all solutions of (1.5) belong to U_H^1. In particular, if we consider the equation

$$(1.11) \qquad C_1^3 y''' + \left[\frac{C_2^2}{x} + \dots + \frac{C_{j_2}^2}{x^{j_2-1}} \right] y'' +$$

$$+ \left[\frac{C_3^1}{x^2} + \dots + \frac{C_{j_1}^1}{x^{j_1-1}} \right] y' + \left[\frac{C_7^0}{x} + \dots + \frac{C_{j_0}^0}{x^{j_0-1}} \right] y = b(x),$$

where $m = 3$, $6C_1^3 = C_2^2 = C_3^1 \neq 0$, $2C_3^2 = C_4^1$, $C_7^0 \neq 0$, $j_2 \geq 2$, $j_1 \geq 3$, $j_0 \geq 7$, then $\bar{B} \in \tilde{s}_2^{\,2}(-6,3)$, $\bar{B}$ is regular matrix, $\|\bar{B}_k\|_2 \geq (k-1)\ |C_3^1 - k(5-k)C_1^3|$ for $k \geq 4$. Hence, for $\Psi \in U_H^2$ and $b_i = 0$ for $i = 1,2,\dots 6$ we have that all solutions of (1.11) belong to U_H^1. The method of finding those solutions we have shown in [2].

2. Let $\langle X, \|\cdot\| \rangle$ be a real linear normed space and $H = (h_n)_{n=1,2,\dots}$ an arbitrary sequence of the elements of the space X (H need not to be ELI). As previously we shall apply the notations H^L and U_H.

Assume that A is a linear operator from X to X and $A\colon H^L \longrightarrow U_H$. Hence, for $j = 1,2,\dots$ there exist numbers a_{ij}, $i = 1,2,\dots$, such that

$$(2.1) \qquad A(h_j) = \sum_{i=1}^{\infty} a_{ij} h_i \in U_H.$$

So, the operator A is connected with the matrix

$$(2.2) \qquad \bar{A} = \left[a_{ij} \right]_{i,j=1,2,\dots},$$

where a_{ij} are arbitrary numbers satysfying (2.1). If H is not ELI, then more matrices of the form (2.2) can exist.

For example, if H is constant i.e. $H = (h_1, h_1, \ldots)$ and $A(h_1) = 2h_1$ we may assume

$$\bar{A} = \begin{bmatrix} 2\,2\,2 \ldots \\ 0\,0\,0 \ldots \\ 0\,0\,0 \ldots \\ \ldots\ldots\ldots \end{bmatrix}, \quad \bar{A} = \begin{bmatrix} 2\,0\,0 \ldots \\ 0\,2\,0 \ldots \\ 0\,0\,2\,\;0 \\ \ldots\ldots\ldots \end{bmatrix} \text{ or } \bar{A} = \begin{bmatrix} 1 & 0 & 0 \ldots \\ \frac{1}{2} & 0 & 0 \ldots \\ \frac{1}{2^2} & 0 & 0 \ldots \\ \ldots & \ldots & \ldots \end{bmatrix}.$$

If $H = (0)_{n=1,2,\ldots}$ i.e. $h_n = 0$ for $n = 1,2,\ldots$, then $\bar{A}$ may be an arbitrary two-dimensional infinite matrix.

If $A\colon H^L \longrightarrow H^L$ and H is linearly independent, then $\bar{A}$ is determined explicitly.

Further, assume that A is H-continuous operator and let $\bar{A} = \left[a_{ij} \right]$ be a fixed matrix of A. Let us consider the equation

$$(2.3) \qquad A\varphi = \Psi,$$

where $\Psi = \sum_{n=1}^{\infty} b_n h_n \in U_H^2$.

Lemma 2.1. If $\bar{A} \in \tilde{s}_2^{\,2} \cap cs^2$, $w_A^0 = 0$ and $\Sigma_A < \infty$, then

(a) if $\inf \|\bar{A}_n\|_2 =: \gamma > 0$ and $\sum_{n=1}^{\infty} \|h_n\| < \infty$, then the equation (2.3) has a solution $\varphi = \sum_{n=1}^{\infty} c_n h_n \in U_H^1$ such that the series $\sum_{n=1}^{\infty} c_n h_n$ is absolutly convergent and

$$\|\bar{\varphi}\|_1 \le \frac{\sqrt{\Pi_A}}{\gamma}\, \|\bar{\Psi}\|_2,$$

where $\bar{\varphi} = (c_n)$, $\bar{\Psi} = (b_n)$, $A_k = \left[a_{nk} \right]_{n=1,2,\ldots}$

(b) if $\sup \|\bar{A}_n\|_2 < \infty$, $\sum_{n=1}^{\infty} \frac{\|h_n\|}{\|A_n\|_2} < \infty$, then the equation (2.3) has a solution $\varphi = \sum_{n=1}^{\infty} c_n h_n \in U_H$ such that the series $\sum_{n=1}^{\infty} c_n h_n$ is absolutly convergent and

$$|c_j| \le \sqrt{\Pi_A}\, \frac{\|\bar{\Psi}\|_2}{\|\bar{A}_j\|_2}, \quad j = 1,2,\ldots$$

If $w_A^+ = 0$ then $\bar{\varphi}$ is the unique solution of the equation

$$\bar{A}\,\bar{\varphi} = \bar{\Psi}. \tag{2.4}$$

Proof. From the proof of Lemma 4.1 and Theorem 4.1 of [2] it implies that the equation (2.4) has exactly one solution $\bar{\varphi} = (c_n)$ such that $c_i = 0$ for $i \le w_A^+$ and fulfilling the inequality from (a) or (b), respectively. We prove, similarly as in Theorem 5.3 of [2], that $\varphi = \sum_{n=1}^{\infty} c_n h_n$ fulfills the equation (2.3) and the series $\sum_{n=1}^{\infty} c_n h_n$ is absolutly convergent. Last part of Lemma 2.1 is obvious. Let us consider the equation

$$B\varphi = \Psi, \tag{2.5}$$

where $\Psi \in X$ and

$$B = c_0 I + c_1 A^{n_1} + \ldots + c_k A^{n_k}, \tag{2.6}$$

$c_0, c_1, \ldots, c_k \in \mathcal{R}$, $c_0 \ne 0$, $n_1, \ldots, n_k \in \mathcal{N}$.

At first take (2.5) with $c_0 = \lambda \ne 0$, $c_1 = -1$, $n_1 = 1$ and $c_i = 0$ for $i = 2,3\ldots k$, thus we consider the equation

$$(\lambda I - A)\varphi = \Psi \quad \text{i.e.} \quad B_\lambda \varphi = \Psi, \tag{2.7}$$

Let $K = (k_n)_{n=1,2,\ldots}$ be an arbitrary sequence of numbers different from zero and $\Psi \neq 0$. Denote

$$(2.8') \quad \begin{cases} h^K_n := k_n A^{n-1}(\Psi)\ , \ h_n := A^{n-1}(\Psi) \quad n = 1,2,\ldots \\ H^K := (h^K_n), \qquad H_\Psi := (h_n). \end{cases}$$

Theorem 2.1. If the operator A is H_Ψ-continuous (in particular A is closed) and there exists a sequence $K = (k_n)$ such that

$$(2.9) \qquad \sum_{n=1}^{\infty}\left[\frac{k_n}{k_{n+1}}\right]^2 < \infty \quad \text{and} \quad \sum_{n=1}^{\infty} |k_n|\ \|A^{n-1}(\Psi)\| < \infty ,$$

then for every numbers $\lambda \neq 0$ the equation (2.7) has the solution $\varphi \in U_H$ of the form

$$(2.10) \qquad \varphi = \sum_{n=1}^{\infty} \frac{A^{n-1}(\Psi)}{\lambda^n} .$$

Proof. From (2.8) and the definition of B_λ we have

$$h^K_1 = k_1\Psi\ ,\ h^K_n = \frac{k_n}{k_{n+1}} A\left[h^K_{n+1}\right] ,\ n = 2,3,\ldots ,$$

and

$$B_\lambda\left[h^K_n\right] = \lambda h^K_n - \frac{k_n}{k_{n+1}} h^K_{n+1} \quad n = 1,2,\ldots .$$

Hence, we get $\bar{B}_\lambda \in \tilde{s}^2_2(0,0) \cap cs^2(2)$ and $\|(B_\lambda)_n\|_2 = \sqrt{\lambda^2 + \frac{k_n}{k_{n+1}}} >$ $> |\lambda| > 0$, $n = 1,2,\ldots$. By (2.9), we infer that

$$\Sigma_{B_\lambda} = \sum_{n=1}^{\infty} \frac{1}{\lambda^2}\left[\frac{k_n}{k_{n+1}}\right]^2 < \infty\ ,\ \sum_{n=1}^{\infty} \|h^K_n\| = \sum_{n=1}^{\infty} |k_n|\ \|A^{n-1}(\Psi)\| < \infty.$$

Since $\Psi = \frac{1}{k_1} h_1^K \in U_H$ and $\overline{\Psi} = \left[\frac{1}{k_1}, 0, 0, \ldots \right]$, we infer, by Lemma 2.1, that the equation (2.7) has a solution φ. Solving the equation $\overline{B}_\lambda \overline{\varphi} = \overline{\Psi}$ we obtain that $\overline{\varphi} = \left[\frac{1}{k_n \lambda^n} \right]$. Hence we have

$$\varphi = \sum_{n=1}^{\infty} \frac{h_n^K}{k_n \lambda^n} = \sum_{n=1}^{\infty} \frac{A^{n-1}(\Psi)}{\lambda^n} .$$

Moreover, by Lemma 2.1, the above series is absolutly convergent.

Now assume that the operator B is determined by (2.6). In the similar way as Theorem 2.1 we can prove the following theorem.

Theorem 2.2. If $\Psi \in X$, $\Psi \neq 0$ the operator A is H-continuous and there exists a sequence $K = (k_n)$ such that

$$(2.11) \quad \begin{cases} \sum_{n=1}^{\infty} \left(\frac{k_n}{k_{n+n_i}} \right)^2 < \infty , \; i = 1,2,\ldots k, \\ \sum_{n=1}^{\infty} |k_n| \, \| A^{n-1}(\Psi) \| < \infty , \end{cases}$$

then for every sequence of numbers $c_0, c_1, \ldots, c_k$, $c_0 \neq 0$ the equation (2.5) has the solution in U_H being the sum of the absolutely convergent series. The coefficients of this series can be found from the equation $\overline{B}\,\overline{\varphi} = \overline{\Psi}$.

Next theorem shows the method of finding of the sequence $K = (k_n)$.

Theorem 2.3. If $\Psi \in X$, $\Psi \neq 0$, $\frac{1}{t} < 2\min(n_1, \ldots, n_k)$,

$k = 1,2,\ldots$ and $g := \lim_{n\to\infty} \sup \sqrt[n]{(n!)^t \|A^{n-1}(\Psi)\|} < \infty$, then for every $k = 0,1,2,\ldots$ and an arbitrary numbers $c_0, c_1, \ldots c_k$, $c_0 \neq 0$ the equation (2.5) has the solution in U_H being the sum of the absolutely convergent series.

Proof. Suppose that $k_n = \frac{(n!)}{(g+1)^n}$, $n = 1,2,\ldots$. Then

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$$\lim_{n\to\infty} \sup \sqrt[n]{|k_n|\ \|A^{n-1}(\Psi)\|} =$$

$$= \frac{1}{g+1} \lim_{n\to\infty} \sup \sqrt[n]{(n!)^t \|A^{n-1}(\Psi)\|} = \frac{1}{g+1} < 1.$$

Moreover, for $i = 1,2,\ldots,k$, we have

$$\sum_{n=1}^{\infty} \left(\frac{k_n}{k_{n+n_i}}\right)^2 = (g+1)^{n_i} \sum_{n=1}^{\infty} \left(\frac{1}{(n+1)(n+2)\ldots(n+n_i)} \right)^{2t} \leq$$

$$\leq (g+1)^{n_i} \sum_{n=1}^{\infty} \frac{1}{n^{2n_i t}} < \infty .$$

Now we may apply Theorem 2.2 what finishes the proof.

Example 2.1. Let X be the space of real functions bounded on the interval $(-1,1)$. Let A be the operator given by

$$Ay = x \int_0^x t y(t) dt \quad \text{for } y \in DA.$$

Assume $\Psi(x) = x$ and consider the equation

(2.12)
$$B\varphi = \Psi ,$$

where

$$B = I - A - A^2 . \tag{2.13}$$

We have at once

$$A^n(\Psi) = \frac{x^{3n+1}}{3^n n!} , \quad \|A^n(\Psi)\| = \frac{1}{3^n n!} , \quad n = 0,1,\ldots$$

If we take t = 1, then $g = \lim\limits_{n\longrightarrow\infty} \sqrt[n]{n!\,\|A^{n-1}(\Psi)\|} = \frac{1}{3}$. Now we can define the sequence K. Assume that $k_n = \frac{3^n n!}{4^n}$, then

$$h_n^K = \frac{3n}{4^n}\,x^{3n-2}, \quad Ah_n^K = \frac{3^2(n+1)^2}{4^{2n+1}}\,x^{3n+1} \quad \text{for } n = 1,2,\ldots .$$

Hence we get

$$Bh_n^K = (I - A - A^2)\,h_n^K =$$

$$= \frac{3n}{4^n}\,x^{3n-2} - \frac{3^2(n+1)^2}{4^{2n+1}}\,x^{3n+1} - \frac{3^3(n+1)^3}{4^{3n+3}}\,x^{3n+4}$$

for n = 1,2,... . Thus $\overline{B} \in \tilde{s}_2{}^2(0,0)$. The sequence $\overline{\varphi} = (d_n)$ can be found from the equation $\overline{B}\,\overline{\varphi} = \overline{\Psi}$, where $\overline{\Psi} = (\frac{4}{3}, 0, 0, \ldots,)$.

After solving this equation (cf. [2]) we have $d_1 = \frac{16}{9}$, $d_2 = \frac{8}{3}$, and $d_n = \frac{3}{4^{n-1}}\left[\frac{3(n-1)^3}{4^{n-2}}\,d_{n-2} + n d_{n-1}\right]$ for n = = 3,4,... . Hence, the solution of (2.12) is of the form

$$\varphi(x) = \sum_{n=1}^{\infty} d_n h_n^K = \sum_{n=1}^{\infty} \frac{3nd_n}{4^n}\,x^{3n-2}.$$

The above series is absolutely convergent on the interval

(-1,1).

Suppose, that for certain $\Psi \in X$, $\Psi \neq 0$, the sequence $H_\Psi = (A^{n-1}(\Psi)_{n=1,2,\dots})$ is linealy dependent and investigate the existence of solutions of the equation (2.7) in U_{H_Ψ}. It is easily verified that if H_Ψ is lineary dependent, then there exists a natural number n such that the sequence

$$\left[\Psi,\ A(\Psi),\ \dots,\ A^n(\Psi),\ \dots,\ A^{n+k}(\Psi) \right],\ k = 1,2,\dots,$$

is linealy dependent. Further we shall denote the sequence $\left[\Psi,\ A(\Psi),\ \dots,\ A^n(\Psi) \right]$ by H^*_Ψ. We can show that $U_{H_\Psi} = U_{H^*_\Psi} = (H^*_\Psi)^L$. Thus, H^*_Ψ is the basis in U_{H_Ψ} and $\dim U_{H_\Psi} = n + 1$. Hence every solution of (2.7) is of the form

$$\varphi = d_0\Psi + d_1 A(\Psi) + \dots + d_n A^n(\Psi). \tag{2.14}$$

Let t_0, t_1, ..., t_n be defined by the equality

$$A^{n+1}(\Psi) = t_0\Psi + t_1 A^1(\Psi) + \dots + t_n A^n(\Psi).$$

We may easily calculate, that φ given by (2.14) fulfils (2.7) if and only if the coefficients d_0, d_1, ..., d_n fulfil the following system of the equations

$$\tag{2.15} \left\{ \begin{aligned} &\lambda d_0 - t_0 d_n = 1 \\ &-d_0 + \lambda d_1 - t_1 d_n = 0 \\ &-d_1 + \lambda d_2 - t_2 d_n = 0 \\ &\dots\dots\dots\dots\dots\dots \\ &-d_{n-1} + \lambda d_n - t_n d_n = 0. \end{aligned} \right.$$

The determinant of (2.15) is of the form

$$D_{n+1}(\lambda) := \lambda^{n+1} - t_n\lambda^{n} - t_{n-1}\lambda^{n-1} - \dots - t_1\lambda - t_0.$$

If $D_{n+1}(\lambda) \neq 0$, then the equation (2.7) has exactly one solution in $U_{H^*_\Psi}$. We can find it from (2.15) and (2.14). If $D_{n+1}(\lambda) = 0$, then (2.7) either has more than one solution or has no solution in $U_{H^*_\Psi}$.

Example 2.2. Consider the equation: $y - y' = x$. We have $H^*_\Psi = (x,1)$, $\varphi(x) = x + 1 \in U_{H^*_\Psi}$. For the equation: $y - y' = e^x$, $H^*_\Psi = (e^x)$, and there does not exist a solution in $U_{H^*_\Psi}$.

Example 2.3. Consider the integral equation

$$\lambda y - \int_0^{\Pi} \cos(x+t)y(t)dt = x \tag{2.16}$$

in the space $\langle X, \|\cdot\| \rangle$ of bounded functions on the interval $\langle 0,\Pi\rangle$. Of course, (2.16) is an equation of the form (2.3) and A is given by $A(\varphi) = \int_0^{\Pi} \cos(x+t)\varphi(t)dt$ for every function $\varphi \in X$, integrable on $\langle 0,\Pi\rangle$, $\Psi(x) = x$.

We see that: $A(\Psi) = -\Pi\sin x - 2\cos x$, $A^2(\Psi) = \frac{\Pi^2}{2}\sin x - \Pi\cos x$, $A^3(\Psi) = -\frac{\Pi^3}{4}\sin x - \frac{\Pi^2}{2}\cos x$. Hence, we have that $H^*_\Psi = \left[\Psi, A(\Psi), A^2(\Psi)\right]$ and $A^3(\Psi) = 0\Psi + \frac{\Pi^2}{4}A(\Psi) + 0A^2(\Psi)$. Thus we get $t_0 = 0$, $t_1 = \frac{\Pi^2}{4}$, $t_2 = 0$, $n = 2$. We see that in this case the system (2.15) has solutions for $\lambda \neq 0$, $\lambda \neq \pm \frac{\Pi}{2}$ and is contradictory for $\lambda = 0$, $\lambda = \pm \frac{\Pi}{2}$.

If λ differs from 0, $\pm \frac{\Pi}{2}$, then the solution $\varphi_\lambda \in U_{H^*_\Psi}$ is of the form

$$\varphi_\lambda(x) = \frac{x}{\lambda} - \frac{\Pi \sin x + 2\cos x}{\lambda^2 - \frac{\Pi^2}{4}} + \frac{\frac{\Pi^2}{2}\sin x - \Pi\cos x}{\lambda\left(\lambda^2 - \frac{\lambda^2}{4}\right)} .$$

If H_Ψ is the linearly dependent sequence we can obtain the analogous result for the equation (2.5). The analysis of existence of its solutions can be reduced to simple algebraic calculations.

REFERENCES

[1] I.Dziubiński, J. Połubiński: Some properties of bases in normed linear spaces, Demonstratio Math. 16 (1983) 561-580.

[2] I.Dziubiński, J. Połubiński: On some method of solving linear equation I, Demonstratio Math. 19 (1986) 77-103.

[3] I.Dziubiński, J. Połubiński: On some method of solving linear equation II, Demonstratio Math. 19 (1986) 791-807, 20 (1987) (in print).

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ON CONNECTIONS ON MANIFOLDS ADMITTING AN F(5,1)-STRUCTURE

Structures on C^{∞}-manifolds, by introducing vector valued linear functions satysfying some algebraic relations, have been studied by many mathematicians under various topics. Such a structure is the f(3,1)-structure ([6]) defined by K. Yano, who jointly with S. Ishihara ([3]) have studied its integrability conditions. Recently, W. Mozgawa and W. Rompała ([4]) have determined for this structure all linear connections satysfying $\nabla f = 0$ and all the torsion tensors of such connections. Then, using these family of linear connections they have calculated its obstructions to the integrability. F. Gouli-Andreou ([1]) defined the f(5,1)-structure and studied ([2]) its integrability conditions. Therefore, it is natural to ask ourselves if we can determine all linear connections satysfying $\nabla f = 0$ and their torsion tensors in the case of the f(5,1)-structure.

Our present objective is to determine the above connections, their torsions tensors and calculate the obstractions to the integrability of an f(5,1)-structure, using these family of linear connections.

1. Connections

Let V_n be an n-dimensional smooth connected differentiable manifold of class C^∞ and let there be given a tensor field f of type (1,1) of class C^∞ and of rank r satysfying the equation

$$f^5 + f = 0. \tag{1.1}$$

Such a structure is called ([1]) an f(5,1)-structure of rank r. A manifold V_n admitting an f(5,1)-structure is said to be an f(5,1)-manifold.

We put

$$l = f^2, \quad m = f^3, \quad v = f^4. \tag{1.2}$$

Definition 1.1. A linear connection on V_n to be an f(5,1)-connection if it satisfies the condition

$$\nabla_k f^i_j = 0. \tag{1.3}$$

From (1.3) we have

$$\partial_k f^i_j + \Gamma^i_{hk} f^h_j - \Gamma^h_{jk} f^i_h = 0.$$

The above relation can be written in the form

$$\Omega^{bi}_{ja} \Gamma^a_{bk} = - f^i_{j/k}, \tag{1.4}$$

where

$$\Omega^{bi}_{ja} = f^b_j \delta^i_a - \delta^b_j f^i_a \quad \text{and} \quad f^i_{j/k} = \partial_k f^i_j. \tag{1.5}$$

Proposition 1 1. For the operator Ω^{bi}_{ja}, we have that the operator

(1.6) $$\overline{\Omega}^{*ec}_{sp} = -f^e_s\,\delta^c_p + \delta^e_s\,f^c_p + \frac{5}{2}\,V^e_s\,f^c_p$$

satisfies the condition

(1.7) $$\Omega^{kt}_{hd}\,\Omega^{sd}_{kc}\,\overline{\Omega}^{*ec}_{sp}\,\Omega^{jp}_{ei}\,\Omega^{bi}_{ja} = \Omega^{bt}_{ha}\,.$$

The proof of the above proposition is a straightforward calculation.

If we put

(1.8) $$\overline{\Omega}^{*jd}_{ki} = \Omega^{sd}_{kc}\,\overline{\Omega}^{*ec}_{sp}\,\Omega^{jp}_{bi} = -m^j_k\,\delta^d_i + \delta^j_k\,m^d_i + 2\,f^j_k\,l^d_i + \frac{1}{2}\,l^j_k\,f^d_i + \frac{5}{2}\,V^j_k\,m^d_i\,,$$

then from Proposition 1.1 we have

Proposition 1.2. For the operator Ω^{bi}_{ja} we have that the operator Ω^{*bi}_{ja} satisfies the condition

(1.9) $$\Omega^{kt}_{hd}\,\Omega^{*jd}_{ki}\,\Omega^{bi}_{ja} = \Omega^{bt}_{ha}\,.$$

Taking account of (1.4) we obtain

Proposition 1.3. For the operators Ω^{bi}_{ja} and Ω^{*jd}_{ki} we have

(1.10) $$\Omega^{kt}_{hd}\,\Omega^{*jd}_{ki}\,f^i_{j/e} = f^t_{h/e}.$$

We recall now the following theorem of Rao—Mitra ([5]).

Theorem 1.1. If A is an arbitrary matrix and A^* is any matrix satisfying the relation $AA^*A = A$, then a necessary and sufficient condition for the existence of a solution of the equation

$$Ax = y \tag{1.11}$$

is that

$$AA^{*}y = y\ . \tag{1.12}$$

If this condition holds, then all solutions are of the form

$$x = A^{*}y + (I - A^{*}A)\, w\ , \tag{1.13}$$

where w is an arbitrary element of the range of A.

By virtue of the above Theorem we have that all solutions of equation (1.4) are given by

$$\Gamma^{a}_{bt} = -\Omega^{*ja}_{bi}\, f^{i}_{j/t} + \left[\, \delta^{j}_{b}\, \delta^{a}_{i} - \Omega^{*da}_{bz}\, \Omega^{jz}_{di} \,\right] \phi^{i}_{jt}\ , \tag{1.14}$$

where ϕ^{i}_{jt} are components of an arbitrary linear connection on V_n. The components of connection (1.14) can be expresssed extensively as follows

$$\begin{aligned} \Gamma^{a}_{bt} = {} & -m^{a}_{i}\, f^{i}_{b/t} + m^{j}_{b}\, f^{a}_{j/t} - 2\, f^{j}_{b}\, l^{a}_{i}\, f^{i}_{j/t} - \\ & - \frac{1}{2}\, l^{j}_{b}\, f^{a}_{i}\, f^{i}_{j/t} - \frac{5}{2}\, v^{j}_{b}\, m^{a}_{i}\, f^{i}_{j/t} + \phi^{a}_{bt} + v^{j}_{b}\, \phi^{a}_{jt} + \\ & + v^{a}_{i}\, \phi^{i}_{bt} + \frac{7}{2}\, f^{j}_{b}\, m^{a}_{i}\, \phi^{i}_{jt} - \frac{3}{2}\, l^{j}_{b}\, l^{a}_{i}\, \phi^{i}_{jt} - \\ & - \frac{3}{2}\, m^{i}_{b}\, f^{a}_{i}\, \phi^{i}_{jt} + \frac{5}{2}\, v^{j}_{b}\, v^{a}_{i}\, \phi^{i}_{jt}\ . \end{aligned} \tag{1.15}$$

We are now in a position to state

Theorem 1.2. If V_n is an f(5,1)-manifold then all f(5,1)-connections on V_n are given by

$$\Gamma^{a}_{bt} = \left(2\, m^{j}_{b}\, \delta^{a}_{i} - 2\, \delta^{j}_{b}\, m^{a}_{i} - \Omega^{*ja}_{bi} \right) \bar{\nabla}_{i} f^{i}_{j} + \phi^{a}_{bt} , \tag{1.16}$$

where $\bar{\nabla}$ denotes the covariant differentiation with respect to ϕ^{a}_{bt}.

Proof. By simple calculation we see that the formula (1.16) is equivalent to formula (1.15).

2. The torsion tensor field of the f(5,1)-connection

Let $/T^{a}_{bc}$ be the components of the torsion tensor field of the connection Γ^{a}_{bc} given by (1.15). If we put

$$\begin{aligned} \theta^{akc}_{dbt} = \frac{1}{2} \Big\{ & \delta^{a}_{d}\, \delta^{k}_{[b} \delta^{c}_{t]} + v^{a}_{d}\, \delta^{k}_{[b} \delta^{c}_{t]} + \delta^{a}_{d}\, v^{k}_{[b} \delta^{c}_{t]} + \\ & + \frac{7}{2} m^{a}_{d}\, f^{k}_{[b} \delta^{c}_{t]} - \frac{3}{2} l^{a}_{d}\, l^{k}_{[b} \delta^{c}_{t]} - \frac{3}{2} f^{a}_{d}\, m^{k}_{[b} \delta^{i}_{t]} + \frac{5}{2} v^{a}_{d}\, v^{k}_{[b} \delta^{c}_{t]} \Big\} , \end{aligned} \tag{2.1}$$

then the equation (1.15) gives.

$$\begin{aligned} \theta^{akc}_{dbt}\, \phi^{d}_{kc} = T^{a}_{bt} + \frac{1}{2} \Big\{ & f^{a}_{j/[b} m^{j}_{t]} + m^{a}_{i}\, f^{i}_{[b/t]} - \\ & - 2\, l^{a}_{i}\, f^{i}_{j/[b} f^{j}_{t]} - \frac{1}{2} f^{a}_{i}\, f^{i}_{j/[b} l^{j}_{t]} - \frac{5}{2} m^{a}_{i}\, f^{i}_{j/[b} v^{j}_{t]} \Big\} . \end{aligned} \tag{2.2}$$

Let θ^{*akc}_{dbt} be the operator defined by

$$\begin{aligned} \theta^{*dut}_{skc} = \frac{1}{2} \Big\{ & \delta^{d}_{s}\, \delta^{u}_{[k} \delta^{t}_{c]} - 2\, \delta^{d}_{s}\, \delta^{u}_{[k} v^{t}_{c]} - l^{d}_{s}\, l^{u}_{[k} v^{t}_{c]} - \\ & - m^{d}_{s}\, f^{u}_{[k} v^{t}_{c]} - f^{d}_{s} m^{u}_{[k} v^{t}_{c]} - \delta^{d}_{s} v^{u}_{[k} v^{t}_{c]} \Big\} . \end{aligned} \tag{2.3}$$

Proposition 2.1. For the operator θ^{akc}_{dbt} we have that the operator θ^{*akc}_{dbt} satisfies the condition

$$\theta^{akc}_{dbq}\, \theta^{*dut}_{skc}\, \theta^{spr}_{fut} = \theta^{apr}_{fbq} \,. \tag{2.4}$$

The proof of the above proposition is a straightforward calculation.

Taking account of Proposition 2.1 we have

Proposition 2.2. For the operators θ^{akc}_{dbt} and θ^{*akc}_{dbt} we have that the operator H^{a}_{bt} defined by

$$H^{a}_{bt} = \theta^{akc}_{dbt}\, \phi^{d}_{kc} \,, \tag{2.5}$$

satisfies the condition

$$\theta^{akc}_{dbq}\, \theta^{*dut}_{skc}\, H^{s}_{ut} = H^{a}_{bq} \,. \tag{2.6}$$

From (2.6) we have that

$$\left(\theta^{akc}_{dbt}\, \theta^{*dgh}_{xuc} - \delta^{a}_{x}\, \delta^{g}_{b}\, \delta^{h}_{t} \right) T^{x}_{gh} = \tag{2.7}$$

$$= - \left(\theta^{akc}_{dbt}\, \theta^{*dgh}_{xuc} - \delta^{a}_{x}\, \delta^{g}_{b}\, \delta^{h}_{t} \right) \left(\frac{1}{2}\, f^{x}_{j/[g} m^{j}_{h]} + \frac{1}{2}\, m^{x}_{i}\, f^{i}_{[g/h]} - \right.$$

$$\left. - l^{x}_{i}\, f^{i}_{j/[g} f^{j}_{h]} - \frac{1}{4}\, f^{x}_{i}\, f^{i}_{j/[g} l^{j}_{h]} - \frac{5}{4}\, m^{x}_{i}\, f^{i}_{j/[g} v^{j}_{h]} \right) .$$

If we put

$$\Sigma^{agh}_{xbt} = \theta^{akc}_{dbt}\, \theta^{*dgh}_{xuc} - \delta^{a}_{x}\, \delta^{g}_{b}\, \delta^{h}_{t} \,, \tag{2.8}$$

then the equation (2.7) gives

$$(2.9)\qquad \Sigma^{agh}_{xbt}\, T^{x}_{gh} = -\Sigma^{agh}_{xbt} \left(\frac{1}{2} f^{x}_{j/[g} m^{j}_{h]} + \frac{1}{2} m^{x}_{i} f^{i}_{[g/h]} - \right.$$

$$\left. - l^{x}_{i} f^{i}_{j/[g} f^{j}_{h]} - \frac{1}{4} f^{x}_{i} f^{i}_{j/[g} l^{j}_{h]} - \frac{5}{4} m^{x}_{i} f^{i}_{j/[g} v^{j}_{h]} \right).$$

Taking account of Proposition 2.1 we have

Proposition 2.3. For the operator Σ^{aut}_{sbq} we have that the operator

$$(2.10)\qquad \Sigma^{*aut}_{sbq} = -\delta^{a}_{s}\, \delta^{u}_{b}\, \delta^{t}_{q}$$

satisfies the condition

$$(2.11)\qquad \Sigma^{akc}_{dbq}\, \Sigma^{*dut}_{skc}\, \Sigma^{spr}_{fut} = \Sigma^{apr}_{fbq}\,.$$

We can easily prove that condition (1.12) for equation (2.9) is trivially satisfied. Hence, taking account of Theorem 1.1 and Proposition 2.2 we have that all skew-symmetric tensors T^{a}_{bc} of type (1.2) which can be assumed as being the torsion tensors of the f(5,1)-connections can be expressed as follows

$$(2.12)\qquad T^{a}_{bc} = -\frac{1}{2} \Sigma^{*akm}_{dbc}\, \Sigma^{drp}_{skm} \left[f^{s}_{j/[r} m^{j}_{p]} + m^{s}_{i} f^{i}_{[r/p]} - \right.$$

$$\left. - 2\, l^{s}_{i} f^{i}_{j/[r} f^{j}_{p]} - \frac{1}{2} f^{s}_{i} f^{i}_{j/[r} l^{j}_{p]} - \frac{5}{2} m^{s}_{i} f^{i}_{j/[r} v^{j}_{p]} \right] +$$

$$+ \left(\delta^{j}_{b}\, \delta^{a}_{i}\, \delta^{z}_{c} - \Sigma^{*aut}_{sbc}\, \Sigma^{sjz}_{iut} \right) w^{i}_{jz}\,,$$

where w^{a}_{bc} is an arbitrary skew-symmetric tensor.

Taking into account of (2.8) and (2.10) we have that

$$(2.13)\quad T^{a}_{bc} = \Sigma^{arp}_{sbc} \Big(\tfrac{1}{2} f^{s}_{j/[r} m^{j}_{p]} + \tfrac{1}{2} m^{s}_{i} f^{i}_{[r/p]} - l^{s}_{i} f^{i}_{j/[r} f^{j}_{p]} -$$

$$- \tfrac{1}{4} f^{s}_{i} f^{i}_{j/[r} l^{j}_{p]} - \tfrac{5}{4} m^{s}_{i} f^{i}_{j/[r} v^{j}_{p]} \Big) + \theta^{apq}_{rbc}\, \theta^{*rut}_{spq}\, w^{s}_{ut} .$$

Thus, using Definitions 2.8 and 2.10 of Σ^{agh}_{xbt} and Σ^{*xkc}_{dgh} respectively, we obtain

Theorem 2.1. If V_n is an f(5,1)-manifold, then the components of all torsion tensors of f(5,1)-connections are given by,

$$(2.14)\qquad T^{a}_{bt} = \tfrac{1}{2} f^{a}_{j/p} v^{p}_{[b} m^{j}_{t]} - \tfrac{3}{4} v^{a}_{i} f^{i}_{j/[b} m^{j}_{t]} +$$

$$+ \tfrac{29}{16} v^{a}_{i} f^{i}_{j/p} v^{p}_{[b} m^{j}_{t]} + \tfrac{23}{8} v^{a}_{i} f^{i}_{j/p} m^{p}_{[b} v^{j}_{t]} -$$

$$- \tfrac{1}{8} v^{a}_{i} f^{i}_{j/p} f^{p}_{[b} l^{j}_{t]} + \tfrac{17}{16} v^{a}_{i} f^{i}_{j/p} l^{p}_{[b} f^{j}_{t]} +$$

$$+ \tfrac{49}{16} m^{a}_{i} f^{i}_{j/p} f^{p}_{[b} m^{j}_{t]} + \tfrac{15}{4} m^{a}_{i} f^{i}_{j/[b} v^{j}_{t]} +$$

$$+ \tfrac{13}{8} m^{a}_{i} f^{i}_{j/p} v^{p}_{[b} v^{j}_{t]} + \tfrac{17}{16} m^{a}_{i} f^{i}_{j/p} m^{p}_{[b} f^{j}_{t]} +$$

$$+ \tfrac{7}{4} m^{a}_{i} f^{i}_{j/p} l^{p}_{[b} l^{j}_{t]} - \tfrac{11}{16} l^{a}_{i} f^{i}_{j/p} m^{p}_{[b} l^{j}_{t]} +$$

$$- l^{a}_{i} f^{i}_{j[b} f^{j}_{t]} + \tfrac{47}{16} l^{a}_{i} f^{i}_{j/p} v^{p}_{[b} f^{j}_{t]} + \tfrac{7}{4} l^{a}_{i} f^{i}_{j/p} m^{p}_{[b} l^{j}_{t]} -$$

$$+ \tfrac{21}{4} l^{a}_{i} f^{i}_{j/p} f^{p}_{[b} v^{j}_{t]} - \tfrac{11}{16} f^{a}_{i} f^{i}_{j/p} m^{p}_{[b} m^{j}_{t]} +$$

$$+ \tfrac{21}{4} f^{a}_{i} f^{i}_{j/[b} l^{j}_{t]} - \tfrac{3}{4} f^{a}_{i} f^{i}_{j/p} v^{p}_{[b} l^{j}_{t]} + \tfrac{103}{16} f^{a}_{i} f^{i}_{j/p} f^{p}_{[b} f^{j}_{t]} -$$

$$- \frac{23}{8} f^a_i f^i_{j/p} l^p_{[b} v^j_{t]} - \frac{1}{2} m^a_i f^i{}_{[b/t]} + \frac{3}{4} m^a_i v^p_{[b} f^i_{t]/p} -$$

$$- \frac{3}{4} f^a_i l^p_{[b} f^i_{t]/p} + \frac{3}{4} v^a_i m^p_{[b} f^i_{t]/p} + \frac{1}{4} \Big(2 \delta^a_s \delta^r_{[b} \delta^p_{t]} - \delta^a_s \delta^r_{[b} v^p_{t]} +$$

$$+ \delta^a_s v^r_{[b} \delta^p_{t]} -2 \delta^a_s v^r_{[b} v^p_{t]} + 2 v^a_s \delta^r_{[b} \delta^p_{t]} + \frac{7}{2} v^a_s \delta^r_{[b} v^p_{t]} +$$

$$+ \frac{5}{2} v^a_s v^r_{[b} \delta^p_{t]} - \frac{5}{2} v^a_s v^r_{[b} v^p_{t]} + \frac{7}{2} v^a_s m^r_{[b} f^p_{t]} - \frac{3}{2} v^a_s l^r_{[b} l^p_{t]} -$$

$$- \frac{3}{2} v^a_s f^r_{[b} m^p_{t]} + \frac{7}{2} m^a_s f^r_{[b} \delta^p_{t]} + \frac{21}{2} m^a_s \delta^r_{[b} f^p_{t]} - \frac{19}{2} m^a_s f^r_{[b} v^p_{t]} +$$

$$+ \frac{7}{2} m^a_s v^r_{[b} f^p_{t]} - \frac{3}{2} m^a_s m^r_{[b} l^p_{t]} - \frac{3}{2} m^a_s l^r_{[b} m^p_{t]} - \frac{3}{2} l^a_s l^r_{[b} \delta^p_{t]} -$$

$$- \frac{9}{2} l^a_s \delta^r_{[b} l^p_{t]} + \frac{1}{2} l^a_s l^r_{[b} v^p_{t]} - \frac{3}{2} l^a_s v^r_{[b} l^p_{t]} - \frac{3}{2} l^a_s m^r_{[b} m^p_{t]} -$$

$$- \frac{7}{2} l^a_s f^r_{[b} f^p_{t]} - \frac{3}{2} f^a_s m^r_{[b} \delta^p_{t]} - \frac{9}{2} f^a_s \delta^r_{[b} m^p_{t]} + \frac{1}{2} f^a_s m^r_{[b} v^p_{t]} -$$

$$- \frac{3}{2} f^a_s v^r_{[b} m^p_{t]} + \frac{3}{2} f^a_s f^r_{[b} l^p_{t]} - \frac{7}{2} f^a_s l^r_{[b} f^p_{t]} \Big) w^s_{rp} .$$

On the other hand from Theorem 1.1 we have that all solutions of equation (2.2) are given by

(2.15) $$\phi^a_{bt} = \theta^{*arp}_{sbt} H^s_{rp} + \Big(\delta^a_s \delta^r_b \delta^p_t - \theta^{*akc}_{dbt} \theta^{drp}_{skc} \Big) \psi^s_{rp} ,$$

where ψ^s_{rp} are components of an arbitrary linear connection.

We denote $\mathcal{T}_{(1,2)}(V_n, f(5,1))$ the set consisting of all tensor fields whose components are given by formula (2.14). Then by Theorem (2.1), the equation (2.2) is consistent iff $T \in \mathcal{T}_{(1,2)}(V_n, f(5,1))$. Therefore, using Definitions (2.3) and

(2.5) of θ^{*arp}_{sbt} and H^s_{rp} respectively and taking account of Theorem (1.2) we obtain the following theorem

Theorem 2.2. If a tensor $T \in \mathcal{T}_{(1,2)}(V_n, f(5,1))$, then the f(5,1)-connections whose torsion tensor is precisely the given tensor T are given by

$$(2.16)\qquad \Gamma^a_{bt} = m^j_b \overline{\nabla}_t f^a_j - m^a_i \nabla_t f^j_b - 2 f^j_b l^a_i \overline{\nabla}_t f^i_j - - \frac{1}{2} l^j_b f^a_i \overline{\nabla}_t f^i_j - \frac{5}{2} v^j_b m^a_i \overline{\nabla}_t f^i_j + \phi^a_{bt} ,$$

where

$$\begin{aligned}
(2.17)\qquad \phi^a_{bt} = {} & T^a_{rp} + v^p_{[b} T^a_{t]p} - \frac{1}{2} T^a_{rp} v^r_{[b} v^p_{t]} - \frac{1}{2} T^s_{rp} m^a_s f^r_{[b} v^p_{t]} - \\
& - \frac{1}{2} T^s_{rp} m^r_{[b} v^p_{t]} - \frac{1}{2} T^s_{rp} l^a_s l^r_{[b} v^p_{t]} + f^a_{j/[b} m^j_{t]} + \\
& + \frac{1}{4} m^a_j f^s_{j/r} f^r{}_{[b} m^j_{t]} + \frac{7}{8} m^a_s f^s_{j/p} v^p{}_{[b} v^j_{t]} + \frac{1}{4} f^a_s f^s_{j/r} m^r{}_{[b} m^j_{t]} + \\
& + \frac{1}{8} f^a_s f^s_{j/p} v^p{}_{[b} l^j_{t]} + \frac{1}{4} l^a_s f^s_{j/r} l^r{}_{[b} m^j_{t]} + \frac{1}{2} l^a_s f^s_{j/p} v^p{}_{[b} f^j_{t]} + \\
& + \frac{1}{2} m^a_i f^i_{[b/t]} + \frac{1}{2} m^a_i v^p{}_{[b} f^i_{t]/p} - 2 m^a_i f^i_{j/[b} v^j_{t]} + \\
& + \frac{3}{8} l^a_i f^i{}_{r/p} f^r{}_{[b} v^p_{t]} - \frac{3}{8} v^a_i f^i_{j/p} m^p_{[b} v^j_{t]} + \frac{1}{4} v^a_i f^i_{j/p} v^p_{[b} m^j_{t]} + \\
& + \frac{3}{8} f^a_i f^i_{j/p} l^j_{[b} v^p_{t]} + \frac{1}{2} f^a_i f^i_{j/p} f^p_{[b} f^j_{t]} - \frac{1}{2} m^a_i f^i_{j/p} m^p_{[b} f^j_{t]} - \\
& - 2 l^a_i f^i_{j/[b} f^j_{t]} - \frac{1}{2} v^a_i f^i_{j/p} l^p_{[b} f^j_{t]} - \frac{1}{8} f^a_i f^i_{j/[b} l^j_{t]} - \\
& - \frac{1}{8} v^a_i f^i_{j/p} f^p_{[b} l^j_{t]} - \frac{1}{8} l^a_i f^i_{j/p} m^p_{[b} l^j_{t]} - \frac{1}{8} m^a_i f^i_{j/p} l^p_{[b} l^j_{t]} +
\end{aligned}$$

$$
\begin{aligned}
&+\Big(\delta^a_s\, \delta^r_{[b}\delta^p_{t]} - \tfrac{1}{2}\, \delta^a_s\, \delta^r_{[b}\delta^p_{t]} + \tfrac{1}{2}\, \delta^a_s\, \delta^r_{[b}v^p_{t]} - \tfrac{1}{2}\, \delta^a_s\, v^r_{[b}\delta^p_{t]} + \\
&+ \tfrac{1}{2}\, \delta^a_s\, v^r_{[b}v^p_{t]} - \tfrac{1}{2}\, v^a_s\, \delta^r_{[b}\delta^p_{t]} - 2\, v^a_s\, v^r_{[b}\delta^p_{t]} + \tfrac{1}{2}\, v^a_s\, \delta^r_{[b}v^p_{t]} + \\
&+ \tfrac{5}{8}\, v^a_s\, v^r_{[b}v^p_{t]} - \tfrac{3}{8}\, v^a_s\, m^r_{[b}f^p_{t]} + \tfrac{7}{8}\, v^a_s\, f^r_{[b}m^p_{t]} + \tfrac{3}{8}\, v^a_s\, l^r_{[b}l^p_{t]} - \\
&- \tfrac{7}{2}\, m^a_s\, f^r_{[b}\delta^p_{t]} + \tfrac{9}{8}\, m^a_s\, f^r_{[b}v^p_{t]} + \tfrac{3}{8}\, m^a_s\, v^r_{[b}f^p_{t]} + \tfrac{3}{8}\, m^a_s\, l^r_{[b}m^p_{t]} + \\
&+ \tfrac{3}{8}\, m^a_s\, l^r_{[b}m^p_{t]} + \tfrac{3}{8}\, m^a_s\, m^r_{[b}l^p_{t]} + \tfrac{3}{4}\, l^a_s\, l^r_{[b}\delta^p_{t]} - \tfrac{1}{8}\, l^a_s\, l^r_{[b}v^p_{t]} + \\
&+ \tfrac{3}{8}\, l^a_s\, v^r_{[b}l^p_{t]} + \tfrac{7}{8}\, l^a_s\, f^r_{[b}f^p_{t]} + \tfrac{3}{8}\, l^a_s\, m^r_{[b}m^p_{t]} + \tfrac{3}{4}\, f^a_s\, m^r_{[b}\, l^p_{t]} - \\
&- \tfrac{1}{8}\, f^a_s\, m^r_{[b}v^p_{t]} + \tfrac{3}{8}\, f^a_s\, v^r_{[b}m^p_{t]} - \tfrac{3}{8}\, f^a_s\, l^r_{[b}f^p_{t]} + \tfrac{7}{8}\, f^a_s\, f^r_{[b}l^p_{t]} \Big)\, \psi^s_{rp}\,,
\end{aligned}
$$

$\bar{\nabla}$ denoting the covariant differentiation with respect to ϕ^a_{bt}.

3. The obstructions to the integrability

We shall calculate the obstructions to the integrability of the f(5,1)-structure using the family (1.16) of the linear connections associated with this structure.

It is known that the Chern-Bernard invariant of a G-structure vanishes iff there exists a torsionless connection of the G-structure. The tensor field (2.19) is a representant of the Chern-Bernard invariant of the f(5,1)-structure and it vanishes iff the Chern-Bernard invariant of the f(5,1)-structure vanishes.

If we consider now the equation (2.12) and put,

$$\equiv^{dbt}_{akc} = \theta^{dmn}_{rkc}\ \theta^{*rbt}_{amn} \tag{3.4}$$

then, equation (2.12) takes the form

$$\equiv^{agh}_{sbt}\ W^{s}_{gh} = -\frac{1}{2}\ \Sigma^{agh}_{sbt} \Big(f^{s}_{j/[b} m^{j}_{t]} + m^{s}_{i}\ f^{i}_{[b/t]} - $$

$$- 2\ l^{j}_{i}\ f^{i}_{j/[b} f^{j}_{t]} - \frac{1}{2}\ f^{s}_{i}\ f^{i}_{j/[b} l^{j}_{t]} - \frac{5}{2}\ m^{j}_{i}\ f^{i}_{j/[b} v^{j}_{t]} \Big) . \tag{3.5}$$

We can easly prove the following proposition.

Proposition 3.1. For the operator $\equiv^{agh}_{sbt}$,

have that the operator

$$\equiv^{*srp}_{kgh} = \delta^{s}_{k}\ \delta^{r}_{g}\ \delta^{p}_{h} \tag{3.6}$$

satisfies the condition

$$\equiv^{agh}_{sbt}\ \equiv^{*skc}_{ugh}\ \equiv^{urp}_{xkc} = \equiv^{arp}_{xbt} . \tag{3.7}$$

If we express the condition (1.12) in terms of the equation (3.5) we obtain the following theorem.

Theorem 3.2. There exists a torsionless f(5,1)-connection if and only if the tensor field,

$$F^{a}_{bt} = \frac{1}{2}\ f^{a}_{j/p}\ v^{p}_{[b} m^{j}_{t]} - \frac{3}{4}\ v^{a}_{i}\ f^{i}_{j/[b} m^{j}_{t]} + \tag{3.8}$$

$$\frac{29}{16}\ v^{a}_{i}\ f^{i}_{j/p}\ v^{p}_{[b} m^{j}_{t]} + \frac{23}{9}\ v^{a}_{i}\ f^{i}_{j/p}\ m^{p}_{[b} v^{j}_{t]} - \frac{1}{8}\ v^{a}_{i}\ f^{i}_{j/p}\ f^{p}_{[b} l^{j}_{t]} +$$

$$+ \frac{17}{16}\ v^{a}_{i}\ f^{i}_{j/p}\ l^{p}_{[b} f^{j}_{t]} + \frac{49}{16}\ m^{a}_{i}\ f^{i}_{j/p}\ f^{p}_{[b} m^{j}_{t]} +$$

$$+ \frac{15}{4} m^a_i f^i_{j/[b} v^j_{t]} + \frac{13}{8} m^a_i f^i_{j/p} v^p_{[b} v^j_{t]} + \frac{17}{16} m^a_i f^i_{j/p} m^p_{[b} f^j_{t]} +$$

$$+ \frac{7}{4} m^a_i f^i_{j/p} l^p_{[b} l^j_{t]} - \frac{11}{16} l^a_i f^i_{j/p} l^p_{[b} m^j_{t]} - l^a_i f^i_{j\,[b} f^j_{t]} +$$

$$+ \frac{47}{16} l^a_i f^i_{j/p} v^p_{[b} f^j_{t]} + \frac{7}{4} l^a_i f^i_{j/p} m^p_{[b} l^j_{t]} +$$

$$+ \frac{21}{4} l^a_i f^i_{j/p} f^p_{[b} v^j_{t]} - \frac{11}{16} f^a_i f^i_{j/p} m^p_{[b} m^j_{t]} + \frac{21}{4} f^a_i f^i_{j/b} l^j_{t]} -$$

$$- \frac{3}{4} f^a_i f^i_{j/p} v^p_{[b} l^j_{t]} + \frac{103}{16} f^a_i f^i_{j/p} f^p_{[b} f^j_{t]} -$$

$$- \frac{23}{8} f^a_i f^i_{j/p} l^p_{[b} v^j_{t]} - \frac{1}{2} m^a_i f^i{}_{[b/t]} +$$

$$+ \frac{3}{4} m^a_i v^p_{[b} f^i_{t]/p} - \frac{3}{4} f^a_i l^p_{[b} f^i_{t]/p} + \frac{3}{4} v^a_i m^p_{[b} f^i_{t]/p}$$

vanishes.

We notice that each term of the components F^a_{bt} involves the partial derivatives of f. However, in [2] is proved the following

Theorem 3.3. A necessary and sufficient condition for an f(5,1)-structure to be integrable is that there exist a coordinate system in which f has the constants components

$$f = \begin{bmatrix} 0 & I_p & 0 & 0 & 0 \\ 0 & 0 & I_p & 0 & 0 \\ 0 & 0 & 0 & I_p & 0 \\ -I_p & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{3.9}$$

where $r = 4p$ being the rank of f and I_p is the $p \circ p$ unit matrix.

Thus we can state

Theorem 3.4. If the $f(5,1)$-structure is integrable, then the tensor field F^a_{bt} vanishes.

REFERENCES

[1] F.Gouli-Andreou: On a structure defined by a tensor field f of type (1,1) satisfying $f^5 + f = 0$, Tensor, N.S., 36 (1982) 79–84.

[2] F.Gouli-Andreou: On integrability conditions of a structure f satisfying $f^5 + f = 0$, Tensor, N.S., 40 (1983) 27–31.

[3] S.Ishihara, K.Yano: On integrability conditions of a structure f satisfying $f^3 + f = 0$, Quart, J. Math. Oxford Ser(2) 15 (1964) 217–222.

[4] W.Mozgawa, W.Rompała: On connections in f-manifolds, Rev. Roumaine Math. Pures Appl. 29 (1984) 219–254.

[5] C.R.Rao, S.K.Mitra: Generalized inverse of matrices and its applications, J.Wiley and Sons, Inc., New York – London – Sydney – Toronto, 1971.

[6] K.Yano: On a structure defined by a tensor field f of type (1,1) satisfying $f^3 + f = 0$. Tensor, N.S. 14 (1963) 99–109.

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FIXED POINT THEOREMS FOR COMMUTING MAPPINGS

Let (X,d) be a metric space. We denote by $CB(X)$ the set of all non-empty closed bounded subsets of (X,d) and by H the Hausdorff-Pompeiu metric on $CB(X)$

$$H(A,B) = \max\left\{ \sup_{x\in A} d(x,B)\ ;\ \sup_{y\in B} d(x,A) \right\}$$

where $A,B \in CB(X)$ and

$$d(x,A) = \inf_{y\in A} \left\{ d(x,y) \right\}.$$

Let $A,B \in CB(X)$ and $k > 1$. In what follows, the following well known fact will be used: For each $a \in A$, there is $b \in B$ such that

$$d(a,b) \le k\ H(A,B).$$

In two recent papers [5], [3] the following theorems have been proved.

Theorem 1. If S and T are mappings of a complete metric space (X,d) into itself satisfying the inequality

$$(1)\quad \left\{ d(Sx,Ty) \right\}^2 \le h \max\left\{ d(x,Sx)\circ d(y,Ty)\ ;\ d(x,Ty)\circ d(y,Sx)\ ;\ ;d(x,Sx)\circ d(x,Ty)\ ;\ d(y,Sx)\circ d(y,Ty)\ ;\ d^2(x,y) \right\}$$

for all x,y in X and $0 \le h < \frac{1}{2}$, then S and T have a unique common fixed point [5].

Theorem 2. Let $T: X \longrightarrow CB(X)$ and let $f: X \longrightarrow X$ and $g: X \longrightarrow X$ be continuous mappings for which $T(X) \subset f(X) \cap g(X)$. Suppose that T commutes with f and g and that there exists an $h \in (0,1)$ such that

$$(2) \qquad H(Tx, Ty) \le hd\,(f(x), g(y));$$

for each $x, y \in X$. Moreover, assume that one of the following holds :

either: (a) $f(x) \ne f^2(x)$ implies $f(x) \cap T(x) = \emptyset$

or (b) $g(x) \ne g^2(x)$ implies $g(x) \cap T(x) = \emptyset$

Then T has a fixed point in X which is also a fixed point of f or g [3].

The purpose of this paper is to extend Theorems 1 and 2 and Theorems 3 from [2].

Theorem 3. Let (X,d) be a complete metric space, f,g continuous mappings from X into itself and S,T continuous multifunctions from X into CB(X). Suppose :

1. Each of f,g commutes with the three of others and $S(X) \cup T(X) \subset fg(X)$;

2.

$$(3) \qquad \left\{ H(Sx, Ty) \right\}^2 \le h \max \left\{ d(fx, Sx) \circ d(gy, Ty) \; ; \; d(fx, Ty) \circ d(gy, Sx) \; ; \; \frac{1}{2} [d(fx, Sx) \circ d(fx, Ty) + d(gy, Sx) \circ d(gy, Ty)] \; ; \; d^2(fx, gy) \right\}$$

for all x,y in X and $0 < h < 1$.

Moreover, assume that one of the following holds:

(a) $f(x) \neq f^2(x)$ implies $f(x) \cap S(x) = \emptyset$, or

(b) $g(x) \neq g^2(x)$ implies $g(x) \cap T(x) = \emptyset$.

Then in the case (a) S and f have a common fixed point and in the case (b) this is true for T and g. In particular, if (a) and (b) hold then there is a unique common fixed point for S,f,T and g.

Proof. Choose a real number k with

$$(4) \qquad 1 < k^2 < \frac{1}{\sqrt{h}} .$$

Take $x_0 \in X$ and let $u = fg$,; using condition 1 we construct a sequence $\{x_n\}$ such that

$$u x_{2n+1} \in Sg x_{2n} ; u x_{2n+2} \in Tf x_{2n+1} ,$$

and

$$d(u x_{2n+1}, u x_{2n+2}) \le k H(Sg x_{2n}, Tf x_{2n+1}),$$

$$d(u x_{2n+1}, u x_{2n}) \le k H(Sg x_{2n}, Tf x_{2n-1}).$$

Now

$$d^2(u x_{2n+1}, u x_{2n+2}) \le k^2 H^2(Sg x_{2n}, Tf x_{2n+1}) \le$$

$$\le k^2 h \max \Big\{ d(fg x_{2n}, Sg x_{2n}) \circ d(gf x_{2n+1}, Tf x_{2n+1}) ;$$

$$; d(fg x_{2n}, Tf x_{2n+1}) \circ d(gf x_{2n+1}, Sg x_{2n}) ; \frac{1}{2} [d(fg x_{2n}, Sg x_{2n}) \circ$$

$$\circ d(fg x_{2n}, Tf x_{2n+1}) + d(gf x_{2n+1}, Sg x_{2n}) \circ d(gf x_{2n+1}, Tf x_{2n+1})];$$

$$; \ d^2(fg\ x_{2n}, gf\ x_{2n+1}) \Big\} \le k^2\ h \max \Big\{ d(u\ x_{2n}, u\ x_{2n+1}) \circ$$

$$\circ\, d(u\ x_{2n+1}, u\ x_{2n+2}) \ ; \ d(u\ x_{2n}, u\ x_{2n+2}) \circ d(u\ x_{2n+1}, u\ x_{2n+1}) \ ;$$

$$; \ \frac{1}{2} [d(u\ x_{2n}, u\ x_{2n+1}) \circ d(u\ x_{2n}, u\ x_{2n+2}) + d(u\ x_{2n+1}, u\ x_{2n+1}) \circ$$

$$\circ\, d(u\ x_{2n+1}, u\ x_{2n+2})] \ ; \ d^2(u\ x_{2n}, u\ x_{2n+1}) \Big\} \le$$

$$\le k^2\ h \max \Big\{ d(u\ x_{2n}, u\ x_{2n+1}) \circ d(u\ x_{2n+1}, u\ x_{2n+2}) \ ; \ 0 \ ;$$

$$; \ \frac{1}{2} d(u\ x_{2n}, u\ x_{2n+1}) \circ [d(u\ x_{2n}, u\ x_{2n+1}) + d(u\ x_{2n+1}, u\ x_{2n+2})];$$

$$; \ d^2(u\ x_{2n}, u\ x_{2n+1}) \Big\} \le k^2\ h\ d(u\ x_{2n}, u\ x_{2n+1}) \circ$$

$$\circ \max \Big\{ d(u\ x_{2n+1}, u\ x_{2n+2}) \ ; \ d(u\ x_{2n}, u\ x_{2n+1}) \ ;$$

$$; \max [d(u\ x_{2n}, u\ x_{2n+1}) \ ; \ d(u\ x_{2n+1}, u\ x_{2n+2})] \Big\} \le$$

$$\le k^2 h\ d(u\ x_{2n}, u\ x_{2n+1}) \circ \max \Big\{ d(u\ x_{2n}, u\ x_{2n+1}) \ ;$$

$$; \ d(u\ x_{2n+1}, u\ x_{2n+2}) \Big\} .$$

We have

$$d^2(u\ x_{2n+1}, u\ x_{2n+2}) \le k^2\ h\ d^2(u\ x_{2n}, u\ x_{2n+1})$$

and

$$d(u\ x_{2n+1}, u\ x_{2n+2}) \le k\ \sqrt{h}\ d(u\ x_{2n}, u\ x_{2n+1}) \le$$

$$\le k^2\ \sqrt{h}\ d(u\ x_{2n}, u\ x_{2n+1})$$

or

$$d^2(u\,x_{2n+1}, u\,x_{2n+2}) \le k^2\, h\, d(u\,x_{2n+1}, u\,x_{2n+2}) \circ d(u\,x_{2n}, u\,x_{2n+1}).$$

Then

$$d(u\,x_{2n+1}, u\,x_{2n+2}) \le k^2\, h\, d(u\,x_{2n}, u\,x_{2n+1}) \le$$

$$\le k^2 \sqrt{h}\; d(u\,x_{2n}, u\,x_{2n+1})$$

because $0 < h < 1$ and $k > 1$ by (4).

Hence

$$d(u\,x_{2n+1}, u\,x_{2n+2}) \le k^2 \sqrt{h}\; d(u\,x_{2n}, u\,x_{2n+1}) .$$

Similarly we have

$$d(u\,x_{2n+1}, u\,x_{2n}) \le (k^2 \sqrt{h})\; d(u\,x_{2n}, u\,x_{2n-1}) .$$

Repeating the above argument, we obtain

$$d(u\,x_n, u\,x_{n+1}) \le (k^2 \sqrt{h})^n\, d(u\,x_0, u\,x_1) ,$$

where $(k^2 \sqrt{h}) < 1$ by (4).

Then by a routine calculation one can show that $\{ux_n\}$ is a Cauchy sequence and since (X,d) is complete, we have $\lim_{n \to \infty} ux_n = v$ for some $v \in X$. Then $fg(x_{2n+1}) \in Sg\,x_{2n}$ implies

$$f(u\,x_{2n+1}) \in f(Sg\,x_{2n}) = S(fg\,x_{2n}) = S(u\,x_{2n}) .$$

Taking the limit as $n \longrightarrow \infty$, we get $f(v) \in S(v)$. Similarly, $g(v) \in T(v)$.

Hence in the event that (a) holds we get

$$f(v) = f^2(v) \in f(Sv) = S(f(v))$$

thus $f(v)$ is a common fixed point of f and S. If (b) holds then $g(v)$ must be a common fixed point of g and T.

If (a) and (b) holds, suppose that $x = f(x) \in S(x)$ and $y = g(y) \in T(y)$. Then we get from (3)

$$H^2(Sx,Ty) \le h\, d^2(x,y) \le h \{ H(Sx,Ty)\}^2 \tag{5}$$

since $d(x,Sx) = d(y,Ty) = 0$ and $d(x,Ty) \circ d(y,Sx) \le d(x,y) \circ d(y,x)$ also $d(x,y) \le H(Sx,Ty)$. Thus $H(Sx,Ty) = 0$, as $h < 1$ and from (5) we get $d(x,y) = 0$ or $x = y$. This means that x is a common fixed point for f,g,S and T. The same reasoning shows that x is unique.

Theorem 4. Let (X,d) be a complete metric space, f,g continuous mappings from X into itself and S,T continuous multifunctions from X into $CB(X)$. Suppose :

1. Each of f,g commutes with the three of others and $S(X) \cup T(X) \subset fg(X)$.
2.

$$\left\{ H(Sx,Ty) \right\}^2 \le h \max \left\{ d(fx,Sx) \circ d(gy,Ty) \;;\; d(fx,Ty) \circ d(gy,Sx) \;;\; d(fx,Sx) \circ d(fx,Ty) \;;\; d(gy,Sx) \circ d(gy,Ty) \;;\; d^2(fx,gy) \right\} \tag{6}$$

for all x,y in X and $0 < h < \frac{1}{2}$.

Moreover, assume that one of the following holds:

(a) $f(x) \ne f^2(x)$ implies $f(x) \cap S(x) = \emptyset$, or

(b) $g(x) \ne g^2(x)$ implies $g(x) \cap T(x) = \emptyset$.

Then in the case (a) S and f have a common fixed point and in the case (b) this is true for T and g. In particular, if (a) and (b) hold then there is a unique common fixed point for S,f,T and g.

Proof.

$$\left\{ H(Sx,Ty) \right\}^2 \le h \max \left\{ d(fx,Sx)\circ d(gy,Ty) \;;\right.$$

$$d(fx,Ty)\circ d(gy,Sx) \;;\; d(fx,Sx)\circ d(fx,Ty) \;;\; d(gy,Sx)\circ d(gy,Ty) \;;$$

$$\left.;\; d^2(fx,gy) \right\} \le 2h \max \left\{ d(fx,Sx)\circ d(gy,Ty) \;;\; d(fx,Ty)\circ d(gy,Sx) ;\right.$$

$$\left.;\; \frac{1}{2}\, d(fx,Sx)\circ d(fx,Ty) \;;\; \frac{1}{2}\, d(gy,Sx)\circ d(gy,Ty) \;;\;;\; d^2(fx,gy) \right\} \le$$

$$\le 2h \max \left\{ d(fx,Sx)\circ d(gy,Ty) \;;\; d(fx,Ty)\circ d(gy,Sx) \;;\right.$$

$$\left.\frac{1}{2}\,[d(fx,Sx)\circ d(fx,Ty) + d(gy,Sx)\circ d(gy,Ty)] \;;\; d^2(fx,gy) \right\},$$

where $0 < 2h < 1$.

The proof of the following theorem is similar to the proof of Theorem 3.

Theorem 5. Let (X,d) be a complete metric space, f,g continuous mappings from X into itself and S,T continuous multifunctions from X into $CB(X)$. Suppose :

1. Each of f,g commutes with the three of others and $S(X) \cup T(X) \subset fg(X)$.

2.

$$\left\{ H(Sx,Ty) \right\}^2 \le b.\, d(fx,Ty)\circ d(fx,Sx) + c.\, d(gy,Sx)\circ d(gy,Ty) \tag{7}$$

for all x,y in X, where $b > 0$, $c > 0$ and

$$\left\{ b + (b^2 + 4b)^{\frac{1}{2}} \right\} \cdot \left\{ c + (c^2 + cb)^{\frac{1}{2}} \right\} < 4 .$$

Moreover, assume that one of the following holds:

(a) $f(x) \neq f^2(x)$ implies $f(x) \cap S(x) = \emptyset$, or

(b) $g(x) \neq g^2(x)$ implies $g(x) \cap T(x) = \emptyset$.

Then in the case (a) S and f have a common fixed point and in the case (b) this is true for T and g. In particular, if (a) and (b) hold then there is a unique common fixed point for S,f,T and g.

Remark. Theorems 3 and 5 are independent.

Example 1. Let $X = \{1,2,3\}$. Let d be the metric for X defined by

$$d(1,2) = 2 \; ; \; d(2,3) = 2 \; ; \; d(1,3) = \frac{3}{2} .$$

Let S and T be the mappings on X such that

$$S(1) = S(2) = S(3) = 1 \; ; \; T(1) = T(3) = 1 \; ;$$

$$T(2) = 3 \; ; \; f = g = id .$$

Then all assumptions of Theorem 3 are fulfilled for $\frac{9}{16} \leq h < 1$ but (7) does not hold (Remark IV [1]).

Example 2. Let $X = \{1,2,3\}$ and $d(x,y) = |x - y|$.

Let S and T be the mappings on X such that

$$S(0) = S(1) = 0 \; , \; S(2) = 1 \; , \; S(3) = 0 \; ; \; T(0) = T(1) = T(2) = 0;$$

$$T(3) = 1 \; ; \; f = g = id .$$

Then all assumptions of Theorem 5 are fulfilled for $b = \frac{1}{2}$, $c = \frac{1}{6}$ but (3) does not hold.

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REFERENCES

[1] M.G.Deshapande, G.G.Hamedani : Some results on common fixed points, J. of M.A.C.T. 17 (1984), 9–15.

[2] B.Fisher : Results on fixed point and common fixed points, Rev. Roum. Math. Pures Appl. 25 (1980) 217–222.

[3] H.Kaneko : A Banach fixed point theorem for multivalued mappings, Kobe J.Math. 1 (1984) 163–165.

[4] V.Popa : Results on common fixed points, Mathematica-Revue d'analise numerique et de la théorie de l'approximation, Mathematica, 26 (49) 1 (1984) 75–79.

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ON ROOTS OF THE EQUATION $f(z) = Q(z)f(a)$ IN THE CLASSES OF FUNCTIONS REPRESENTED BY THE STIELTJES INTEGRAL

1. Let E denote any class of functions defined and holomorphic in the disc $\Delta = \{ z \in \mathbb{C} : |z| < 1 \}$; $a \in \Delta$ is an arbitrarily fixed number, and Q is an arbitrarily fixed non-constant function holomorphic in the disc Δ . We assume that the class E is compact in the topology of almost uniform convergence and connected (cf. [1], [3]).

Consider the equation

$$f(z) = Q(z)\, f(a), \tag{1}$$

where $f \in E$. For a fixed function $f \in E$, equation (1) has at least one solution z_f in the disc Δ or is inconsistent in this disc. Let

$$D_f = D_f(a,Q) = \Big\{ z_f \in \Delta : f(z_f) = Q(z_f)\, f(a) \Big\}.$$

Assume that, for a chosen number $a \in \Delta$ and a fixed function Q, there exists a function $f \in E$ such that $D_f \neq 0$.

We shall consider the problem consisting in the determination of the set $D_E = D_E(a,Q) = \bigcup_{f \in E} D_f$. Such a problem was studied earlier by L. Burshteyn in [4] in the case of the equation of the form

$$f(z) = \gamma f(a) \tag{2}$$

where γ, a are fixed numbers, $\gamma \in \mathbb{C} \setminus \{0,1\}$, $a \in \Delta$, $f \in E$. For instance, one knows ([4]) the set D_{S^*} for equation (2), where S^* denotes the well-known class of functions starlike in the disc Δ.

Another question connected with equations (1), (2) is the search of the largest disc centred at the point $z = 0$ which contains no solution z_f. In other words, this question lies in determining a solution z_E of (1) or (2) such that

$$|z_E| = \min_{f \in E} |z_f|.$$

Papers [11], [6], [7], [8], [13], [4] are devoted to this question. The results obtained concern, for the most part, several classes of functions univalent in the disc Δ, in particular, the well-known class of functions f holomorphic and univalent in Δ, for which $f(0) = f'(0) - 1 = 0$. Some of the known results concern also the classes of functions which are not, in general, univalent. The solution z_{TR} of equation (2) for the class TR of functions typically-real in the disc Δ may serve as an example here.

Note that the equation of the form (1) is more general that that investigated in [6], where the function Q is a polynomial of the first degree.

If one knows the set D_E for (1) or (2), then, of course, the search of the solution z_E in the given class of functions is considerably facilitated.

According to the classical Hurwitz theorem, the functional $f \longrightarrow z_f$, where $f \in E$ and the arbitrarily chosen $z_f \in D_f$, is continuous. Hence (cf. [1], [3]) the set D_E is closed in the disc Δ treated as a subspace of $\mathbb{C}$. Moreover D_E is an intersection of the disc Δ and the connected set on the plane.

Let L denote the boundary of the set D_E of solutions of equation (1). In accordance with [9], z_0 is called a non-singular boundary point of the set D_E if there exists a point $d \notin D_E$ such that

$$|z_f - d| \geq |z_0 - d| \tag{3}$$

for any $z_f \in D_E$. It is known taht if L' is the set of non-singular boundary points, then $\overline{L'} = L$, and thus, in order to define the set D_E , it is enough to find L'. In view of (3), the search of the set L' reduced to the determination of the minimal value of the expression $|z_f - d|$ in the class E for all possible points $d \notin D_E$.

A function $f_0 \in E$ for which, with a fixed $d \notin D_E$, $|z_f - d|$ attains its minimum is called a boundary function. The solutions z_{f_0} of equation (1) are therefore non-singular boundary points of the set D_E .

2. Let $\mathcal{M}(\alpha, \beta)$, $\alpha < \beta$, be the class of functions μ non-decreasing in $\langle \alpha; \beta \rangle$ and such that $\int_{\alpha}^{\beta} d_{\mu}(t) = 1$. Next, let $g : \Delta \times \langle \alpha, \beta \rangle \longrightarrow \mathbb{C}$ stand for a function that satisfies the

conditions :

(i) for any $z \in \Delta$, there exists a derivative g'_t continuous with respect to $t \in \langle \alpha, \beta \rangle$,

(ii) for any $t \in \langle \alpha, \beta \rangle$, the function g is holomorphic with respect to $z \in \Delta$.

Denote by E_g (cf. [5], p. 505) the following class of functions : $f \in E_g$ if and only if there exists a function $\mu_f \in \in \mathcal{M}(\alpha, \beta)$ such that

$$f(z) = \int_{\alpha}^{\beta} g(z,t)\, d_{\mu_f}(t) \ , \ z \in \Delta \ . \tag{4}$$

It is known that E_g is a compact (see [12]) and connected class of functions holomorphic in the disc Δ.

We shall define one more class of functions to be considered in the paper. Let Φ be an arbitrarily fixed non-constant entire function. Denote by K the class of functions f of the form $f = \Phi \circ u_f$, where $u_f \in E_g$. This is also a compact and connected class of functions holomorphic in the disc Δ.

In the paper we shall find necessary conditions for functions of the classes E_g and to be boundary; we shall also give an example of applying one of the theorems obtained to the determination of the set D_K in some special case of equation (1) and the class K. The above example can also be solved employing the results obtained in [10] or [13].

Similarly as L. Burshteyn in [4], in the proofs of the necessary conditions (Th. 1 and Th. 2) we shall use Goluzin's variational method [5] for the classes of functions represented by the Stieltjes integrals. The qualitative differences between the forms of equations (1) and (2) have caused some indispensable modifications of the procedure applied in [4].

3. Let $z_0 \in \Delta$ be the non-singular boundary point of the set D_{E_g}, corresponding to a point $d \notin D_{E_g}$, whereas $f_0 \in E_g$ a boundary function for which

$$f_0(z_0) = Q(z_0)\, f_0(a) . \tag{5}$$

From the first variational formula of Goluzin in the class E_g we have

$$f_*(z) = f_0(z) + \int_{t_1}^{t_2} g_t'(z,t)\, |\mu_{f_0}(t) - c|\, dt ,$$

$$-1 \le \lambda \le 1 , \quad \alpha \le t_1 < t_2 \le \beta ;$$

it is known that $f_* \in E_g$.

For the function f_*, equation (1) takes the form

$$F(\lambda, z^*) = 0 , \tag{6}$$

where

$$F(\lambda, z^*) = f_0(z^*) - Q(z^*)\, f_0(a) + \tag{7}$$

$$+ \lambda \int_{t_1}^{t_2} \Big[g_t'(z^*, t) - Q(z^*)\, g_t'(a,t) \Big]\, |\mu_{f_0}(t) - c|\, dt .$$

We shall examine the solution $z^* = z^*(\lambda)$ of equation (6).

Let us first assume that

$$F'_{z^*}(0,z_0) = f_0'(z_0) - Q'(z_0)\, f_0(a) \neq 0 . \tag{8}$$

In view of (5), $F(0,z_0) = 0$; thus, in a sufficiently small neighbourhood of the point $\lambda = 0$, equation (6) has a solution of the form

$$z^* = z_0 + \lambda h^* + O(\lambda^2) , \tag{9}$$

where $O(\cdot)$ is the Landau symbol of the order of an infinitesimal.

If we take (9) into consideration, equation (6) can be represented in the form

$$f_0(z_0) - Q(z_0)f_0(a) + (f_0'(z_0) - Q'(z_0)f_0(a))\, h^*\lambda +$$

$$+ \lambda \int_{t_1}^{t_2} \Big[g_t'(z_0,t) - Q(z_0)\, g_t'(a,t) \Big] \, |\mu_{f_0}(t) - c|\, dt + O(\lambda^2) = 0$$

so, in view of (5),

$$(f_0'(z_0) - Q'(z_0)f_0(a))\, h^* +$$

$$+ \int_{t_1}^{t_2} \Big[g_t'(z_0,t) - Q(z_0)\, g_t'(a,t) \Big] \, |\mu_{f_0}(t) - c|\, dt + O(\lambda) = 0 .$$

Hence, after passing to the limit with $\lambda \longrightarrow 0$, we get

$$h^* = \Big[f_0'(z_0) - f(a)Q'(z_0) \Big]^{-1} \int_{t_1}^{t_2} \Big[Q(z_0)\, g_t'(a,t) - \tag{10}$$

$$- g_t'(z_0,t) \Big] \, |\mu_{f_0}(t) - c| dt.$$

Consider the expression $|z^* - d|^2$. Of course, we have

$$|z^* - d|^2 - |z_0 - d|^2 = \lambda \Big[2\mathrm{Re} \Big((\overline{z_0} - \overline{d})h^* \Big) + O(\lambda) \Big], \tag{11}$$

$$-1 \le \lambda \le 1.$$

If we suppose that $\mathrm{Re}\Big((\overline{z_0} - \overline{d})h^* \Big) \neq 0$, it then follows from (11) that $|z_0 - d|^2$ is not the minimal value of the expression $|z^* - d|^2$. Thus,. we obtain a contradiction with the assumption that z_0 is a non-singular boundary point. Consequently,

$$\mathrm{Re}\Big((\overline{z_0} - \overline{d})h^* \Big) = 0, \tag{12}$$

From (10) and (12), after simple transformations, we obtain the equation

$$\int_{t_1}^{t_2} \Big\{ \Big[(z_0 - d) (f_0'(z_0) - f_0(a)Q'(z_0)) \Big]^{-1} \times \tag{13}$$

$$\times \Big[Q(z_0) \, g_t'(a,t) - g_t'(z_0,t) \Big] \Big\} \, |\mu_{f_0}(t) - c| dt = 0 .$$

From (13) it follows at once that if the interval $(t_1;t_2)$ contains no roots of the equation

$$\mathcal{F}(t) = 0 , \tag{14}$$

where

$$(15)\quad \mathcal{F}(t) = \operatorname{Re}\left\{ \left[(z_0 - d)\,(f_0'(z_0) - f_0(a)Q'(z_0)) \right]^{-1} \times \right.$$

$$\left. \times \left[Q(z_0)\, g_t'(a,t) - g_t'(z_0,t) \right] \right\},$$

then $|\mu_{f_0}(t) - c| \equiv 0$ in this interval, whence $\mu_{f_0}(t) \equiv c$. We have thus obtained the following corollary: the function μ_{f_0} is piecewise constant in $\langle\alpha;\beta\rangle$, whereas its points of discontinuity may only be the points α,β and the roots of equation (14). It is, be the way, obvious that functions of the class $\mathcal{M}(\alpha,\beta)$ may have at most points of discontinuity of the first kind.

We shall now make us of the second variational formula of Goluzin in the class E_g. Namely, it is known [5] that the function

$$(16)\qquad f_{**}(z) = f_0(z) + \lambda\Big[g(z,t_1) - g(z,t_2) \Big]$$

for sufficiently small real values of λ belongs to the class E_g; we assume that t_1, t_2 are any points of discontinuity of the function μ_{f_0}, such that $\alpha \le t_1 < t_2 \le \beta$.

For function (16), equation (1) takes the form

$$(17)\qquad G(\lambda, z^{**}) = 0\ ,$$

where

$$(18)\qquad G(\lambda, z^{**}) = f_0(z^{**}) - Q(z^{**})f_0(a) +$$

$$+\left[g(z^{**},t_1) - g(z^{**},t_2) - Q(z^{**}) \left[g(a,t_1) - g(a,t_2) \right] \right] \lambda.$$

Note that $G'_{z^{**}}(0,z_0) = F'_{z^{*}}(0,z_0)$; moreover, $G(0,z_0) = 0$. Hence, under condition (8), there exists a sufficiently small neighbourhood of the point $\lambda = 0$ in which the solution z^{**} of (17) of the form

$$z^{**}(\lambda) = z_0 + \lambda h^{**} + O(\lambda^2) \tag{19}$$

is determined. From (17), (18), (19) we have

$$f_0(z_0) - Q(z_0)f_0(a) + \left[f_0'(z_0) - Q'(z_0)f_0(a) \right] h^{**}\lambda +$$

$$+\left[g(z_0,t_1) - g(z_0,t_2) - Q(z_0)\left[g(a,t_1) - g(a,t_2)\right]\right]\lambda + O(\lambda^2) = 0$$

thus, in view of (5),

$$\left[f_0'(z_0) - Q'(z_0)f_0(a) \right] h^{**} + g(z_0,t_1) - g(z_0,t_2) +$$

$$- Q(z_0)\left[g(a,t_1) - g(a,t_2)\right] + O(\lambda) = 0 ;$$

applying the passing to the limit with $\lambda \longrightarrow 0$, we shall obtain

$$h^{**} = \left[f_0'(z_0) - Q'(z_0)f_0(a) \right]^{-1} \left[g(z_0,t_2) - g(z_0,t_1) + \right. \tag{20}$$

$$\left. + Q(z_0)\left[g(a,t_1) - g(a,t_2)\right] \right].$$

The expression $|z^{**} - d|^2$ attains its minimum for $z^{**} = z_0$.

Hence, analogously to (12), we get the condition

$$(21) \qquad \operatorname{Re}\left[(\overline{z_0} - \bar{d})h^{**} \right] = 0 .$$

In consequence, from (20) and (21), after some easy transformations, we have

$$\begin{aligned} &\operatorname{Re}\left\{ \left[(z_0 - d)\,(f_0'(z_0) - Q'(z_0)f_0(a))\right]^{-1} \left[g(z_0,t_2) + \right.\right. \\ &\left.\left. - Q(z_0)g(a,t_2) \right] \right\} = \operatorname{Re}\left\{ \left[(z_0 - d)\,(f_0'(z_0) - Q'(z_0)f_0(a))\right]^{-1} \times \right. \\ &\qquad\qquad \left. \times \left[g(z_0,t_1) - Q(z_0)g(a,t_1) \right] \right\} , \end{aligned}$$

which proves that the function

$$\begin{aligned} (22) \quad \mathfrak{g}(t) = \operatorname{Re}&\left\{ \left[(z_0 - d)\,(f_0'(z_0) - Q'(z_0)f_0(a))\right]^{-1} \times \right. \\ &\qquad \left. \times \left[Q(z_0)g(a,t) - g(z_0,t) \right] \right\} \end{aligned}$$

takes equal values at any points t_1, t_2 of discontinuity of the function μ_{f_0}.

We shall now proceed to the examination of the case $F'_{z^*}(0,z_0) = 0$ (cf. (8)).

Note that, if for each $k \in \mathbb{N}$, $F^{(k)}_{z^{*k}}(0,z_0) = f_0^{(k)}(z_0) + - Q^{(k)}(z_0)f_0(a) = 0$, then equation (1) is the identity in the disc Δ. So, assume that there exists a number $m \in \mathbb{N}$ such that

$F^{(k)}_{z^{*k}}(0,z_0) = 0$ for $k = 1,2,\dots,m-1$, and $F^{(m)}_{z^{*m}}(0,z_0) \neq 0$. Then in some neighbourhood of the point $\lambda = 0$ there exists a solution of (6) of the form (cf. [14], 14.2 ch. III, 14.2 ch. VI)

$$z^* = z_0 + s^*\lambda^{\frac{1}{m}} + O(\lambda^{\frac{2}{m}}) \; ; \tag{23}$$

$\lambda^{\frac{1}{m}}$ takes the values of the principal branch of the m-th root, defined in a radially cut neighbourhood of the point $\lambda = 0$.

Let us represent equation (6) in the form

$$\frac{1}{m!}\left[f_0^{(m)}(z_0) - Q^{(m)}(z_0)f_0(a) \right](z^* - z_0)^m + \dots +$$

$$+ \lambda \int_{t_1}^{t_2} \left[g_t'(z_0,t) - Q(z_0)\, g_t'(a,t) \right] |\mu_{f_0}(t) - c|dt +$$

$$+ \lambda(z^* - z_0) \int_{t_1}^{t_2} \left[g''_{tz^*}(z_0,t) - Q'(z_0)\, g_t'(a,t) \right] \times$$

$$\times\, |\mu_{f_0}(t) - c|dt + \dots = 0 \, .$$

Hence, taking account of (23), we get

$$\frac{1}{m!}\left[f_0^{(m)}(z_0) - Q^{(m)}(z_0)f_0(a) \right]\lambda s^{*m} +$$

$$+ \lambda \int_{t_1}^{t_2} \left[g_t'(z_0,t) - Q(z_0)\, g_t'(a,t) \right] |\mu_{f_0}(t) - c|dt + O(\lambda^{1+\frac{1}{m}}) = 0$$

and, in consequence, after some easy transformations and the passing to the limit with λ ⟶ 0, we have

$$(24)\qquad s^{*} = \left[\frac{m!}{f_0^{(m)}(z_0) - Q^{(m)}(z_0) f_0(a)} \times \right.$$

$$\left. \times \int_{t_1}^{t_2} \Big[Q(z_0)\, g_t'(a,t) - g_t'(z_0,t) \Big] \, |\mu_{f_0}(t) - c| \, dt \right]^{\frac{1}{m}}.$$

Since, in view of (23) and (24),

$$(25)\qquad |z^{*} - d|^2 - |z_0 - d|^2 =$$

$$= 2\mathrm{Re} \left\{ \left[(\overline{z_0} - \overline{d}) \left[\frac{m!\lambda}{f_0^{(m)}(z_0) - Q^{(m)}(z_0) f_0(a)} \times \right. \right. \right.$$

$$\left. \left. \times \int_{t_1}^{t_2} \Big[Q(z_0)\, g_t'(a,t) - g_t'(z_0,t) \Big] \, |\mu_{f_0}(t) - c| \, dt \right]^{\frac{1}{m}} \right\} + O(\lambda^{\frac{2}{m}})$$

and $|z^{*} - d|^2$ attains its minimum for $z^{*} = z_0$, therefore, by (25) we have

$$(\overline{z_0} - \overline{d}) \left[\frac{m!\lambda}{f_0^{(m)}(z_0) - Q^{(m)}(z_0) f_0(a)} \times \right.$$

$$\left. \times \int_{t_1}^{t_2} \Big[Q(z_0)\, g_t'(a,t) - g_t'(z_0,t) \Big] \, |\mu_{f_0}(t) - c| \, dt \right]^{\frac{1}{m}} = 0 ,$$

but $z_0 - d \neq 0$, thus

$$\int_{t_1}^{t_2} \Big[Q(z_0)\, g_t'(a,t) - g_t'(z_0,t) \Big] \, |\mu_{f_0}(t) - c|\,dt \Big]^{\frac{1}{m}} \Big\} = 0 .$$

This equality proves that if the interval (t_1,t_2) contains no roots of the equation

$$\mathcal{F}_1(t) = 0 , \tag{26}$$

where

$$\mathcal{F}_1(t) = Q(z_0)\, g_t'(a,t) - g_t'(z_0,t) , \tag{27}$$

then $|\mu_{f_0}(t)-c| \equiv 0$ in this interval, whence $\mu_{f_0}(t) \equiv$ const. We have thus found that the function μ_{f_0} is piecewise constant in $\langle\alpha;\beta\rangle$, whereas its only points of discontinuity may be the points α, β and the roots of equation (26).

We shall now apply the second variational formula for the Stieltjes integrals, still assuming that $F^{(m)}_{z^{**}m}(0,z_0) \neq 0$, $m > 1$. Note that, for functions (7) and (18), $F^{(k)}_{z^{**}k}(0,z_0) = G^{(k)}_{z^{**}k}(0,z_0)$ for any $k \in \mathbb{N}$. Hence $G^{(\nu)}_{z^{**}\nu}(0,z_0) = 0$ for $\nu = 1,\dots,m-1$, and $G^{(m)}_{z^{**}m}(0,z_0) = f_0^{(m)}(z_0) - Q^{(m)}(z_0)f_0(a) \neq 0$. Consequently, in some neighbourhood of the point $\lambda = 0$ there is a solution of equation (17) (cf. (23))

$$z^{**} = z_0 + s^{**}\lambda^{\frac{1}{m}} + O(\lambda^{\frac{2}{m}}) . \tag{28}$$

Let us represent equation (17) in the form

$$\frac{1}{m!}\left[\, f_0^{(m)}(z_0)-Q^{(m)}(z_0)f_0(a)\,\right](z^{**}-z_0)^m +$$

$$+\Big\{ g(z_0,t_1)-g(z_0,t_2)-Q(z_0)(g(a,t_1)-g(a,t_2))+$$

$$+\left[\, g'_{z^{**}}(z_0,t_1)-g'_{z^{**}}(z_0,t_2)-Q'(z_0)(g(a,t_1)-g(a,t_2))\,\right]\times$$

$$\times(z^{**}-z_0)+\dots\Big\}\lambda=0\,.$$

Hence, in view of (28), we have

$$\frac{1}{m!}\left[\, f_0^{(m)}(z_0)-Q^{(m)}(z_0)f_0(a)\,\right]\lambda s^{**m}+$$

$$+\lambda\left[\, g(z_0,t_1)-g(z_0,t_2)-Q(z_0)(g(a,t_1)-g(a,t_2))\,\right]+$$

$$+\,O(\lambda^{1+\frac{1}{m}})=0$$

thus after passing to the limit with λ ⟶ 0, we obtain

$$(29)\qquad s^{**}=\Bigg\{\frac{m!}{f_0^{(m)}(z_0)-Q^{(m)}(z_0)f_0(a)}\Big[\,Q(z_0)(g(a,t_1)+$$

$$-\,g(a,t_2))-\left[\,g(z_0,t_1)-g(z_0,t_2)\,\right]\Big]\Bigg\}^{\frac{1}{m}}.$$

From (28) we have

$$|z^{**}-d|^2=|\overline{z_0}-\overline{d}|^2+2\mathrm{Re}\left[\,(\overline{z_0}-\overline{d})s^{**}\lambda^{\frac{1}{m}}\,\right]+O(\lambda^{\frac{2}{m}})\,;$$

since z_0 is a boundary point, therefore $(z_0 - d)s^{**}\lambda^{\frac{1}{m}} = 0,$

whence, in view of (29),

$$g(z_0,t_2) - Q(z_0)g(a,t_2) = g(z_0,t_1) - Q(z_0)g(a,t_1) .$$

Hence we infer that the function

$$\mathcal{G}_1(t) = Q(z_0)g(a,t) - g(a,t) \tag{30}$$

takes equal values at all points of discontinuity of the function μ_{f_0}.

The analysis we have carried out allows us to formulate the following result.

Theorem 1. Let z_0 be any non-singular boundary point of the set D_{E_g} of solutions of equation (1), whereas $f_0 \in E_g$ is the boundary function corresponding to the z_0. Then

(i) if $f'_0(z_0) - Q'(z_0)f_0(a) \neq 0$, then the function μ_{f_0} of the class $\mathcal{M}(\alpha,\beta)$ is piecewise constant and has points of discontinuity of the first kind only in those roots of the equation

$$\operatorname{Re}\left\{ \left[(z_0 - d)\,(f'_0(z_0) - Q'(z_0)f_0(a))\right]^{-1} \times \left[Q(z_0)g'_t(a,t) - g'_t(z_0,t) \right] \right\} = 0 \tag{31}$$

in which function (22) takes equal values,

(ii) if $f'_0(z_0) - Q'(z_0)f_0(a) = 0$, then the function μ_{f_0} of the class $\mathcal{M}(\alpha,\beta)$ is piecewise constant and has points of discontinuity of the first kind only in those roots of the equation

$$Q(z_0)g_t'(a,t) - g_t'(z_0,t) = 0 \tag{32}$$

in which function (30) takes equal values.

To complement Th. 1, let us notice that, in each interval $\langle t_1;t_2\rangle$ whose endpoints are the succesive points of discontinuity of the function μ_{f_0}, function (15) has at least one zero. Namely, it suffices to find that (15) is a derivative of function (22), the latter taking equal values at the points t_1,t_2. Obviously, there will be most points of discontinuity of the function μ_{f_0} if, between the succesive points t_1,t_2 of discontinuity, there occurs exactly one zero of function (15). If n denotes the number of solutions of equation (31) belonging to the interval $(\alpha;\beta)$, then the function μ_{f_0} has at most k points of discontinuity, where

$$k = \begin{cases} \frac{n+2}{2} & \text{for n even,} \\ \frac{n+3}{2} & \text{for n odd.} \end{cases}$$

Remark. If the equation (31) has n solutions in $(\alpha;\beta)$, then the piecewise costant function μ_{f_0} of the class $\mathcal{M}(\alpha,\beta)$ has at most $k = \left[\frac{n+3}{2}\right]$ points of discontinuity in $\langle\alpha;\beta\rangle$. then the boundary function f_0 of the class E_g (cf. (4)) is of the form

$$f_0(z) = \sum_{\nu=1}^{k} \lambda_\nu g(z,t_\nu) , \tag{33}$$

where $t_\nu \in \langle\alpha;\beta\rangle$ for $\nu = 1,2,\ldots,k$ are points of discontinuity of the function μ_{f_0}. Since functions of the class $\mathcal{M}(\alpha,\beta)$ are non-decreasing and $\int_\beta^\alpha d_\mu(t) = 1$, therefore $\lambda_\nu \geq 0$ for $\nu = 1,2,\dots,k$ and $\sum_{\nu=1}^{k} \lambda_\nu = 1$.

An analogous remark can be formulated with reference to equation (32).

4. For functions of the class K, we have the following result.

Theorem 2. Let z_0 be any non-singular boundary point of the set D_K of solutions of equation (1), whereas $f_0 \in K$ is the boundary function corresponding to the point z_0.

Then

(i) if $f_0'(z_0) - Q'(z_0)f_0(a) \neq 0$, then the function $\mu_{u_{f_0}}$ of the class $\mathcal{M}(\alpha,\beta)$ is piecewise constant and has points of discontinuity of the first kind only in those roots of the equation

$$\operatorname{Re}\left\{ \left[(z_0 - d)(f_0'(z_0) - Q'(z_0)f_0(a))\right]^{-1} \times \right. \tag{34}$$

$$\left. \times \left[Q(z_0)g_t'(a,t)\Phi_u'[u_{f_0}(a)] - g_t'(z_0,t)\Phi_u'[u_{f_0}(z_0)]\right] \right\} = 0$$

in which function

$$\mathcal{K}(t) = \operatorname{Re}\left\{ \left[(z_0 - d)(f_0'(z_0) - Q'(z_0)f_0(a))\right]^{-1} \times \right. \tag{35}$$

$$\times \left[Q(z_0)g_t'(a,t)\Phi_u'(u_{f_0}(a)) - g(z_0,t)\Phi_u'(u_{f_0}(z_0)) \right] \Big\}$$

takes equal values,

(ii) if $f_0'(z_0) - Q'(z_0)f_0(a) = 0$, then the function $\mu_{u_{f_0}}$ of the class $\mathcal{M}(\alpha,\beta)$ is piecewise constant and has points of discontinuity of the first kind only in those roots of the equation

$$Q(z_0)g_t'(a,t)\Phi_u'(u_{f_0}(a)) - g_t'(z_0,t)\Phi_u'(u_{f_0}(z_0)) = 0 \tag{36}$$

in which function

$$\mathcal{G}_1(t) = Q(z_0)g(a,t)\Phi_u'(u_{f_0}(a)) - g(z_0,t)\Phi_u'(u_{f_0}(z_0)) \tag{37}$$

takes equal values.

The proof runs similarly as in the case of Theorem 1.

It is worth stressing that Th. 1 and Remark 2 resulting from it, included in [4], are limit cases of the above Theorems 1 and 2, respectively. For a given equation (2) and a constant γ, $\gamma \notin \{0,1\}$, occurring in it, it suffices to construct a sequence of functions $\{Q_n(z)\}$, $Q_n \neq \text{const}$ for $n \in \mathbb{N}$ (cf. (1)), such that $Q_n(z) \longrightarrow \gamma$ almost uniformly in the disc Δ.

5. To illustrate an application of Theorem 2, we shall determine the set $D_K = D_K(a,Q)$ of solutions of the equation

$$f(z) = Q(z)f(a) \ , \tag{38}$$

where $a \in \Delta$ is an arbitrarily fixed number, $f \in K$, while

$\mathcal{Q}(z) = \frac{1}{1-|a|}(e^{-i\arg a} z - 1) + 2$. If $a = 0$, then $\arg a$ denotes an arbitrarily fixed real number. K denotes in this case the class of functions of the form $f(z) = \sqrt{\frac{\varphi(z)}{z}}$, $z \in \Delta$, $\sqrt{\frac{\varphi(z)}{z}} = 1$ for $z = 0$, where $\varphi \in S^*$ is any starlike function.

Since for $f \in K$

$$f(z) = \exp\left[-\int_{-\Pi}^{\Pi} \log(1 - e^{it}z)\, d\mu(t)\right], \tag{39}$$

where $\mu \in \mathcal{M}(-\Pi,\Pi)$, $\log 1 = 0$, thus $\Phi(u) = \exp u$ and $u \in E_g$ for $g(z,t) = -\log(1 - e^{it}z)$ (cf. (4)).

The class K thus defined is compact and connected ; moreover, $D_K \neq 0$.

Note that the determination of D_K for (38) reduces to the determination of the set $D_K' = D_K(a, Q_1)$ of solutions of the equation

$$f(z) = Q_1(z) f(a), \tag{40}$$

where $f \in K$, $a \in (0;1)$, $Q_1(z) \frac{1}{1-a}(z-1) + 2$.

Since (36) has no solutions, therefore the boundary functions will be determined by the case (i) of Th. 2.

Considering (34) and (35) we have obtained that the function $\mu_f \in \mathcal{M}(-\Pi,\Pi)$ may have only one point of discontinuity. Hence

$$f(z) = \frac{1}{1 - e^{-it}z}$$

and, in consequence, equation (40) will take the form

$$e^{-it}(zQ_1(z) - a) = Q_1(z) - 1 .$$

Thus, the non-singular boundary points of the set D'_K lie on a curve with the equation

$$|zQ_1(z) - a| = |Q_1(z) - 1| ,$$

i.e. on the circle $|z - a + 1| = 1$.

Using now the well-known inequality ([2])

$$\left| \sqrt{\frac{z}{\varphi(z)}} - 1 \right| \le |z| \quad , \varphi \in S^*, \ z \in \Delta ,$$

and (40) we get

$$D_K(a, Q_1) = \Delta \cap \{z \in \mathbb{C} : \ |z - a + 1| \ge 1 \ \} \ , a \in \langle 0;1) .$$

Finally, if $\Delta_1 = \{z \in \mathbb{C} : \ |z - e^{i \arg a} (|a| - 1)| \ge 1 \}$, $a \in \Delta$, then for the equation (38) $D_K = \Delta \cap \Delta_1$. Consequently, we have $z_k = a$.

This result can be obtained on the basis of result in [10] or [13] as well.

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REFERENCES

[1] И.А.Александров: Параметрические продолжения в теории однолистных функций, Москва 1976.

[2] И.Я.Ашневиц, Г.В.Ульина: Об областях значений

аналитических функций представимых интеграгом Стилътъеса, Вестник ЛГУ, 11, 4 (1955) 31–42.

[3] J.Baranowicz, J.Bąkowska: O zwartości i spójności rodziny funkcji holomorficznych, Zesz. Nauk. UŁ, ser. II, 52 (1973) 109–117.

[4] Л.Х.Бурштейн: О корнях уравнения $f(z) = \alpha f(a)$ для функций представимых с помощъю интеграла Стилътъеса, Вестник ЛГУ, 13 (1969) 7–19.

[5] Г.М.Голузин: Геометрическая теория функций комплексново переменново, Москва, 1966.

[6] M.Iovanow: On the equation $F(z) = (\beta z + \alpha)f(a)$ in the class of univalent functions, Mathematica, 13 (1984) 33–43.

[7] J.Kaczmarski: Sur l'équation $f(z) = pf(c)$ dans la classe des fonctions étoilées, Bull. Acad. Polon. Sci., 15 (1967) 455–464.

[8] J.Kaczmarski: Sur l'équation $f(z) = pf(c)$ dans la famille des fonctions univalentes à coefficients réels, Bull. Acad. Polon. Sci., 15 (1967) 245–251.

[9] Н.А.Лебедев: Мажорантная областъ для выражения $I = \ln |z^{\lambda}f'(z)^{1-\lambda}/f(z)^{\lambda}|$ в классе S, Вестник ЛГУ, 8 (1955) 29–41.

[10] Z.Lewandowski: Starlike majorants and subordination, Ann Univ. Mariae Curie Skłodowska, Sect A 15 (1961) 79–84.

[11] P.T.Mocanu: On the equation $f(z) = \alpha f(a)$ in the class of univalent functions, Mathematica, 6 (1964) 63–79.

[12] И.П.Натансон: Теория функций вещественной переменной, Москва 1957.

[13] M.O.Reade, E.J.Złotkiewicz: On the equation f(z) = pf(a) in certain classes of analytic functions, Mathematica, 13 (1971) 281-286.

[14] S.Saks, A.Zygmund: Funkcje analityczne, Warszawa 1959.

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EXISTENCE OF SOLUTION OF NONCONVEX OPTIMIZATION IN BANACH SPACE

1. Introduction

We shall consider the problem of the minimum of the functional

$$I(x,u) = \int_{t_0}^{t_1} f(t,x(t),u(t))\,dt$$

under the constraints

$$\frac{d}{dt}\,x(t) = \varphi(t,x(t),u(t)),$$

$$x(t_0) = x_0,$$

$$x(t_1) \in A,$$

where X,Y are separable reflexive Banach spaces, x_0 is a fixed point in X, A is a subset of X, $f : [t_0,t_1] \times X \times Y \longrightarrow R$, $\varphi : [t_0,t_1] \times X \times Y \longrightarrow X$, $x : [t_0,t_1] \longrightarrow X$, $u : [t_0,t_1] \longrightarrow Y$.

We give here a sufficient condition for the existence of a solution of the above problem. The idea of the proof of the main result consists in reducing the optimization problem to a certain problem of the calculus of variations. Our method is in the spirit of Ioffe and Tikhomirov ([4] ch. IX). We present a different approach to the above problem than Cesari in [2] and Cesari and Suryanarayana in [3].

2. Preliminaries

Let be a separable reflexive real Banach space with the norm $\|\circ\|$, Z^* – the dual connected with Z by the scalar product $\langle \circ | \circ \rangle$. We shall consider in Z not only the norm topology Z_n but also the sequentially weak topology denoted by Z_{sw}.

If $g : Z \longrightarrow (-\infty, +\infty]$ is a function, then by g^* we mean the Young–Fenchel conjugation to g, i.e.

$$g^* : Z^* \ni z^* \longrightarrow \sup (\langle z, z \rangle - g(z) : z \in Z).$$

By B_Z we denote the σ–algebra of subsets of Z, generated by Borel sets in Z_n. By $\mathcal{L}$ we mean the σ–algebra of Lebesgue measurable subsets of a given Lebesgue subset T of the real line. By $\mathcal{L} \otimes B_Z$ we denote the σ–algebra of subsets of $T \times Z$, generated by products of sets in $\mathcal{L}$ and B_Z.

A function $g : T \times Z \longrightarrow (-\infty, +\infty]$ is a called a normal integrand if g is $\mathcal{L} \otimes B_Z$ –measurable and, for almost every (a.e.) t in T, $g(t, \circ) : Z_n \longrightarrow (-\infty, +\infty]$ is lower semicontinuous (l.s.c.) and not identically $+\infty$. These conditions are equivalent [1] to the following one: the graph of $t \longrightarrow \operatorname{epi} g_t$ belongs to $\mathcal{L} \otimes B_R \otimes B_Z$ and, for a.e. t in T, the set $\operatorname{epi} g_t$ is closed in $R \times Z_n$ and nonempty, where

$$\operatorname{epi} g_t = \left\{ (r, z) \in R \times Z : g(t, z) \le r \right\}.$$

A normal integrand $g : T \times Z \longrightarrow (-\infty, +\infty]$ is said to satisfy the growth condition if, for each function q from $L^{\infty}(T,Z^{*})$, the function $t \longrightarrow g^{*}(t,q(t))$ is summable, where g^{*} is the Young–Fenchel conjugation to g with respect to the second variable.

3. Measurability and summability

Let S be a Lebesque measurable subset of the real line and let Z be a separable reflexive Banach space.

Proposition 3.1. Let $g : S \times Z \longrightarrow (-\infty,+\infty\,]$ be a function such that

(3.1) for each y in Z, $t \longrightarrow g(t,y)$ is Lebesgue measurable,

(3.2) there exists y_0 in Z such that, for any t in S and y_1, y_2 in Z, we have that $g(t,y_1) \le g(t,y_2)$ if $\|y_1 - y_0\| \le \|y_2 - y_0\|$.

Then there exists a normal integrand $\overline{g} : S \times Z \longrightarrow (-\infty,+\infty\,]$ such that

$$\overline{g}(t,y) \le g(t,y) \quad \text{for any } t \text{ in } S \text{ and } y \text{ in } Z,$$

$$\overline{g}^{*}(t,p) = g^{*}(t,p) \text{ for any } t \text{ in } S \text{ and } p \text{ in } Z,$$

where g^{*} (resp. $\overline{g}^{*}$) denotes the Young–Fenchel conjugation to g (resp. $\overline{g}^{*}$) with respect to the second variable.

Proof. Define $\overline{g}$ as follows

$$\overline{g}(t,y) = \liminf_{z \longrightarrow y} g(t,z). \tag{3.3}$$

This immediately implies that, for all t in S, $\overline{g}(t,\circ)$ is l.s.c. in Z_n and, for each y in Z, $\overline{g}(t,y) \le g(t,y)$. It is easy to see that $\overline{g}^{*}(t,p) = g^{*}(t,p)$ for each p in Z. Indeed, for any t in S and p in Z, we have

$$g^{*}(t,p) \le \overline{g}^{*}(t,p) = \sup\,(\, \langle y|p\rangle - g(t,y) : y \text{ in } Z\,) =$$

$$= \sup\left\{ \limsup_{z\to y}\,(\, \langle y|p\rangle - g(t,z)) : y \text{ in } Z \right\} \le$$

$$\le \sup\,(\, \langle z|p\rangle - g(t,z) : z \text{ in } Z\,) = g^{*}(t,p)\ .$$

Let $\{ y_n : n = 1,2,\ldots\}$ be dense in X_n, let $\{ a_k : k = 1,2,\ldots\}$ be dense in $[\,0,+\infty)$. Define the $\mathcal{L}$-measurable functions

$$y_{nk} : S \ni t \longrightarrow y_n\ ,$$

$$r_{nk} : S \ni t \longrightarrow g(t,y_0) + a_k\ ,$$

$$w_{nk} : S \ni t \longrightarrow (r_{nk}(t), y_{nk}(t))\ ,\ \text{for } n,k = 1,2\ldots\ .$$

Let (r,y) belong to epi g_t and let $\varepsilon > 0$ be fixed. It is possible to choose, by (3.2), y_n and a_k such that

$$\|y_n - y_0\| \le \|y - y_0\|\ ,\ \|y_n - y\| \le \frac{\varepsilon}{2}$$

and

$$r \le g(t,y_0) + a_k \le r + \frac{\varepsilon}{2}\ .$$

This implies that

$$\|w_{nk}(t) - (r,y)\| \le \varepsilon$$

and (see (3.2))

$$g(t,y_{nk}(t)) \le g(t,y) \le r \le r_{nk}(t)\ ,$$

so $w_{nk}(t)$ belongs to epi g_t. These and (3.1) imply that $W(t) = \left\{ w_{nk}(t) : n,k = 1,2,\ldots \right\}$ is dense in epi g_t. Hence, from (3.3) it follows that $W(t)$ is dense in epi $\bar{g}_t$ [1]. Therefore [1], g is $\mathcal{L} \otimes B_X$ –measurable, which ends the proof.

Proposition 3.2. If $\gamma : S \times Z \longrightarrow (-\infty, +\infty]$ is a normal integrand satisfying the growth condition, then, for every real number k,

$$\int_0^T \sup \left[\gamma^*(t,p) : p \in Z^* \text{ and } \|p\| \leq k \right] dt \text{ is finite.}$$

The proof is an immediate consequence of Theorem 2 in [5].

4. The main result

Let X, Y be separable reflexive Banach spaces and let $T = [t_0, t_1]$ be an interval of R. let

$$\varphi : T \times X \times Y \longrightarrow X \quad \text{and}$$

$$f : T \times X \times Y \longrightarrow R$$

be functions such that

(4.1) $f(t,\circ,\circ) : X_n \times Y_n \longrightarrow R$ is continuous for a.e. t in T,

(4.2) $f(t,\circ,\circ) : X_{sw} \times Y_{sw} \longrightarrow R$ is l.s.c. for a.e. t in T,

(4.3) $\varphi(t,\circ,\circ) : X_{sw} \times Y_{sw} \longrightarrow X_n$ is continuous for a.e. t in T,

(4.4) $f(\circ,x,u) : T \longrightarrow R$ is $\mathcal{L}$–measurable for all x in X and u in Y,

(4.5) $\varphi(\circ,x,u) : T \longrightarrow R$ is $\mathcal{L}$–measurable for all x in X and u in Y, (4.6) the function $T \ni t \longrightarrow \sup\ (\ |f(t,x,u)| + \|\varphi(t,x,u)\| : (x,u) \in C\ )$ is summable for each bounded subset C of $X \times Y$.

Remark 4.1. Assumptions (4.1), (4.3), (4.4) and (4.5) imply that f and φ are $\mathcal{L} \otimes B_X \otimes B_Y$ –measurable functions [1].

By $\mathcal{A}(T,X)$ we shall denote the set of all absolutely continuous functions $x : T \longrightarrow X_n$. It is well known [8] that, for each $x \in \mathcal{A}(T,X)$, there exists $\mathring{x}(t) = \frac{d}{dt}\ x(t)$ for a.e. t in T, and

$$x(t) = x(t_0) + \int_{t_0}^{t} \mathring{x}(s)ds \quad \text{for each } t \text{ in } T.$$

By $\mathcal{U}(T,U)$ we shall mean the set of all $\mathcal{L}$–measurable functions $u : T \longrightarrow U$ where U is a subset of Y.

Let A be closed in X_{sw}, U – closed in Y_{sw} and x_0 – any fixed point in X.

The following problem (P) will be considered :

$$\text{minimize } I(x,u) = \int_{t_0}^{t_1} f(t,x(t),u(t))dt,$$

where

(a) x belongs to $\mathcal{A}(T,x)$,

(b) $x(t_0) = x_0$, $x(t_1) \in A$,

(c) $\mathring{x}(t) = \varphi(t,x(t),u(t))$ for a.e. t in T,

(d) u belongs to $\mathcal{U}(T,U)$.

The pair of functions (x,u) is said to be admissible in (P) if it satisfies all conditions (a)–(d).

Let us define the function $h : T \times X \times X \longrightarrow (-\infty, +\infty]$ as follows

$$
(4.7) \quad h(T,x,y) = \begin{cases} \inf(\, f(t,x,u) : u \in U \text{ and } \varphi(t,x,u) = y) : & : y \in \varphi(t,x,U) \\ +\infty & : y \notin \varphi(t,x,U) \end{cases}
$$

Along with problem (P) we shall consider problem (Q) :

$$
\text{minimize } I_h(x) = \int_{t_1}^{t_0} h(t,x(t),\dot{x}(t))dt \ ,
$$

where x satisfies conditions (a) and (b). In this case, the function x is admissible in (Q).

Theorem 4.1. ([6], Theorem 4). Let $L : T \times X \times X \longrightarrow (-\infty, +\infty]$ be an $\mathcal{L} \otimes B_X \otimes B_X$ - measurable function , let $g : T \times X \longrightarrow (-\infty, +\infty]$ be a normal integrand satisfying the growth condition. Let :

$L(t,\circ,\circ) : X_{sw} \times X_{sw} \longrightarrow (-\infty, +\infty]$ be l.s.c. for all t in T,

$L(t,x,\circ) : X \longrightarrow (-\infty, +\infty]$ be convex for all t in T and x in X,

$L(t,x,y) \geq g(t,y)$ for all t in T and x,y in X.

If there exists a function $\tilde{x}$ in $\mathcal{A}(t,X)$ satisfying (b), such that

$$
\int_{t_0}^{t_1} L(t,\tilde{x}(t),\dot{\tilde{x}}(t))dt < +\infty \ ,
$$

then there exists

$$\min \left[\int_{t_0}^{t_1} L(t, x(t), \dot{x}(t))dt \; : \; x \in \mathscr{A}(T,X), \; x(t_0) = x_0, \; x(t_1) \in A \right].$$

We state the assumptions which will be used in the main theorem

(A) there exists an admissible pair $(\tilde{x}, \tilde{u})$ in problem (P) such that

$$\int_{t_0}^{t_1} f(t, \tilde{x}(t), \tilde{u}(t))dt < +\infty,$$

(B) there exists a normal integrand $\psi : T \times Y \longrightarrow (-\infty, +\infty]$ satisfying the growth condition, such that

$f(t,x,u) \geq \psi(t,u)$ for all t in T, x in X and u in Y,

(C) there exists a positive number k and a summable function $r : T \longrightarrow (-\infty, +\infty]$, such that

$\|\varphi(t,x,u)\| \leq k\|u\| + r(t)$ for all t in T, x in X, u in Y,

(D) the function h defined in (4.7) is convex with respect to the last variable.

Theorem 4.2. If all assumptions (A)–(D) are satisfied, then problem (P) has at least one solution.

The proof of this theorem is based on the following four lemmas:

Lemma 4.1. For a.e. t in T, the following property holds :

if $h(t,x,y) < +\infty$, then there exists u in U such that $h(t,x,y) = f(t,x,u)$ and $\varphi(t,x,u) = y$.

Lemma 4.2. For a.e. t in T, the function

$$h(t,\circ,\circ) : X_{sw} \times X_{sw} \longrightarrow (-\infty, +\infty]$$

is l.s.c.

Lemma 4.3. The function h is $\mathcal{L} \otimes B_X \otimes B_X$-measurable.

Lemma 4.4. There exists a normal integrand $\bar{g} : T \times X \longrightarrow (-\infty, +\infty]$ satisfying the growth condition, such that $h(t,x,y) \geq \bar{g}(t,y)$ for all t in T and x,y in X.

Proof of Lemma 4.1. Applying the Young - Fenchel inequality to the integrand ψ with respect to the second variable, we obtain that

$$\psi(t,u) + \psi^*(t,p) \geq \langle u|p\rangle \quad \text{for all } t \text{ in } T,\ u \text{ in } Y \text{ and } p \text{ in } Y.$$

This implies that

$$\psi(t,u)+\sup\,(\psi^*(t,p) : \|p\| \leq 1) \geq \|u\| \text{ for all } t \text{ in } T \text{ and } u \text{ in } Y.$$

From the last inequality and Proposition (3.2) it follows that, for a.e. t in T, the set $\{ u \in Y : \psi(t,u) \leq c \}$ is bounded for every number c.

Hence, from assumptions (B), (4.2), (4.3) we obtain that, for a.e. t in T and all x,y in X, if $h(t,x,y) < +\infty$, then the set

$$Q(t,x,y) = \Big\{ u \in U : \varphi(t,x,u) = y \text{ and } f(t,x,u) \leq h(t,x,y) + 1 \Big\}$$

is bounded in Y, nonempty and closed in Y_{sw}. Thus ([7], Theorem 9.2), for a.e. t in T, if $h(t,x,y) < +\infty$, then there exists u in U such that $\varphi(t,x,u) = y$ and $h(t,x,y) = f(t,x,u)$.

Proof of Lemma 4.2. Let us consider the Lebesgue set $W_c = \Big\{ (x,y) \in X \times X : h(t,x,y) \leq c \Big\}$ where c is a fixed real number, t is as in Lemma 1. We have to show that W_c is closed in $X_{sw} \times X_{sw}$. Let (x_n), (y_n) be countable sequences in X such that, for each n = 1,2,... , the pair (x_n, y_n) belongs to W_c and (x_n) converges weakly to x_0 in X, (y_n) converges weakly to y_0 in X. From Lemma 4.1 we obtain that, for each n = = 1,2,..., there exists a point u_n in U such that

$$f(t, x_n, u_n) = h(t, x_n, y_n) \quad \text{and} \quad \varphi(t, x_n, u_n) = y_n.$$

From (B) and the first part of the proof of Lemma 4.1 we see that the sequence (u_n) is bounded in Y. Since U is closed in Y_{sw}, there exists a subsequence (u_{n_k}) and the point u_0 in U, such that (u_{n_k}) converges weakly to u_0.

Hence, from (4.2) and (4.3) we obtain that

$$f(t, x_0, u_0) \leq c \quad \text{and} \quad \varphi(t, x_0, u_0) = y_0 ,$$

so $h(t, x_0, y_0) \leq c$ and, in consequence, (x_0, y_0) belongs to W_c, as asserted.

Proof of Lemma 4.3. Let $\{ x_m : m = 1,2,... \}$ be a dense subset of X_n, let $\{ u_i : i = 1,2,... \}$ be a dense subset of $U \subset Y_n$ and $\{ a_j : j = 1,2,... \}$ be a dense subset of $[0, +\infty)$. Take

$$x_{mij}(t) = x_m \qquad \text{for } t \text{ in } T ,$$

$$y_{mij}(t) = \varphi(t, x_m, u_i) \qquad \text{for } t \text{ in } \dot{T} ,$$

$$r_{mij}(t) = f(t, x_m, u_i) + a_j \quad \text{for } t \text{ in } T ,$$

$w_{mij}(t) = (r_{mij}(t), x_{mij}(t), y_{mij}(t)) \in R \times X \times X$ for t in T. It is easy to see, analogously to ([4], ch.8.1), that the set $\left\{ w_{mij}(t) : m,i,j = 1,2,\dots \right\}$ is a dense in epi h_t, so epi h_t is a measurable multifunction [1]. This and Lemma 4.2 imply [1] that h is $\mathcal{L} \otimes B_X \otimes B_X$ – measurable.

P r o o f of Lemma 4.4. Let $g : T \times X \longrightarrow (-\infty, +\infty]$ be defined as follows :

$$g(t,y) = \inf (\psi(t,u) : u \in U \text{ and } k\|u\| + r(t) \geq \|y\|).$$

From ([1], Lemma III. 39) we conclude that $t \longrightarrow g(t,y)$ is Lebesgue measurable for each y in X. Since $g(t,y_1) \leq g(t,y_2)$ whenever $\|y_1\| \leq \|y_2\|$, therefore Proposition 3.1 implies that there exists a normal integrand $\overline{g} : T \times X \longrightarrow (-\infty, +\infty]$ such that, for each t in T,

$$\overline{g}^*(t,p) = g^*(t,p) \quad \text{for each p in } X^*,$$

$$\overline{g}(t,y) \leq g(t,y) \quad \text{for each y in X.}$$

Hence, by (B) and (4.7), analogously to ([4], ch.9.2), we can show that $h(t,x,y) \geq \overline{g}(t,y)$ for all t in T and x,y in X, and that, for each $q \in L^\infty(T,X^*)$, there exists a constant M such that $\overline{g}^*(t,q(t)) \leq M\,r(t) + \sup (\psi^*(t,p) : p \in Y^* \text{ and } \|p\| \leq k\,M)$ for a.e. t in T. From proposition 3.2 we conclude that $\overline{g}$ satisfies the growth condition.

P r o o f of Theorem 4.2. From Lemma 4.2 it follows that there exists a function $L: T \times X \times X \longrightarrow (-\infty, +\infty]$ and a subset N of T with Lebesgue measure zero, such that

$$L(t,x,y) = \begin{cases} h(t,x,y) & : \ t \in T\setminus N \\ +\infty & : \ t \in N\,. \end{cases}$$

From Lemmas 4.3, 4.4 and assumptions (A),(D) it follows that L satisfies all assumptions of Theorem 1. Thus there exists an admissible function $\bar{x}$ in problem (Q) such that

$$\int_{t_0}^{t_1} h(t,\bar{x}(t),\mathring{\bar{x}}(t))dt = \int_{t_0}^{t_1} L(t,\bar{x}(t),\mathring{\bar{x}}(t))dt =$$

$$= \min\left\{ \int_{t_0}^{t_1} L(t,x(t),\mathring{x}(t))dt \ : \ x \in \mathscr{A}(T,X), x(t_0) = x_0, x(t_1) \in A \right\} =$$

$$= \min\left\{ \int_{t_0}^{t_1} h(t,x(t),\mathring{x}(t))dt \ : \ x \in \mathscr{A}(T,X), x(t_0) = x_0, x(t_1) \in A \right\}.$$

Taking

$$b : T \times Y \ni (t,u) \longrightarrow f(t,\bar{x}(t),u) \ ,$$

$$\Sigma : T \ni t \longrightarrow \left\{ u \in U : \varphi(t,\bar{x}(t),u) = \dot{\bar{x}}(t) \right\},$$

$$H : T \ni t \longrightarrow h(t,\bar{x}(t),\dot{\bar{x}}(t)),$$

we see that, by (4.4), (4.5), Lemma 4.1 and ([1]. Theorem III. 38), there exists a Lebesgue measurable function $\bar{u} : T \longrightarrow U$ such that, for a.e. t in T, we have

$$f(t,\bar{x}(t),\bar{u}(t)) = h(t,\bar{x}(t),\mathring{\bar{x}}(t)) \text{ and } \varphi(t,\bar{x}(t),\bar{u}(t)) = \mathring{\bar{x}}(t)\,.$$

Hence $(\bar{x},\bar{u})$ is admissible in (P) and

$$\int_{t_0}^{t_1} f(t,\bar{x}(t),\bar{u}(t))dt = \int_{t_0}^{t_1} h(t,\bar{x}(t),\mathring{\bar{x}}(t))dt \le$$

$$\le \int_{t_0}^{t_1} h(t,x(t),\mathring{x}(t))dt \le \int_{t_0}^{t_1} f(t,x(t),u(t))dt$$

for all admissible pairs (x,u) in (P). This ends the proof.

REFERENCES

[1] C.Castaing, M.Valadier: Convex analysis and measurable multifunctions, Springer-Verlag 1977.

[2] L.Cesari: Nonlinear optimization, in Nonlinear analysis and optimization, proceedings, Bologna 1982, Springer-Verlag 1984, pp. 63-81.

[3] L.Cesari, M.B.Suryanarayana: On recent existence theorems in the theory of optimization, Optimization Theory Appl. 31 (1980) 397-415.

[4] A.D.Ioffe, W.M.Tikhomirov: Theory of extremal problems, Moscow 1974 (in Russian).

[5] A.Nowakowski: Note on convex integral functionals, to appear in Demonstratio Math.

[6] A.Nowakowski, D.Zakrzewska: Variational problems and differential equations in Hilbert spaces, to appear in Demonstratio Math.

[7] M.M. Vainberg : Variational method and the method of monotone operators in the theory of nonlinear equations, Moscow 1972 (in Russian).

[8] H. Brézis : Opérateurs maximaux monotones et semi-groupes de contractions dans les espaces de Hilbert, North Holland 1973.

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EXISTENCE THEOREM FOR SOME MINIMAX PROBLEM

1. Introduction

In the paper the problem of minimizing a maximum functional is considered. The necessary conditions for optimality for such problems, in the form of the maximum principle, are given, among others, in papers [1], [2], [4], [5], [6], [8], [14]. In [9], [13] the existential theorems for the minimax problems considered therein are proved. The aim of the considerations included in the present paper is to find conditions for the existence of a solution for the problem being examined.

Let $C^n\langle 0,T\rangle$ denote the space of n-dimensional vector functions continuous on the interval $[0,T]$ and $L^k_\infty\langle 0,T\rangle$ the space of k-dimensional functions measurable and essentially bounded on $[0,T]$, where $T > 0$ is a fixed number. We shall consider the extremal problem in which the dynamics is described by the system of equations

(1)
$$\overset{\circ}{x} = f(x,u,z),$$

under the boundary conditions

(2)
$$x(0) = c,$$

$$x(T) = d, \tag{3}$$

where c and d are fixed points of the space R^n. The functions $x(\cdot) \in C^n(0,T)$, $u(\cdot) \in L^r_\infty(0,T)$, $z(\cdot) \in L^s_\infty(0,T)$ are called a state of the object, a control and a disturbance, respectively. We assume that

$$u(t) \in U \text{ for almost all } t \in [0,T], \tag{4}$$

where $U \subset R^r$ is a compact convex set and $\operatorname{Int} U \neq 0$.

Let $D \neq 0$ denote the set admissible controls. We define the functional

$$J(u(\cdot),z(\cdot)) = \int_0^T f^0(x(t),u(t),z(t))dt \tag{5}$$

on the set of solutions of the system of differential equations corresponding to the admissible controls. The assumptions concerning the scalar function f^0, the vector function f as well as further assumptions about the classes of control functions and disturbances will be given in our further considerations.

Let $Z \subset L^s_\infty(0,T)$ be a set sequentially compact itself.

The aim of the considerations is to find conditions under which there exists an admissible control realizing the minimum in the expression

$$\inf_{u(\cdot)\in D} \sup_{z(\cdot)\in Z} J(u(\cdot),z(\cdot)) . \tag{6}$$

The problem of minimum of the functional (6), where $J(u(\cdot),z(\cdot))$ is of the form (5), with conditions (1)–(4) in the sequel will be called problem A.

Many papers (e.g. [1], [2], [4], [5], [6], [8], [14]) are devoted to the investigations of various versions of the extremal problem formulated above. What was mainly sought for in them were the necessary conditions for the existence of solutions, in the form of the maximum principle. It is natural that, in connection with those investigations, there appeared problems of the existence of optimal solutions. These problems, in the case of D being the set of admissible controls in the sense of Pontryagin, were dealt with by, among others, Podimovskiy V.V. [9] and Vasilev N.S. [13].

In paper [12] the author considers the problem of optimal control without disturbances in which the admissible controls are functions with bounded variation. He proves there an existential theorem and gives necessary conditions for the existence of a solution for a minimax problem in the class of functions with bonded variation are given in [8]. THe ideals and results included in [12] and [13] will be made use of in our further considerations.

2. Auxiliary information and the existence theorem

Denote by V^k the set of k-dimensional vector functions $g : [0,T] \longrightarrow R^k$ with finite variation on [0,T] and by V the set of scalar functions with finite variation on [0,T]. It is known [10] that V^k is a Banach space with the norm $\|g\|^2 = \sum_{i=1}^{k} \|g^i\|^2$, where $g^i \in V$ and $\|g^i\| = |g^i(0)| + \overset{T}{\underset{o}{V}} g^i$

The symbol $\overset{T}{\underset{o}{V}} g^i$ denotes the variation of the function g^i on

$[0,T]$.

Let P^k be a compact subset of the positive cone in the space R^k. Assume that it is normal with respect to each of the axes, i.e. if $(p^1,\dots,p^i,\dots,p^k) \in P^k$, then $(p^1,\dots,\lambda p^i,\dots,p^k) \in P^k$, with any $\lambda \in [0,1]$, $i = 1,2,\dots,k$. This is equivalent to the condition that, together with any point $a = (a^1,\dots,a^k)$, the set P^k contains the rectangular parallelepiped $P^k_a := \{ p \in R^k : 0 \le p^i \le a^i,\ i = 1,2,\dots,k \}$, (cf. [11] and [12]).

We shall now adopt the assumptions announced above, concerning the class of disturbances and the class of admissible controls. Let Z stand for the class of s-dimensional vector functions $z(\cdot)$ with finite variation on $[0,T]$ and such that their values $z(t)$ belong to a set $Z_1 \subset R^s$ sequentially compact in itself. Let the variations of functions of Z belong to the set P^s described above, that is $\overset{T}{\underset{0}{V}} z := (\overset{T}{\underset{0}{V}} z^1,\dots \overset{T}{\underset{0}{V}} z^s) \in P^s$. In [12] it was shown that the set Z thus defined is a subset of the space of all functions with finite variation, sequentially compact in itself.

Put $D = \{ u \in V^r : u(t) \in U,\ \overset{T}{\underset{0}{V}} u \in P^r \}$, where $U \subset R^r$ is a closed bonded rectangular parallelepiped with non-empty interior. Let D_0 denote a subset of the set D, consisting of elements u such that, for each $z(t) \in Z_1$, there exists a solution $x(\cdot)$ of problem (1)–(3), such that $x(t) \in X$ for each $t \in [0,T]$, where X is a compact subset of the space R^n. The set D_0 is called the set of admissible controls and its elements – admissible controls. In [12] it was proved (Lemma 1, p. 90) that the set D_0 is sequentially compact in the topology of pointwise convergence.

Remark. In paper [8], by using the Dubovitsky – – Milutyn method ([3]) and some results from [12], the local maximum principle is proved for non–autonomous problem of type A. This result was obtained under the following assumptions : $u(\circ) \in D$, $z(\circ) \in Z$, $x(\circ) \in C^n(0,T)$, the functions f, f^0 are continuous with respect to (x,u,z) in the set of their admissible values, the function f^0 is bounded for bounded $x(t)$, $u(t)$, $z(t)$, the derivatives f_x, f_u, f^0_x, f^0_u are continuous with respect to (x,u,z) in the set of their admissible values and bounded for bounded $x(t)$, $u(t)$, $z(t)$, the derivatives f_u, f^0_u satisfy the Lipschitz condition with respect to (x,u,z).

Before formulating the main result we shall prove the following lemma.

Lemma. If $D_0 \neq \emptyset$, the set $X \times U \times Z_1$ is compact in R^{n+r+s}, the functions f and f^0 are continuous with respect to (x,u,z) on the set $X \times U \times Z_1$, then for any $u(\circ)$ from the set D_0,

$$\max_{z(\circ)\in Z} J(u(t),z(t))$$

is realized for some disturbance $z(\circ)$.

Proof. Let $u(\circ)$ be a momentarily fixed admissible control. Consider a sequence of disturbances $z_k(t)$, $t \in [0,T]$, $k = 1,2,\ldots$. The family of solutions $x_k(t)$, $t \in [0,T]$, $k = 1,2,\ldots$, of system (1) with the functions $z_k(t)$, $k = 1,2,\ldots$, is commonly bounded, in view of the compactness of the set X. We shall show that this family is equicontinuous. This follows immediately from the continuity of the function f and from the inequalities

$$|x_k(t_2) - x_k(t_1)| \leq \int_{t_1}^{t_2} |f(x_k(t), u(t), z_k(t))| dt \leq$$

$$\leq \max_{X \times U \times Z} |f(x,u,z)| \circ |t_2 - t_1| \ , \ k = 1,2,\dots \ .$$

Consequently, in virtue of the Arzela theorem ([7]), each sequence of this family contains a subsequence $(x_{k_n}(\circ))$ uniformly convergent to a function $\bar{x}(\circ)$ from the very family. Without loss of generality, for the subsequence $(x_{k_n}(\circ))$, we shall apply the notation $(x_k(\circ))$. At the same time, from the sequential compactness of the set Z it follows that $z_k(\circ) \longrightarrow z(\circ) \in Z$, since, for each $t \in [0,T]$, $z_k(\circ) \longrightarrow \bar{z}(\circ) \in$ $\in z_1$ as $k \longrightarrow \infty$. In view of the continuity of the function f, for each $t \in [0,T]$, we have

$$\lim f_i(x_k(t), u(t), z_k(t)) = f_i(\bar{x}(t), u(t), \bar{z}(t)), \ i = 1,2,\dots,n,$$

and

$$|f_i(x_k(t), u(t), z_k(t))| \leq \max_{X \times U \times Z} |f_i(x,u,z)|, \ i = 1,2,\dots,n.$$

Making use of the Lebesque theorem on passing with the limit under the integral sign, we get

$$x_k(t) = c + \int_0^t f(x_k(\tau), u(\tau), z_k(\tau)) d\tau \longrightarrow \bar{x}(t) =$$

$$= c + \int_0^t f(\bar{x}(\tau), u(\tau), \bar{z}(\tau))d\tau,$$

which means that $\bar{x}(\circ)$ is a solution of system (1) with the disturbance $\bar{z}(\circ)$. From the continuity of the function f^0 and from the facts $x_k(t) \longrightarrow \bar{x}(t)$, $z_k(t) \longrightarrow \bar{z}(t)$ it follows that the right-hand side of the inequality

$$|J(u(t), z_k(t)) - J(u(t), \bar{z}(t))| \le$$

$$\le \int_0^T |f^0(x_k(t), u(t), z_k(t)) - f^0(\bar{x}(t), u(t), \bar{z}(t))|dt,$$

tends to zero as $k \longrightarrow \infty$.

Hence we obtain the continuity of the functional $J(u(\circ), z(\circ))$ with respect to the disturbance $z(\circ)$ which, together with the fact that $\bar{z}(\circ)$ belongs to the sequentially compact set Z, proves our lemma, by Weierstrass theorem.

The main result is as follows.

Theorem. Let for problem A, the assumptions about the classes of admissible controls and disturbances be satisfied. If

1^0 the functión f is continuous with respect to (x,u,z) and continously differentiable with respect to x and the condition $x^{tr}f(x,u,z) \le C[1 + \|x\|^2]$ is satisfied for some constant C and for all $x(t)$, $u(t) \in U$, $z(t) \in Z_1$.

2^0 the function f^0 is continuous, convex with respect to u with fixed (x,z) and satisfies the Lipschitz condition with respect to (x,u) with the constant L_2, when z is fixed,

3^0 the set D_0 is non-empty,

then there exists $\bar{u}(\cdot) \in D_0$ realizing the minimum in the expression

$$\inf_{D_0} \sup_{Z} J(u(t),z(t)).$$

Proof. Reasoning analogously as in paper [13], one can prove that if $u_k(t)$ is a sequence of admissible controls, converging to $\bar{u}(t) \in D_0$ in the sense defined above, then, in view of the adopted assumptions, its corresponding sequence of solutions of problem (1)-(3) converges uniformly to the solution $\bar{x}(t)$ on the interval $[0,T]$, and this solution does not come out of the set X. It follows from our lemma, with any admissible control, there exists

$$\sup_{z(\cdot)\in Z} J(u(t),z(t)) = J(u(t)). \tag{7}$$

We shall show that functional (7) is lower semicontinuous on the set D_0. For this purpose, consider a sequence of admissible controls $u_k(t)$, $k = 1,2,\ldots$, convergent to $\bar{u}(t)$ in the adopted sense, and its corresponding sequence of trajectories $x_k(t)$, $k = 1,2,\ldots$, uniformly convergent to $\bar{x}(t)$.

Let us take into consideration the functional

$$\dot{I}(u(t)) = \int_0^T f^0(\bar{x}(t),u(t),z(t))dt, \tag{8}$$

where $z(\cdot)$ is a fixed disturbance. It can be demonstrated that, under assumptions 2^0, functional (8) is lower semicontinuous on the set of functions $u(\cdot)$ with values in the set U (cf. [13], p. 69). Consequently, we get

$$\lim_{k\to\infty} \inf I(u_k(t)) \geq I(\bar{u}(t)). \tag{9}$$

Besides, we have

$$|J(u_k(t),z(t)) - I(u_k(t))| \leq$$

$$\leq \int_0^T |f^0(x_k(t),u_k(t),z_k(t)) - f^0(\bar{x}(t),u_k(t),z(t))|dt \leq$$

$$\leq L_2\int_0^T |x_k(t) - \bar{x}(t)|dt \longrightarrow 0 \text{ as } k \longrightarrow \infty .$$

Hence

$$\lim_{k\to\infty} \inf J(u_k(t),z(t)) = \lim_{k\to\infty} \inf I(u_k(t)).$$

By the definition of functional (8),

$$J(\bar{u}(t),z(t)) = I(\bar{u}(t)).$$

Hence and from (9) we have

$$\lim_{k\to\infty} \inf J(u_k(t),z(t)) \geq J(\bar{u}(t),z(t)) \tag{10}$$

for each fixed $z(\cdot) \in Z$.

Note that since

$$\sup_{z(\cdot)\in Z} J(u_k(t),z(t)) \geq J(u_k(t),z(t)),$$

also the inequality

$$(11)\quad \lim_{k\to\infty}\ \inf\ \sup_{z(\circ)\in Z} J(u_k(t),z(t)) \geq \lim_{k\to\infty}\ \inf J(u_k(t),z(t))$$

is true for each $z(\circ) \in Z$. Since inequality (10) holds, in particular, for $z(\circ)$ which realizes $\sup\limits_{Z} J(\bar{u}(t),z(t))$, therefore on the ground of (11), we obtain

$$\lim_{k\to\infty}\ \inf\ \sup_{z(\circ)\in Z} J(u_k(t),z(t)) \geq \sup_{Z} J(\bar{u}(t),z(t)).$$

This inequality yields the announced lower semicontinuity of functional (7) on the set D_0. This and the sequential compactness of the set of admissible controls imply the proposition of our theorem.

REFERENCES

[1] V.V.Alsevith: Neobkhodimye uslovija optimalosti v minimaksnykh zadachakh optimazatsii, Differentialnye Uravnenija 12 (1976) 1384–1391.

[2] V.F.Damianov: Paketnyi printsip minimaksa, Vestnik Leningrad. Univ. Math 19 (1976) 35–39.

[3] A.Ya.Dubovitskiy, A.A.Milyutin: Zadachi na extremum pri nalichii ogranichenii, Dokl. Akad. Nauk. SSSR 149 (1963) 395–453.

[4] R.Gabasov, F.M.Kirillova: Printsip maxima v teoriy optimalnogo upravleniya, Moskva 1974.

[5] L.G.Gurin, E.M.Stolyarova: Printsip maxima v odnoy minimaksnoy zadachi, Zh. Vychisl. Math. and Math.

Phys. 13 (1973) 1175–1185.

[6] N.N.Moiseev : Chislennya metody v teorii optimalnyh sistem, Nauka, Moskva 1971.

[7] J.Musielak : Wstęp do analizy funkcjonalnej, PWN, Warszawa 1976.

[8] W.Paprocki : Warunki konieczne optymalności w zadaniach ekstremalnych z zakłóceniami, Doctoral thesis, Łódź 1977.

[9] V.V.Podimovskiy : O sushchestvovanii reshenii minimaksnykh zadachi optimalnykh procesov, Izv. Akad. Nauk. SSSR Techn. Kiberenеb. (1968) 121–136.

[10] R.Sikorski : Funkcje rzeczywiste, PWN, Warszawa 1959.

[11] J.Sztajnic : Necessary conditions for existence of extremum for controls with bounded variation, Bull. Soc. Sci. Lettres Łódź, 28 (1978) 1–13.

[12] S.Walczak : Euler–Lagrange's conditions for controls with bounded variation, Control Cybernet. 8 (1979) 87–100

[13] N.S.Vasilev : Theorema sushchestvovaniya v odnoy minimaksnoy zadachi upravleniya, Zh. Vychisl. Math. and Math. Phys. 17 (1977) 64–71.

[14] T.K.Vinogradova, V.F.Demianov: O printsipie minimaksa v zadachakh optimalnego upravleniya, Dokl. Akad. Nauk. SSSR 213 (1973) 512–518.

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ON THE ASYMPTOTIC BEHAVIOUR OF SOLUTIONS OF LINEAR SYSTEMS OF TWO ORDINARY EQUATIONS OF FIRST ORDER

1 Introduction

Let

$$(1.1)\qquad \begin{cases} \overset{\circ}{\underset{\sim}{x}}_1 = \underset{\sim}{a}_{11}\underset{\sim}{x}_1 + \underset{\sim}{a}_{12}\underset{\sim}{x}_2, \\ \overset{\circ}{\underset{\sim}{x}}_2 = \underset{\sim}{a}_{21}\underset{\sim}{x}_1 + \underset{\sim}{a}_{22}\underset{\sim}{x}_2, \end{cases}$$

be a system of two linear homogenous differential equations each of first order.

A more general system is the nonhomogenous system

$$(1.2)\qquad \begin{cases} \overset{\circ}{\underset{\sim}{x}}_1 = \underset{\sim}{a}_{11}\underset{\sim}{x}_1 + \underset{\sim}{a}_{12}\underset{\sim}{x}_2 + \underset{\sim}{f}_1, \\ \overset{\circ}{\underset{\sim}{x}}_2 = \underset{\sim}{a}_{21}\underset{\sim}{x}_1 + \underset{\sim}{a}_{22}\underset{\sim}{x}_2 + \underset{\sim}{f}_2. \end{cases}$$

where the functions $\underset{\sim}{a}_{ij}$, $\underset{\sim}{f}_i$; $i,j = 1,2$ are assumed to be defined on the set $\overline{R}^+ := [0;+\infty)$.

We shall prove that under some additional assumptions (see assumptions $S_1, S_2, S_3(\varepsilon)$ and S_4) all the solutions of the system (1.1) tend to zero exponentially (for $t \longrightarrow 0$) and that all the solutions of the system (1.2) grow not too rapidly.

In the case when the functions $\underset{\sim}{a}_{ij}$ are constants (that is in the case when the system (1.1) is a linear system with constant coefficients), and the assumptions S_1, S_2 and $S_3(0)$ hold the point (0,0) is a focus.

The used method is due to K. Tatarkiewicz – see [1]. Also in this paper, the reader can find the intuitive reasoning which led to the used method in the proof. As in [1], $\underset{\sim}{a}_{12}$ is equal to $\underset{\sim}{1}$ and here we have $\underset{\sim}{a}_{12}(t) \leq 0$, the results of the paper [1] are not a special case of the results of the present paper.

2. The assumptions

Let $a_{ij} = \underset{\sim}{a}_{ij}$; i, j = 1,2 be four given functions. Put

$$
(2.1)\qquad
\begin{cases}
\underset{\sim}{p}(t) := \frac{1}{2}\,[\underset{\sim}{a}_{11}(t) + \underset{\sim}{a}_{22}(t)], \\[2ex]
\underset{\sim}{h}(t) := \frac{1}{2}\,[\underset{\sim}{a}_{22}(t) - \underset{\sim}{a}_{11}(t)], \\[2ex]
\underset{\sim}{d}(t) := \underset{\sim}{h}^2(t) + \underset{\sim}{a}_{12}(t) \circ \underset{\sim}{a}_{21}(t), \\[2ex]
\qquad \underset{\sim}{v}(t) := \sqrt{\underset{\sim}{d}(t)}.
\end{cases}
$$

Define

$$
(2.2)\qquad \underset{\sim}{b}_1(t,\varepsilon) := \underset{\sim}{v}(t)\,[\underset{\sim}{p}(t) - (\underset{\sim}{\ln}(\underset{\sim}{a}_{21}(t)))^{\circ} - \varepsilon],
$$

$$
(2.3)\qquad \underset{\sim}{b}_2(t) := -\frac{\underset{\sim}{h}(t)}{2} \circ (\underset{\sim}{\ln}\, \frac{|\underset{\sim}{h}(t)|}{\underset{\sim}{a}_{21}(t)})^{\circ} =
$$

$$= \frac{\underset{\sim}{h}(t)}{2}\,[(\ln \underset{\sim}{a}_{21}(t))^{\circ} - (\ln |\underset{\sim}{h}(t)|)^{\circ}],$$

(2.4) $$\underset{\sim}{b}_3(t,\varepsilon) := \underset{\sim}{v}(t)\,[\underset{\sim}{p}(t) - \ln \underset{\sim}{v}(t)^{\circ} - \varepsilon].$$

We are making the following assumptions :

Assumption S_1. We have $C^1 \ni \underset{\sim}{a}_{ij} : \overline{R}^{+} \longrightarrow R$ for $i,j = 1,2$.

Assumption S_2°. There exist five positive constants $\underset{\sim}{a}_{21}$, $\bar{\underset{\sim}{a}}_{21}$, $\underset{\sim}{a}_{12}$, $\underline{d}$, and $\bar{d}$ such that

(2.5) $$0 < \underline{a}_{21} \le \underset{\sim}{a}_{21}(t) \le \bar{a}_{21},$$

(2.6) $$-\underline{a}_{12} \le \underset{\sim}{a}_{12}(t)$$

and

(2.7) $$-\underline{d} \le \underset{\sim}{d}(t) \le -\bar{d} < 0$$

for all $t \in R^{+}$.

Assumption S_3. There exists a constant $\varepsilon \in R$ such that

(2.8) $$\underset{\sim}{p}(t) - \frac{d}{dt}(\ln \underset{\sim}{a}_{21}(t)) < -\varepsilon$$

and

(2.9) $$\underset{\sim}{b}_1(t,\varepsilon) - \underset{\sim}{b}_3(t,\varepsilon) - \underset{\sim}{b}_2^2(t) > 0.$$

From (2.6) it follows that there exist two constants $\underline{v}$ and $\bar{v}$ such that

(2.10) $$0 < \underline{v} \le \underset{\sim}{v}(t) \le \bar{v}$$

and from (2.5) and (2.7) it follows that there exists

$\bar{a}_{12}(:= \dfrac{d}{\bar{a}_{21}}$ such that

$$(2.11) \qquad \underset{\sim}{a}_{12}(t) \le -\bar{a}_{12} < 0.$$

We have also

$$(2.12) \qquad 0 < -\underline{a}_{12} + \bar{a}_{12} \le \underset{\sim}{a}_{12}(t) - \underset{\sim}{a}_{12}(t) \le \bar{a}_{12} + \underline{a}_{12}.$$

From (2.6) and (2.11) we have

$$-\underline{a}_{12} \le \underset{\sim}{a}_{12}(t) \le \bar{a}_{12} < 0.$$

It follows from (2.1) and (2.7) that

$$\underset{\sim}{h}^{2}(t) \le -\bar{d} + \underset{\sim}{a}_{12}(t)\circ\underset{\sim}{a}_{21}(t) \le -\bar{d} + \underline{a}_{12}a_{21} =: h^{2}$$

(where $h > 0$) and

$$(2.13) \qquad |\underset{\sim}{h}(t)| < h.$$

3. The tubes

Let $\varepsilon \in R$ be a fixed number and $r \in R^{+} := (0; +\infty.)$ be a number. We will consider a family of tubes $T = T(r,\varepsilon)$ given by the parametric formulas

$$(3.1) \qquad \begin{cases} x_1 = \underset{\sim}{y}_1(t,u) := re^{\varepsilon t}\,[-\underset{\sim}{h}(t)\,\underset{\sim}{\cos}\, u - \underset{\sim}{v}(t)\,\underset{\sim}{\sin}\, u], \\ x_2 = \underset{\sim}{y}_2(t,u) := re^{\varepsilon t}\,\underset{\sim}{a}_{21}\,\underset{\sim}{\cos}\, u. \end{cases}$$

for $u \in [0,2\Pi)$ and $t \in \overline{R}^{+}$.

We will show that for a fixed $t_0 \ge 0$ (and fixed $r > 0$, ε) the intersection of the plane $t = t_0$ and the tube $T(r,\varepsilon)$ is an ellipse.

To simplify the calculations we introduce the following abbreviation : for fixed $r > 0$, and all $t \in \overline{R}^{+}$, we put

$$\underset{\sim}{w}_{r\varepsilon}(t) := re^{\varepsilon t}. \tag{3.2}$$

Eliminating u from the two equations of (3.1), taking into account (3.2) and dividing the result by $\underset{\sim}{a}_{21}(t)$, we get

$$\underset{\sim}{a}_{21}(t)x_1^2 + 2\underset{\sim}{h}(t)x_1x_2 - \underset{\sim}{a}_{12}(t)x_2^2 = \underset{\sim}{a}_{21}(t)\underset{\sim}{w}_{r\varepsilon}(t)\underset{\sim}{v}^2(t). \tag{3.3}$$

Under the assumption S_2 we have $\underset{\sim}{a}_{21}(t) \geq a_{21} > 0$ and

$$\begin{vmatrix} \underset{\sim}{a}_{21}(t) & \underset{\sim}{h}(t) \\ \underset{\sim}{h}(t) & -\underset{\sim}{a}_{12}(t) \end{vmatrix} = -\underset{\sim}{a}_{21}(t)\underset{\sim}{a}_{12}(t) - \underset{\sim}{h}^2(t) = -\underset{\sim}{d}(t) > 0.$$

Thus for any constant $t \in [0,+\infty)$ the formula (3.3) represents an ellipse. We will now evaluate its axes.

The solutions m_+ and m_- of the following pseudo-secular equation (for a constant t it is a secular equation)

$$\begin{vmatrix} \underset{\sim}{a}_{21}(t) - m & \underset{\sim}{h}(t) \\ \underset{\sim}{h}(t) & -\underset{\sim}{a}_{12}(t) - m \end{vmatrix} = 0$$

are

$$2\underset{\sim}{m}_{+,-} = (\underset{\sim}{a}_{21} - \underset{\sim}{a}_{12}) \pm \sqrt{(\underset{\sim}{a}_{21} - \underset{\sim}{a}_{12})^2 + 4(\underset{\sim}{a}_{12}\underset{\sim}{a}_{21} + h^2} =$$

$$= (\underset{\sim}{a}_{21} - \underset{\sim}{a}_{12})^2 \pm \sqrt{(\underset{\sim}{a}_{21} - \underset{\sim}{a}_{12})^2 + (\underset{\sim}{a}_{22} - \underset{\sim}{a}_{11})^2}\,.$$

It follows that the axes of the ellipse (3.3) are $\underset{\sim}{l}_{+,-}(t)\underset{\sim}{w}_{r\varepsilon}(t)$, where

$$\underset{\sim}{l}_{+,-} := \sqrt{(\underset{\sim}{a}_{21}\underset{\sim}{v}) : \underset{\sim}{m}_{+,-}} .$$

From (2.6) and (2.10) it results that there exists two positive constans $\underline{m}$ and $\overline{m}$ such that

$$0 < \underline{m} \le \underset{\sim}{a}_{21}(t) \circ \underset{\sim}{v}(t) \le \overline{m} \quad \text{for } t \in \overline{R}^{+}.$$

Also from (2.7) and (2.12) it results that there exist two positive numbers n_- and n_+ such that

$$0 < n \le \underset{\sim}{m}_{-}(t) \le \underset{\sim}{m}_{+}(t) \le n_{+}$$

and this in turn implies the existence of two positive numbers $\underline{l}$ and $\overline{l}$ such that

$$0 < \underline{l} \le \underset{\sim}{l}_{-}(t) \le \underset{\sim}{l}_{+}(t) \le \overline{l}.$$

Thus we have proved the following lemma

Lemma. The tube $T = T(r,\varepsilon)$ is contained in the surface of revolution

(3.4) $$x_1^2 + x_2^2 = \overline{l}^2 \underset{\sim}{w}_{r\varepsilon}^2(t), \quad t \in \overline{R}^{+}$$

and contains the surface of revolution

(3.5) $$x_1^2 + x_2^2 = \underline{l}^2 \underset{\sim}{w}_{r\varepsilon}^2(t), \quad t \in \overline{R}^{+}$$

4. The normal vector of the tube

Let $N = \underset{\approx}{N}(t,u)$ denote the outward normal vector of the surface (3.1). It is known that this vector is given by the formula

$$\underset{\approx}{N} = \left[\frac{\partial \underset{\sim}{y}_2}{\partial u},\ -\frac{\partial \underset{\sim}{y}_1}{\partial u},\ \frac{\partial \underset{\sim}{y}_1}{\partial u}\frac{\partial \underset{\sim}{y}_2}{\partial t} - \frac{\partial \underset{\sim}{y}_2}{\partial u}\frac{\partial \underset{\sim}{y}_1}{\partial t} \right].$$

From (3.1) it results

$$\frac{\partial \underset{\sim}{y}_1}{\partial u}(t,u) = re^{r\varepsilon}\,[\underset{\sim}{h}(t)\ \underset{\sim}{\sin}\ u - \underset{\sim}{v}(t)\ \underset{\sim}{\cos}\ u],$$

$$\frac{\partial \underset{\sim}{y}_2}{\partial u}(t,u) = -re^{r\varepsilon}\ \underset{\sim}{a}_{21}\ \underset{\sim}{\sin}\ u,$$

$$\frac{\partial \underset{\sim}{y}_1}{\partial t}(t,u) = \varepsilon re^{r\varepsilon}\,[-\underset{\sim}{h}(t)\ \underset{\sim}{\cos}\ u - \underset{\sim}{v}(t)\ \underset{\sim}{\sin}\ u] +$$

$$+ re^{r\varepsilon}\,[-\overset{\circ}{\underset{\sim}{h}}(t)\ \underset{\sim}{\cos}\ u - \overset{\circ}{\underset{\sim}{v}}(t)\ \underset{\sim}{\sin}\ u],$$

$$\frac{\partial \underset{\sim}{y}_2}{\partial t}(t,u) = \varepsilon re^{r\varepsilon}\ \underset{\sim}{a}_{21}\ \underset{\sim}{\cos}\ u + re^{r\varepsilon}\ \overset{\circ}{\underset{\sim}{a}}_{21}\ \underset{\sim}{\cos}\ u.$$

5. The homogenous case

Let $n = \underset{\approx}{n}(t,u)$ be the vector field of the system (1.1) on the surface of the tube (3.1). We have

$$\underset{\approx}{n}(t,u) = (\overset{\circ}{\underset{\sim}{x}}_1, \overset{\circ}{\underset{\sim}{x}}_2, 1)\ \Big|_{(3.1)} =$$

$$= (a_{11}x_1 + a_{12}x_2, a_{21}x_1 + a_{22}x_2, 1)\ \Bigg|_{\begin{array}{l} a_{ij} = \underset{\sim}{a}_{ij}(t) \\ x_i = \underset{\sim}{y}_i(t,u) \end{array}}.$$

Making the scalar product $\underset{\approx}{N} \cdot \underset{\approx}{n}$ and noting the definitions

(2.2), (2.3) and (2.4) we get

$$(5.1)\qquad \underset{\sim}{g}_0(t,u) := \frac{\underset{\approx}{n}(t,u)\circ \underset{\approx}{N}(t,u)}{\underset{\sim}{a}_{21}(t)v^2e^{2\varepsilon t}} =$$

$$= \underset{\sim}{b}_1(t,\varepsilon)\ \underset{\sim}{\cos}^2 u + 2b_2(t)\underset{\sim}{\cos}\ u\ \underset{\sim}{\sin}\ u + \underset{\sim}{b}_3(t,\varepsilon)\ \underset{\sim}{\sin}^2 u.$$

Let us suppose that the hypothesis $S_3(\varepsilon_0)$ where $\varepsilon_0 < 0$ is verified.

By (2.10) we have $\underset{\sim}{v}(t) > 0$ and by our hypothesis $S_3(\varepsilon_0)$ the inequalities (2.8) and (2.9) are verified for $\varepsilon > \varepsilon_0$ and for all $t \in \overline{R}^+$. Then of course $\underset{\sim}{b}_3(t,\varepsilon) < 0$ and the quadratic form $\underset{\sim}{g}_0$ of the two variables $\underset{\sim}{\cos}\ u$ and $\underset{\sim}{\sin}\ u$ is negative definite.

It means that under our assumptions we have $\underset{\approx}{n}(t,u)\circ\underset{\approx}{N}(t,u) > 0$ for some $\varepsilon > 0$ – that is all the points of the family of the tubes $T(r,\varepsilon)$ corresponding to these $\varepsilon < 0$ will be entrance points of the solutions of the system (1.1). For any given solution $(\underset{\sim}{x}_1,\underset{\sim}{x}_2)$ of the system (1.1) if

$$r > \overline{\Gamma}\,[\underset{\sim}{x}_1^2(0) + \underset{\sim}{x}_2^2(0)]^{\frac{1}{2}}$$

then the initial point $(\underset{\sim}{x}_1(0),\underset{\sim}{x}_2(0),0)$ of this solution lies in the tube $T(r,\varepsilon)$ and this solution beginning at the considered initial point will exist in the same tube for all $t \ge 0$.

Thus, we have proved the following theorem

Theorem I. Under the assumptions S_1, S_2 and $S_3(\varepsilon)$

for an $\varepsilon > 0$, all the solutions of the system (1.1) tend exponentially to zero (for $t \longrightarrow +\infty$).

More precisely and more generally : If the assumptions S_1, S_2 and $S_3(\varepsilon)$ are verified for all $\varepsilon > \varepsilon_0$, then for any solution $(\underset{\sim}{x}_1, \underset{\sim}{x}_2)$ of the system (1.1), we have

$$\lim_{t \longrightarrow \infty} [\overline{\underset{\sim}{x}}_1^2(t) + \overline{\underset{\sim}{x}}_2^2(t)]\, e^{-2\varepsilon t} = 0$$

for all $\varepsilon > \varepsilon_0$.

6. The non homogenous case

We make one more assumption

Assumption S_4. We have

$$C^0 \ni \underset{\sim}{f}_i : \overline{R}^+ \longrightarrow R \text{ for } i = 1,2.$$

Now, we will consider the non homogenous system of two linear differential equations (1.2). Let $m = \underset{\approx}{m}(t,u)$ be the vector field of this system on the surface of the tube (3.1). We have here

$$\underset{\approx}{m}(t,u) = (\mathring{\underset{\sim}{x}}_1, \mathring{\underset{\sim}{x}}_2, 1)\Big|_{(3.1)} =$$

$$= (\underset{\sim}{a}_{11}x_1 + \underset{\sim}{a}_{12}x_2 + \underset{\sim}{f}_1(t); \underset{\sim}{a}_{21}x_1 + \underset{\sim}{a}_{22}x_2 + \underset{\sim}{f}_2(t), 1)\Bigg|_{\substack{a_{ij} = \underset{\sim}{a}_{ij}(t) \\ x_i = \underset{\sim}{y}_i(t,u)}}$$

Calculating the scalar product $\underset{\approx}{m} \cdot \underset{\approx}{N}$, we get

$$\underset{\sim}{g}_1(t,u) := \underset{\approx}{m}(t,u) \cdot \underset{\approx}{N}(t,u) =$$

$$= \underset{\sim}{g}_0(t,u)\underset{\sim}{a}_{21}(t)r^2e^{2\varepsilon t} + \underset{\sim}{f}(t,u)re^{\varepsilon t},$$

where the function $\underset{\sim}{g}_0$ is given by the formula (5.1), and we have put

$$\underset{\sim}{f}(t,u) := -\underset{\sim}{f}_1(t)\underset{\sim}{a}_{21}(t)\ \underset{\sim}{\sin}\ u + \underset{\sim}{f}_2(t)\ [-\underset{\sim}{h}(t)\ \underset{\sim}{\sin}\ u + \underset{\sim}{v}(t)\ \underset{\sim}{\cos}\ u].$$

We have

$$\underset{\sim}{g}_1(t,u) = re^{2\varepsilon t}\underset{\sim}{g}_2(t,u),$$

where

$$\underset{\sim}{g}_1(t,u) := r\underset{\sim}{a}_{21}(t)\underset{\sim}{g}_0(t,u) + \underset{\sim}{f}_0(t,u;\varepsilon)$$

and

$$\underset{\sim}{f}_0(t,u;\varepsilon) := \underset{\sim}{f}_1(t)e^{-\varepsilon t}\underset{\sim}{a}_{21}(t)\underset{\sim}{\sin}\ u +$$

$$+ \underset{\sim}{f}_2(t)\ [-\underset{\sim}{h}(t)\ \underset{\sim}{\cos}\ u + v(t)\ \underset{\sim}{\sin}\ u].$$

Hence

(6.1) $$\underset{\sim}{\operatorname{sgn}}\ \underset{\sim}{m}(t,u) \cdot \underset{\approx}{N}(t,u) = \underset{\sim}{\operatorname{sgn}}\ \underset{\sim}{g}_2(t,u).$$

We have already proved in the preceding section the quadratic form $\underset{\sim}{g}_0$ of the two variables $\underset{\sim}{\cos}\ u$ and $\underset{\sim}{\sin}\ u$ is negative definite. It means that there exists a number $\bar{g}$ such that

$$g_0(t,u) \le -\bar{g} < 0$$

for all $t \in \overline{R}^+$ and $u \in [0,2\Pi)$.

From (2.5), (2.9) and (2.12) it follows

$$|\underset{\sim}{f}_0(t,u;\varepsilon)| \le \bar{a}_{21}|\underset{\sim}{f}_1(t)|e^{-\varepsilon t} + (\bar{v} + k)|\underset{\sim}{f}_2(t)|e^{-\varepsilon t}$$

We have

$$(6.2) \quad \underset{\sim}{g}_2(t,u) \le -r\bar{a}_{21}\bar{g} + [\bar{a}_{21}|f_1(t)| + (v + h)|\underset{\sim}{f}_2(t)|e^{-\varepsilon t}$$

7. The bounded solutions

Let both the functions $\underset{\sim}{f}_1$ and $\underset{\sim}{f}_2$ be bounded. It means that there exists a constant $m > 0$ such that $|\underset{\sim}{f}_i(t)| \le m$, $i = 1,2$ for all $t \in \bar{R}^+$ Let $\varepsilon = 0$. From (2.6) we get

$$\underset{\sim}{g}_2(t,u) \le -r\underline{a}_{21}\bar{g} + m_2, \quad t \in \bar{R}^+, \ u \in [0,2\Pi),$$

where

$$m_2 := (\bar{a}_{21} + \bar{v} + h)m.$$

Let $(\underset{\sim}{x}_1, \underset{\sim}{x}_2)$ be a solution of the system (1.2). We can choose $r \ge \sqrt{\underset{\sim}{x}_1^2(0) + \underset{\sim}{x}_2^2(0)}$ such that

$$-r\underline{a}_{21}\bar{g} + m_2 < 0.$$

Inasmuch we have, all the points of the tube $T(r,0)$ will be entrance points. As the solution $(\underset{\sim}{x}_1, \underset{\sim}{x}_2)$ is beginning $(t = 0)$ in this tube, it will rest in it for all $t \in \bar{R}^+$.

Thus we have proved the following theorem.

Theorem II. Let the assumptions $S_1, S_2, S_3(0)$ and S_4 be verified. If both the functions $\underset{\sim}{f}_1$ and $\underset{\sim}{f}_2$ are bounded, then all the solutions of the system (1.2) are bounded.

8. The ε-bounded solutions

Let both the functions $\underset{\sim}{f}_1$ and $\underset{\sim}{f}_2$ are ε-bounded for $\varepsilon > \varepsilon_0$, where $\varepsilon_0 \geq 0$ (see [1], p. 22). It means that we have

$$(8.1) \qquad \lim_{t \longrightarrow \infty} \underset{\sim}{f}_i(t) e^{-\varepsilon t} = 0, \ i = 1,2.$$

for all $\varepsilon > \varepsilon_0$, that is for any $\varepsilon > \varepsilon_0 \geq 0$, there exists an $\underset{\sim}{n}(\varepsilon) > 0$ such that if $t \geq \underset{\sim}{n}(\varepsilon)$ then $|\underset{\sim}{f}_i(t)| e^{-\varepsilon t} \ll 1$.

Let

$$\max \left[1, \sup_{t \in [0;\underset{\sim}{n}(\varepsilon)]} |\underset{\sim}{f}_1(t)|, \sup_{t \in [0;\underset{\sim}{n}(\varepsilon)]} |\underset{\sim}{f}_2(t)| \right] =: \underset{\sim}{s}(\varepsilon).$$

Then

$$|\underset{\sim}{f}_i(t)| e^{-\varepsilon t} \leq \underset{\sim}{s}(\varepsilon), \ i = 1,2$$

for all $\varepsilon > \varepsilon_0$.

By (6.2) we get

$$\underset{\sim}{g}_2(t,u) \leq -r \, \underline{a}_{21} \overline{g} + \underset{\sim}{s}(\varepsilon) \, (\overline{a}_{21} + \overline{v} + h), \ t \in \overline{R}^+, \ u \in [0,2\Pi).$$

Repeating the reasoning which have led us to Theorem II, we see that (choosing an r large enough) for a given $\varepsilon > \varepsilon_0 \geq 0$ any solution $(\underset{\sim}{x}_1, \underset{\sim}{x}_2)$ of the system (1.2) remains in the tube $T(r,\varepsilon)$ for all $t \in \overline{R}^+$. It means that

$$(8.2) \qquad \underset{\sim}{x}_1^2(t) + \underset{\sim}{x}_2^2(t) \leq r^2 e^{2\varepsilon t}$$

for all $t \in \overline{R}^+$.

As (8.2) is true for any $\varepsilon > \varepsilon_0 \geq 0$, we have

$$\lim_{t \longrightarrow \infty} \underset{\sim}{x}_i(t) e^{-\varepsilon t} = 0, \ i = 1,2$$

for all $\varepsilon > \varepsilon_0 \geq 0$. We have got the following theorem

Theorem III. Let $\varepsilon_0 > 0$ be a constant and let the assumptions $S_1, S_2, S_3(\varepsilon)$ for all $\varepsilon > \varepsilon_0$ and S_4 be verified. If both the functions $\underset{\sim}{f}_1$ and $\underset{\sim}{f}_2$ are ε-bounded for all $\varepsilon > \varepsilon_0$, then all the solutions of the system (1.2) are ε-bounded for all $\varepsilon > \varepsilon_0$ (that is verify the condition (8.3) for all $\varepsilon > \varepsilon_0$).

A small modification of the proof of Theorem II gives us the theorem which states that : if the functions $\underset{\sim}{f}_i$; $i = 1,2$ are ε_0-bounded (that is the formula (8.1) is verified for $\varepsilon = \varepsilon_0$) for a given $\varepsilon_0 \geq 0$ and the assumptions $S_1, S_2, S_3(\varepsilon)$ and S_4 are verified, then all the solutions of the system (1.2) are also bounded for that value ε_0.

9. Other results

Let us make a new assumption.

Assumption $S_3^+(\varepsilon)$. There exists a constant $\varepsilon > 0$ such that

(2.8')
$$\underset{\sim}{p}(t) - \frac{d}{dt}(\ln \underset{\sim}{a}_{21}(t)) > 0$$

and (2.9) for all $t \in \overline{R}^+$.

Under the assumptions S_1, S_2 and $S_3^+(0)$ the quadratic form (5.1) is positive definite and from (3.5) we will get the following theorem.

Theorem I'. Under the assumptions S_1, S_2 and $S_3^+(0)$ for an $\varepsilon > 0$ all the non trivial solutions of the homogenous system (1.1) tend exponentially to $+\infty$ for $t \longrightarrow +\infty$, that is, there exists a constant $\varepsilon > 0$ such that

$$\lim_{t\longrightarrow\infty} [\underset{\sim}{x}_1^2(t) + \underset{\sim}{x}_2^2(t)]e^{-2\varepsilon t} = +\infty$$

and there exists, a constant $\varepsilon_0 > 0$ such that they are ε_0-bounded.

We can get similar results for the non homogeneous system (1.2), namely : If the functions $\underset{\sim}{f}$; $i = 1,2$ are bounded (ε_0-bounded, ε_0-bounded for $\varepsilon > \varepsilon_0 \geq 0$) and the assumptions S_1, S_2, $S_3(0)$ ($S_3(\varepsilon)$, $S_3(\varepsilon)$ for $\varepsilon > \varepsilon_0 \geq 0$) and S_4 are verified, then one solution of the system (1.2) is bounded (ε_0-bounded, ε_0-bounded for $\varepsilon > \varepsilon_0$ respectively) and all the other solutions tend exponentially to infinity.

REFERENCES

[1] K. Tatarkiewicz : Sur l'allure asymptotique des solutions de l'équation différentielle du second ordre, Ann. Univ. M. Curie - Skłodowska (A) 7 (1953) 19-81.

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A GOURSAT PROBLEM FOR A POLYVIBRATING EQUATION OF MANGERON

1. M.N. Oğuztöreli [9] (cf. also [10]) examined a Goursat problem for the so-called polyvibrating equation of Mangeron (cf. [7]).

$$L^p u(x,y) = c(x,y), \tag{1}$$

where $L = D_x D_y$ and p is a positive integer, with the boundary conditions containing the operator L and given on two curves emanating from a common point.

A. Borzymowski [1]–[4] dealt with a Goursat problem for equation (1) in case when the boundary conditions were given on 2p curves and did not contain the operator L.

The aim of this paper is to generalize both of the aforesaid results. We study a Goursat problem for equation (1) with the boundary conditions given on 2q curves (q being a positive divisor of p) and containing those from papers [4] and [9] as particular cases. The functions given in the problem take their values in a Banach space. We reduce the problem to a system of differential-functional equations and then solve this system by using the method similar to that in paper [4]. As a result, we find a solution of our problem in the form of a series.

2. Let E be a Banach space with norm $\|\cdot\|$, Ω the rectangle

$$\Omega = \left\{ (x,y) \in \mathbb{R}^2 : 0 \le x \le A < \infty , \ 0 \le y \le B < \infty \right\}$$

and $k := \frac{p}{q}$.

Let us consider a system of $2q$ curves with equations $y = f_i(x)$ and $x = h_i(y)$, respectively, $i = 1,2,\dots,q$, placed in Ω and passing through the origin of the coordinates system.

We examine the following Goursat problem (G) :

Find a function $u : \Omega \longrightarrow E$ that is the solution of equation (1) in Ω (cf. [4], p. 254) and satisfies the boundary conditions

$$(2) \qquad L^{jq}u(x,y) = \begin{cases} M_{i,j}(x) & \text{for } y = f_i(x) \\ N_{i,j}(y) & \text{for } x = h_i(y) , \end{cases}$$

$((x,y) \in \Omega;\ j = 0,1,\dots,k-1;\ i = 1,2,\dots,q)$.

Let $\overset{*}{f}_i := f_i'(0)$ and $\overset{*}{h}_i := h_i'(0)$.

We make the following assumptions :

I. The functions $f_i : [0,A] \longrightarrow [0,B]$ and $h_i : [0,B] \longrightarrow [0,A]$, $i = 1,2,\dots,q$, are of class C^p such that $f_i(0) = h_i(0) = 0$ and satisfy the following conditions

$$(3) \quad \min(\overset{*}{f}_1, \overset{*}{h}_1) > 0, \ g_0 := \overset{*}{f}_q \overset{*}{h}_q < q^{\frac{-2q}{\beta_0}}, \ \max(\overset{*}{f}_q, \overset{*}{h}_q) \le 1,$$

where $\beta_0 \in [0,1]$,

$$(4) \quad \begin{cases} \min\limits_{2 \le i \le q} (\overset{*}{f}_i - \overset{*}{f}_{i-1}) > [q(1+\varepsilon)]^{-1} \overset{*}{f}_q, \\[2ex] \min\limits_{2 \le i \le q} (\overset{*}{h}_i - \overset{*}{h}_{i-1}) > [q(1+\varepsilon)]^{-1} \overset{*}{h}_q \end{cases}$$

when $q \geq 2$, where ε is such a number that

$$0 < \varepsilon < (q^q \sqrt{g_0})^{\frac{1}{1-q}} - 1. \tag{5}$$

Moreover, the curves $y = f_i(x)$ and $x = h_i(y)$, $i = 1,2,\ldots,q$, do not intersect one another in $\Omega \setminus \{(0,0)\}$.

II. The functions $M_{i,j} : [0,A] \longrightarrow E$ and $N_{i,j} : [0,B] \longrightarrow E$, $i = 1,2,\ldots,q$; $j = 0,1,\ldots,k-1$, are of class C^{p-jq}, respectively, and the relations[1]

$$\tag{6} \left\{ \begin{aligned} & \| M_{i,j}^{(m)}(x) \| \leq \text{const}\ x^{2(p-jq)-m-1+\beta_0} \\ & \| N_{i,j}^{(m)}(y) \| \leq \text{const}\ y^{2(p-jq)-m-1+\beta_0} \end{aligned} \right.$$

$((x,y) \in \Omega$; $i = 1,2,\ldots,q$; $j = 0,1,\ldots,k-1$; $m = 0,1,\ldots,p-jq)$ hold true, where β_0 is the number introduced in Assumption I.

III. The function $c : \Omega \longrightarrow E$ is continuous.

3. Let $\mathbb{N}$ denote the set of all positive integers. We adopt the following notation (cf. [4])

$$\tag{7} \left| \begin{aligned} & z_{\overrightarrow{i}_{(2m+2)}}(x) = h_{i_{2m+2}} \circ f_{i_{2m+1}} \circ z_{\overrightarrow{i}_{(2m)}}(x), \\ & z_{\overrightarrow{i}_{(2)}}(x) = h_{i_2} \circ f_{i_1}(x), \end{aligned} \right.$$

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[1] Here and in the sequel const denotes a positive constant.

$$(7)\quad \begin{cases} \tilde{z}_{\vec{i}_{(2m+2)}}(y) = f_{i_{2m+2}} \circ h_{i_{2m+1}} \circ \tilde{z}_{\vec{i}_{(2m)}}(y), \\ \\ \tilde{z}_{\vec{i}_{(2)}}(y) = f_{i_2} \circ h_{i_1}(y), \\ \\ \overset{*}{z}_{\vec{i}_{(2m+1)}}(x) = f_{i_{2m+1}} \circ z_{\vec{i}_{(2m)}}(x),\ \overset{*}{z}_{\vec{i}_{(1)}}(x) = \\ \\ \overset{**}{z}_{\vec{i}_{(2m+1)}}(y) = h_{i_{2m+1}} \circ \tilde{z}_{\vec{i}_{(2m)}}(y),\ \overset{**}{z}_{\vec{i}_{(1)}}(y) \end{cases}$$

where $m \in \mathbb{N}$; $1 \le i_r \le q$ for $r \in \mathbb{N}$ and $\vec{i}_{(s)} = (i_1, i_2, \dots, i_s)$ for $s \in \mathbb{N}$. Furthermore, we denote

$$(8)\qquad w_i(x) = \begin{cases} \prod\limits_{\substack{r=1 \\ r \ne i}}^{q} [f_r(x) - f_i(x)]^{-1} & \text{for } q \ge 2, \\ \\ 1 & \text{for } q = 1, \end{cases}$$

$$(9)\qquad e_i^s(x) = \begin{cases} \sum\limits_{\substack{1 \le r_1 < \dots < r_{q-s} \le q \\ r_1 \ne i, \dots, r_{q-s} \ne i}} \prod\limits_{l=1}^{q-s} f_{r_l}(x) \\ \qquad \text{for } s = 1, 2, \dots, q-1, \text{ when } q \ge 2, \\ \\ 1 \qquad\qquad \text{for } s = q, \end{cases}$$

$$(10)\qquad \begin{cases} G_{r,i}^{s}(x) = (-1)^s w_i(x) e_i^s(x) x^{r-1}, \\ \\ \tilde{G}_{r,i}^{s}(y) = (-1)^s \tilde{w}_i(y) \tilde{e}_i^s(y) y^{r-1}, \end{cases}$$

$(x \in (0,A];\ y \in (0,B];\ i,r,s = 1,2,\dots,q)$, where $\tilde{w}_i$ and $\tilde{e}_i^s$ are given by formulae (8) and (9), respectively, with the replacement of f by h and x by y.

It follows from paper [4] (cf. Lemma 1 and the argument on pp. 257-259) that if u is a solution of the (G)-problem, then

$$(11)\qquad u(x,y) = R_p(x,y) + \sum_{m=1}^{p} [y^{m-1}\varphi_m(x) + x^{m-1}\psi_m(y)]$$

$((x,y) \in \Omega)$, where

$$(12)\qquad R_p(x,y) = \int_0^x \int_0^y \frac{[(x-\xi)(y-\eta)]^{p-1}}{[(p-1)!]^2}\, c(\xi,\eta)\, d\xi\, d\eta$$

and the functions $\varphi_1,\varphi_2,\dots,\varphi_p$ and $\psi_1,\psi_2,\dots,\psi_p$ $(\varphi_m : [0,A] \longrightarrow E$, $\psi_m : [0,B] \longrightarrow E$ for $m = 1,2,\dots,p)$ are of class C^p, satisfy the initial conditions

$$(13)\qquad \varphi_m^{(l)}(0) = \psi_m^{(l)}(0) = 0 \quad \text{for } 0 \le l \le m-2 \le p-2$$

and the following system of differential-functional equations

$$(14)\qquad \varphi^{(jq)}_{jq+s}(x) = V_{jq+s}(x) + \frac{(s-1)!}{(s+jq-1)!} \sum_{i,r=1}^{q} \frac{(r+jq-1)!}{(r-1)!} G^s_{r,i}(x) \times$$

$$\times\ \psi^{(jq)}_{jq+s}(x) \circ f_i(x),$$

$$(15)\qquad \psi^{(jq)}_{jq+s}(y) = W_{jq+s}(y) + \frac{(s-1)!}{(s+jq-1)!} \sum_{i,r=1}^{q} \frac{(r+jq-1)!}{(r-1)!} \tilde{G}^s_{r,i}(y) \times$$

$$\times\ \varphi^{(jq)}_{jq+s}(y) \circ h_i(y),$$

($x \in (0,A]$; $y \in (0,B]$; $s = 1,2,\dots,q$; $j = 0,1,\dots,k-1$), where[2/]

$$(16)\quad V_{jq+s}(x) = (-1)^{s-1}\frac{(s-1)!}{(s+jq-1)!}\sum_{i=1}^{q} w_i(x)e_i^{s}(x)\Big\{M_{i,j}(x) - \sum_{m=(j+1)q+1}^{p}\frac{(m-1)!}{(m+jq-1)!}\Big[[f_i(x)]^{m-jq-1}\varphi_m^{(jq)}(x) + x^{m-jq-1}\psi_m^{(jq)}\circ f_i(x)\Big] - R_{p-jq}[x,f_i(x)]\Big\},$$

$$(17)\quad W_{jq+s}(y) = (-1)^{s-1}\frac{(s-1)!}{(s+jq-1)!}\sum_{i=1}^{q} \tilde{w}_i(y)\tilde{e}_i^{s}(y)\Big\{N_{i,j}(y) - \sum_{m=(j+1)q+1}^{p}\frac{(m-1)!}{(m+jq-1)!}\Big[y^{m-jq-1}\varphi_m^{(jq)}\circ h_i(y) + [h_i(y)]^{m-jq-1}\psi_m^{(jq)}(y)\Big] - R_{p-jq}[h_i(y),y]\Big\},$$

with R_m ($m \in \mathbb{N}$) denoting expression (12) with p replaced by m. Moreover, the derivative $D^{\alpha}u$ (with $\alpha=(\alpha_1,\alpha_2)$, $0 \le \alpha_1, \alpha_2 \le p$) does not depend on order in which the differentiation is performed.

For a fixed value of j, $0 \le j \le k-1$, the system of differential - functional equations (14), (15) is, with respect to the functions $\tilde{\varphi}_s = \varphi^{(jq)}_{jq+s}$ and $\tilde{\psi}_s = \psi^{(jq)}_{jq+s}$, of the same type as the system of functional equations examined in paper [4].

In further reasoning we will need the following notation

2/ $\sum_{r=l}^{m} a_r = 0$ and $\prod_{r=l}^{m} a_r = 1$ for $l > m$.

$$(18)\quad \begin{cases} F_{jq+s,n} = \begin{cases} V_{jq+s} & \text{for } n \text{ even},\\ W_{jq+s} & \text{for } n \text{ odd},\end{cases}\\[2ex] \tilde{F}_{jq+s,n} = \begin{cases} W_{jq+s} & \text{for } n \text{ even},\\ V_{jq+s} & \text{for } n \text{ odd},\end{cases}\end{cases}$$

$(s = 1,2,\dots,q;\ j = 0,1,\dots,k-1;\ n \in \mathbb{N})$,

$$(19)\quad \begin{cases} \zeta_{\vec{i}_{(n)}} = \begin{cases} z_{\vec{i}_{(n)}} & \text{for } n \text{ even}\\ \overset{*}{z}_{\vec{i}_{(n)}} & \text{for } n \text{ odd}\end{cases}\\[2ex] \tilde{\zeta}_{\vec{i}_{(n)}} = \begin{cases} \tilde{z}_{\vec{i}_{(n)}} & \text{for } n \text{ even}\\ \overset{**}{z}_{\vec{i}_{(n)}} & \text{for } n \text{ odd}\end{cases}\end{cases}$$

and

$$(20)\quad \begin{aligned} Q^{s}_{\vec{i}_{(n)}\vec{r}_{(n)}}(x) = {} & G^{s}_{r_1,i_1}(x) \prod_{t=1}^{\lceil (n-1)/2\rceil} G^{r_{2t}}_{r_{2t+1},i_{2t+1}} \circ z_{\vec{i}_{(2t)}}(x) \times \\ & \times \prod_{t=1}^{\lceil n/2\rceil} \tilde{G}^{r_{2t-1}}_{r_{2t},i_{2t}} \circ z_{\vec{i}_{(2t-1)}}(x), \end{aligned}$$

$$(20)\left\{\begin{aligned}\tilde{Q}^{s}_{\overrightarrow{i}_{(n)}\overrightarrow{r}_{(n)}}(y) &= \tilde{G}^{s}_{r_1,i_1}(y)\prod_{t=1}^{[(n-1)/2]}\tilde{G}^{r_{2t}}_{r_{2t+1},i_{2t+1}}\circ\tilde{z}_{\overrightarrow{i}_{(2t)}}(y)\ \times\\ &\times\prod_{t=1}^{[n/2]}G^{r_{2t-1}}_{r_{2t},i_{2t}}\circ\overset{**}{z}_{\overrightarrow{i}_{(2t-1)}}(y),\end{aligned}\right.$$

($\overrightarrow{i}_{(n)} = (i_1, i_2, \ldots, i_n)$; $\overrightarrow{r}_{(n)} = (r_1, r_2, \ldots, r_n)$; $n, i_m, r_m \in \mathbb{N}$; $1 \leq i_m$, $r_m \leq q$; $m = 1, 2, \ldots, n$; $s = 1, 2, \ldots, q$), where $[a]$ is the greatest integer not exceeding a.

Lemma. If the Assumptions I–III are satisfied, then the set of functions $\varphi_1, \varphi_2, \ldots, \varphi_p$; $\psi_1, \psi_2, \ldots, \psi_p$ given by the formulae[3]

$$(21)\ \varphi_{jq+s}(x) = \begin{cases}(1-\delta_{0j})\displaystyle\int_0^x \frac{(x-\xi)^{jq-1}}{|jq-1|!}S_{j,s}(\xi)d\xi + \delta_{j0}S_{0,s}(x) & \text{for } x \in (0, A],\\ 0 & \text{for } x = 0,\end{cases}$$

---

[3] For $j = k-1$ the expression $s_{j,s}$ and $\tilde{s}_{j,s}$ (cf. (23), (24)) do not depend on the functions φ_m and ψ_m, $m = 1, 2, \ldots, p$, and hence the functions φ_{p-q+s} and ψ_{p-q+s}, $s = 1, 2, \ldots, q$, are given by (21), (22) with (23), (24). The remaining functions φ_{jq+s} and ψ_{jq+s}, $j = k-j_*-1$; $j_* = 1, 2, \ldots, k-1$; $s = 1, 2, \ldots, q$, can be obtained recursively.

$$(22)\quad \psi_{jq+s}(y) = \begin{cases} (1-\delta_{0j}) \displaystyle\int_0^y \frac{(y-\eta)^{jq-1}}{|jq-1|!} \tilde{S}_{j,s}(\eta)\,d\eta + \delta_{j0}\tilde{S}_{0,s}(y) & \text{for } y \in (0,B], \\ 0 & \text{for } y = 0, \end{cases}$$

($s = 1,2,\ldots,q$; $j = 0,1,\ldots,k-1$; δ_{0j} – Kronecker's symbol; $\alpha(\pm\infty) := 0$), where .

$$(23)\quad S_{j,s}(x) = V_{jq+s}(x) + \sum_{n=1}^{\infty} \sum_{\substack{i_1,\ldots,i_n=1 \\ r_1,\ldots,r_n=1}}^{q} \frac{(s-1)!(r_n+jq-1)!}{(s+jq-1)!(r_n-1)!} \times$$

$$\times\, Q^{s}_{\vec{i}_{(n)}\vec{r}_{(n)}}(x) F_{jq+r_n,n} \circ \zeta_{\vec{i}_{(n)}}(x),$$

$$(24)\quad \tilde{S}_{j,s}(y) = W_{jq+s}(y) + \sum_{n=1}^{\infty} \sum_{\substack{i_1,\ldots,i_n=1 \\ r_1,\ldots,r_n=1}}^{q} \frac{(s-1)!(r_n+jq-1)!}{(s+jq-1)!(r_n-1)!} \times$$

$$\times\, \tilde{Q}^{s}_{\vec{i}_{(n)}\vec{r}_{(n)}}(y) \tilde{F}_{jq+r_n,n} \circ \tilde{\zeta}_{\vec{i}_{(n)}}(y).$$

is a solution of the system (14), (15) in the set Ω. The functions φ_m and ψ_m, $m = 1,2,\ldots,p$, are of class C^p.

It is the only solution of the system (14), (15) in the class $\mathcal{K}$ of all sets of functions φ_m, ψ_m, $m = 1,2,\ldots,p$, possessing continuous derivatives up to $\lceil (m-1)/q \rceil$th order, respectively, and such that the relations

$$(25)\qquad \begin{cases} \| \varphi_m^{(l)}(x) \| \le \text{const } x^{2p-m-l+\beta_0}, \\ \| \psi_m^{(l)}(y) \| \le \text{const } y^{2p-m-l+\beta_0}, \end{cases}$$

$(x \in (0,A]\ ;\ y \in (0,B]\ ;\ m = 1,2,\ldots p;\ l = 0,1,\ldots,\lceil (m-1)/q \rceil q)$ hold true.

Furthermore, the functions φ_m and ψ_m given by the formulae (21), (22) respectively, fulfil the relation (13) with $m = 1,2,\ldots,p$ and $l = 0,1,\ldots,p$.

Proof. It can be observed that there exists a positive number $\delta \le \max(A,B)$ so small that (cf. [4], Lemma 2 and relations (30), (31))

$$(26)\qquad |w_i(x)| \le [q(1+\varepsilon)]^{q-1}\,[f_q(x)]^{1-q}$$

and

$$(27)\qquad |e_i^s(x)| \le q^{q-1}\,[f_q(x)]^{q-s},$$

for $x \in (0,\delta)$; $i,s = 1,2,\ldots,q$.

Definition (10) and inequalities (26), (27) immediately yield the following estimate

$$(28)\qquad |G^s_{r,i}(x)| \le [q^2(1+\varepsilon)]^{q-1}\,[f_q(x)]^{1-s}\,x^{r-1},$$

$(x \in (0,\delta)$; $i,r,s = 1,2,\ldots,q)$.

Moreover, it can be proved (cf. [4], Lemma 4) that $\zeta_{\vec{i}(n)}$ tends to zero (for $n \longrightarrow \infty$) uniformly with respect to $x \in (0,A]$ and hence there is a positive integer n_0 such that for each $n \ge n_0$ and each $x \in (0,A]$ the relation $\zeta_{\vec{i}(n)}(x) \in (0,\delta)$ holds true.

Let us consider the system (14), (15) with $j = k-1$ and observe that the expressions V_{p-q+s} and W_{p-q+s}, $s = 1,2,\ldots,q$, do not depend on the functions φ_m and ψ_m, $m = 1,2,\ldots,p$, and, moreover, due to the Assumptions I–III and relations (16), (17), (26), (27) we have the following estimates

$$\tag{29} \begin{cases} \|V_{p-q+s}(x)\| \le \text{const } x^{2q-s+\beta_0}, \\ \\ \|W_{p-q+s}(y)\| \le \text{const } y^{2q-s+\beta_0}, \end{cases}$$

$(x \in (0,\delta);\ s = 1,2,\ldots,q)$.

Denote by

$$\tag{30} a_n^{p-q-s}(x) = \sum_{\substack{i_1,\ldots,i_n=1 \\ r_1,\ldots,r_n=1}}^{q} \frac{(s-1)!(r_n+p-q-1)!}{(s+p-q-1)!(r_n-1)!} \times$$

$$\times\, Q^{s}_{\vec{i}(n)\vec{r}(n)}(x)\, F_{p-q+r_n,n} \circ \zeta_{\vec{i}(n)}(x),$$

$(x \in (0,A];\ s = 1,2,\ldots,q;\ n \in \mathbb{N})$ the nth term of the series appearing on the right–hand side of formula (23) with $j = k-1$. Assuming that $n \ge n_0$, basing on Assumptions I–III and relations (20), (28), (29), and using an argument analogous to that in [4], pp. 265 – – 267, we obtain the inequality

$$\tag{31} \|a_n^{p-q-s}(x)\| \le \text{const } b^n\, x^{2q-s+\beta_0},$$

$(x \in (0,A];\ s = 1,2,\ldots,q;\ n \ge n_0)$, where const depends on n_0 but is independent of n and b is a fixed number in $(0,1)$.

Hence the series $\sum_{n=1}^{\infty} a_n^{p-q+s}$ (cf. (30)) is uniformly convergent in the interval $(0,A]$ and the functions $S_{k-1,s}$, $s = 1,2\dots,q$, are continuous and the estimate (cf. (29), (31))

$$(32) \qquad \|S_{k-1,s}(x)\| \leq \text{const}\ x^{2q-s+\beta_0},$$

$(x \in (0,A];\ s = 1,2,\dots,q)$ holds good, which immediately implies

$$(33) \qquad \|\varphi^{(l)}_{p-q+s}(x)\| \leq \text{const}\ x^{p+q-s-l+\beta_0},$$

$(x \in (0,A];\ s = 1,2,\dots,q;\ l = 0,1,\dots,p-q)$.

Differentiating the expression a_n^{p-q+s} and using an argument similar to that above (cf. [4], pp. 270–275), we can prove that the functions $S_{k-1,s}$, $s = 1,2,\dots,q$, are of class C^q in $(0,A]$ and the estimate

$$(34) \qquad \|S^{(l)}_{k-1,s}(x)\| \leq \text{const}\ x^{2q-s-l+\beta_0},$$

$(x \in (0,A];\ s,l = 1,2,\dots,q)$ holds good whence the relation (33) is valid for $l = p-q+1,\dots,p$. Basing on inequality (34), we get

$$(35) \qquad \lim_{x \to 0+} \varphi^{(l)}_{p-q+s}(x) = 0,$$

$(s = 1,2,\dots,q;\ l = 0,1,\dots,p)$. As a consequence, the functions φ_{p-q+s}, $s = 1,2,\dots,q$, are of class C^p in $[0,A]$.

By a similar argument one can prove that the functions ψ_{p-q+s}, $s = 1,2,\dots,q$, are of class C^p in $[0,B]$.

A direct calculation shows that the set of functions

$\varphi_{p-q+s},\dots,\varphi_p;\ \psi_{p-q+s},\dots,\psi_p$ is a solution of the system (14), (15) with $j = k-1$ in the set Ω.

To prove the uniqueness of a solution in the class K_x, let us note that if the functions φ_{p-q+s} and ψ_{p-q+s}, $s = 1,2,\dots,q$, satisfy the system (14), (15) with $j = k-1$, then for each positive integer m the equality

$$(36)\qquad \varphi^{(p-q)}_{p-q+s}(x) = V_{p-q+s}(x) + \sum_{n=1}^{m} a_n^{p-q+s}(x) + r_m^{p-q+s}(x),$$

$(x \in (0,A];\ s = 1,2,\dots,q)$ holds good, where

$$(37)\qquad r_m^{p-q+s}(x) = \sum_{\substack{i_1,\dots,i_{m+1}=1\\ r_1,\dots,r_{m+1}=1}}^{q} \frac{(s-1)!(r_{m+1}+p-q-1)!}{(s+p-q-1)!(r_{m+1}-1)!} \times$$

$$\times\, Q^{s}_{\vec{i}_{(m)}\vec{r}_{(m)}}(x)\ \overset{*}{G}{}^{r_m}_{r_{m+1},i_{m+1}}\circ\zeta_{\vec{i}_{(m)}}(x)\ \overset{*}{F}_{p-q+r_{m+1},m+1}\circ\zeta_{\vec{i}_{(m+1)}}(x),$$

with

$$\overset{*}{G}{}^{r_m}_{r_{m+1},i_{m+1}} = \begin{cases} G^{r_m}_{r_{m+1},i_{m+1}} & \text{for } n \text{ even},\\ \tilde{G}^{r_m}_{r_{m+1},i_{m+1}} & \text{for } n \text{ odd}, \end{cases}$$

$$\overset{*}{F}_{p-q+r_{m+1},m+1} = \begin{cases} \varphi_{p-q+r_{m+1}} & \text{for } n \text{ even},\\ \psi_{p-q+r_{m+1}} & \text{for } n \text{ odd}. \end{cases}$$

Basing on the definition of the class $\mathcal{K}$ and using an argument similar to that in the proof of relation (31), we can assert that r_m^{p-q+s} tends to zero $(m \longrightarrow \infty)$ uniformly with respect to $x \in (0,A]$. Hence, applying Taylor's, formula we observe that the functions φ_{p-q+s}, $s = 1,2,\ldots,q$, are given by formula (21). Analogous result can be obtained for the functions ψ_{p-q+s}, $s = 1,2,\ldots,q$.

Thus, the proof is completed in case when $k = 1$ $(p = q)$. Assume that $k > 1$ and that there exists a positive integer $j_0 < k-1$ such that for all integer j, $j_0+1 \le j \le k-1$, the functions φ_{jq+s} and ψ_{jq+s} belong to C^p on $[0,A]$ and $[0,B]$, respectively, and satisfy the inequalities

$$(38) \qquad \begin{cases} \|\varphi^{(l)}_{jq+s}(x)\| \le \text{const } x^{2p-jq-l-s+\beta_0}, \\ \\ \|\psi^{(l)}_{jq+s}(y)\| \le \text{const } y^{2p-jq-l-s+\beta_0}, \end{cases}$$

$((x,y) \in \Omega;\ s = 1,2,\ldots,q;\ l = 0,1,\ldots,p)$.

Let us reconsider the system (14), (15) with $j = j_0$. In virtue of the aforesaid assumption, one can prove that the assertion holds true for $j = j_0$.

An application of the mathematical induction principle ends the proof of our Lemma.

Basing on the above-obtained results, we can formulate the final statement

Theorem. Under the Assumptions I–III, the (G) – problem has a solution given by formula (11), where the functions φ_m and ψ_m, $m = 1,2,\ldots,p$, are given by formulae (21) and (22) (cf. Lemma). This solution is unique in the set $\mathcal{Z}$ of functions u of the form (11), where the set $\varphi_1,\varphi_2,\dots,\varphi_m;$ $\psi_1,\psi_2,\dots,\psi_m$ belongs to the class $\mathcal{K}$.

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REFERENCES

[1] A. Borzymowski: A Goursat problem for a polyvibrating equation of D. Mangeron, Funkcial. Ekvac., 23 (1980) 1–16.

[2] A. Borzymowski: A non-linear Goursat problem for a 4th order polyvibrating equation, Proc. Roy. Soc. Edinburgh Sect. A, 91 (1982) 153–164.

[3] A. Borzymowski: A Goursat problem for some partial differential equation of order 2p, Bull. Acad. Polon. Sci. Math., 32 (1984) 577–580.

[4] A. Borzymowski: Concerning a Goursat problem for some partial differential equation of order 2p, Demonstratio Math., 18 (1985) 253–277.

[5] M. Cinquini – Cibrario, S. Cinquini: Equazioni a derivate parziali di tipo iperbolico, Edizioni Cremonese Roma, 1965.

[6] M. Kuczma: Functional equations in a single variable, PWN Warszawa, 1968.

[7] D. Mangeron, L.E. Krivoshein: Sopra i

problemi al contorno per le equazioni integro - - differenziali lineari a derivate parziali con derivate d'ordine superiore di Picone, Rend. Sem. Mat. Univ. Padova, 33 (1963) 226-266.

[8] D. Mangeron , L. E. Krivoshein : Sulla risoluzione di una classe di problemi al contorno le equazioni integro-differenziali a derivate parziali, ibidem, 34 (1964) 344-368.

[9] M.N.Oǧuztöreli : Sur le problème de Goursat pour une équation de Mangeron d'ordre supérieur I, II, Acad. Roy. Belg. Bull. Cl. Sci., 58 (1972) 464-471, 577-582.

[10] M.N.Oǧuztöreli , S.Easwaran : A Goursat problem for a high order Mangeron equation, Atti Acad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur., 50 (1971) 650-653.

[11] O.Sjöstrand : Sur le problème de M.Goursat pour les équations aux dérivées partielles du seconde ordre ou de l'ordre supérieur, Göteborg, 1929.

[12] W.Walter : Differential and integral inequalities, Springer - Verlag Berlin, 1970.

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ON NONOSCILLATORY LINEAR SYSTEMS OF DIFFERENCE EQUATIONS

1. Introduction

Nonoscillation of difference equations is investigated by many authors. In particular, some properties of nonoscillatory solutions of linear equations and their systems were considered in [6]–[8]. In the present paper, we investigate number of sign changes of nonoscillatory solutions of linear systems of difference equations. This problem for linear systems of differential equations was considered in [1]–[5]. But it is not a complete analogy. A classic condition guaranteeing a constant sign of a component and at most one sign change of the second component of an arbitrary nontrivial solution of the system of differential equations

$$x_i'(t) = \sum_{j=1}^{2} \alpha_{ij}(t)x_j(t), \ i = 1,2,$$

where $\alpha_{ij} \in C[t_0, \infty)$ for $i,j = 1,2$, has the form : $\alpha_{12}\alpha_{21} > 0$ (see [1], [3]). In the theory of difference equations such condition is not true. For example one of solutions of the system

$$x_1(n+1) = x_2(n), \quad x_2(n+1) = x_1(n)$$

has the form

$$u_1(n) = (-1)^n, \quad u_2(n) = (-1)^{n+1}.$$

Let $N := \{0,1,2,\dots\}$, $N_0 := \{n_1,n_2,\dots,n_r\} \subset N$ with $n_i < n_{i+1}$ for $i = 1,2,\dots,r-1$ and $R := (-\infty, \infty)$. Let us take the linear system of m difference equations $(m \geq 2)$

$$(1.1) \qquad x_i(n+1) = \sum_{j=1}^{m} a_{ij}(n)x_j(n) + f_i(n), \quad i = 1,2,\dots,m,$$

where $a_{ij}, f_i : N \longrightarrow R$ for $i,j = 1,2,\dots,m$.

By a solution of this system we mean an arbitrary function $N \longrightarrow R^m$ satisfying its equations. We give a relationship between numbers of sign changes of the first component of solutions of the system (1.1) and a corresponding linear system of m-1 first - order difference equations. In particular, we obtain a sufficient condition under which the first component of all solutions of (1.1) has a bounded number of sign changes. Analogous relationships for the remaining components can be given.

We will use solutions of the linear homogeneous system

$$(1.2) \qquad y_i(n+1) = \sum_{j=1}^{m} a_{ij}(n)y_j(n), \quad i = 1,2,\dots,m,$$

corresponding to the system of difference equations (1.1).

A function $f : N \longrightarrow R$ is said to change sign p times if

(i) there exist $\nu_1,\nu_2,\dots,\nu_{p+1} \in N$ such that $\nu_i < \nu_{i+1}$ and $f(\nu_i)f(\nu_{i+1}) < 0$ for $i = 1,2,\dots,p$,

(ii) there do not exist $\tilde{\nu}_1,\tilde{\nu}_2,\dots,\tilde{\nu}_{p+1} \in N$ such that $\tilde{\nu}_i < \tilde{\nu}_{i+1}$ and $f(\tilde{\nu}_i)f(\tilde{\nu}_{i+1}) < 0$ for $i = 1,2,\dots,p+1$.

A function $f : N \longrightarrow R$ is called nonoscillatory if there exists an $n_0 \in N$ such that either $f|_{(n_0,\infty)\cap N} > 0$ or $f|_{(n_0,\infty)\cap N} < 0$.

For a set $M \subset N$, we denote $M-1 := \{ n-1 : n \in M \}$.

2. **Lemma 1.** Assume that

$$a_{11}(n) \ge 0,\ a_{1j}(n) \ne 0,\ a_{1j}(n+1)a_{j1}(n) \ge 0, \tag{2.1}$$

$$a_{1j}(n+1)a_{1i}(n)a_{ji}(n) \ge 0 \text{ for } i,j = 2,3,\dots,m,\ n \in N,$$

$$a_{11}(n+1) + \sum_{j=2}^{m} |a_{j1}(n)| \ne 0 \text{ for } n \in N. \tag{2.2}$$

a) If $(\xi_1,\xi_2,\dots,\xi_m)$ is a solution of the system of difference equations (1.2) such that

$$\xi_1(0) > 0,\ a_{1i}(0)\xi_i(0) \ge 0 \text{ for } i = 2,3,\dots,m, \tag{2.3}$$

$$a_{11}(0) + \sum_{j=2}^{m} |\xi_j(0)| \ne 0, \tag{2.4}$$

then $\xi_1(n) > 0$ for $n \in N$.

b) If $(\xi_1,\xi_2,\dots,\xi_m)$ is a solution of the system of difference equations (1.2) such that

$$\xi_1(0) < 0,\ a_{1i}(0)\xi_i(0) \le 0 \text{ for } i = 2,3,\dots,m \tag{2.5}$$

and (2.4) holds, then $\xi_1(n) < 0$ for $n \in N$.

Proof. Let $(\xi_1,\xi_2,\dots,\xi_m)$ be a solution of the system (1.2) satisfying the conditions (2.3) and (2.4). Hence

$$\xi_1(1) = \sum_{j=1}^{m} a_{1j}(0)\xi_j(0) > 0,$$

because all terms on the right – hand side are nonnegative and at least one term is positive. Applying mathematical induction, suppose that for a certain $k \in N$, $k \geq 1$, we have

$$\xi_1(k-1) > 0,\ a_{1i}(k-1)\xi_i(k-1) \geq 0,\ i = 2,3,\dots,m,\ \xi_1(k) > 0.$$

We are going to show that

$$a_{1i}(k)\xi_i(k) \geq 0,\ i = 2,3,\dots,m,\ \xi_1(k+1) > 0.$$

By the assumption (2.1), there is

$$a_{1i}(k)\xi_i(k) = a_{1i}(k) \sum_{j=1}^{m} a_{ij}(k-1)\ \xi_j(k-1) =$$

$$= a_{1i}(k)a_{i1}(k-1)\xi_1(k-1) + \sum_{j=2}^{m} \frac{1}{a_{1j}^2(k-1)} \times$$

$$\times \left[a_{1i}(k)a_{ij}(k-1)a_{1j}(k-1) \right] \cdot \left[a_{1j}(k-1)\xi_j(k-1) \right] \geq 0$$

for $i = 2,3,\dots,m$. Moreover, taking into account (2.2) we have

$$\xi_1(k+1) = a_{11}(k)\xi_1(k) + \sum_{i=2}^{m} a_{1i}(k)\xi_i(k) > 0.$$

By the principle of mathematical induction, $\xi_1(n) > 0$ for $n \in N$.

The proof for a solution of (1.2) satisfying the conditions (2.5) and (2.4) is analogous.

3. **Lemma 2.** Assume that

$$a_{11}(n) \begin{cases} \leq 0 \text{ if } n \in N_0, \\ \geq 0 \text{ if } n \in N \setminus N_0, \end{cases} \quad a_{1j}(n) \neq 0 \text{ for } n \in N,$$

$$a_{1j}(n+1)a_{j1}(n)\begin{cases}\le 0 \text{ if } n\in[N_0\setminus(N_0-1)]\cup[N\cap(N_0-1)\setminus N_0],\\ \ge 0 \text{ if } n\in N\setminus\{[N_0\setminus(N_0-1)]\cup[N\cap(N_0-1)\setminus N_0]\},\end{cases}$$

$$a_{1j}(n+1)a_{1i}(n)a_{ji}(n)\begin{cases}\le 0 \text{ if } n\in N\cap(N_0-1),\\ \ge 0 \text{ if } n\in N\setminus(N_0-1)\end{cases}$$

for $i,j = 2,3,\ldots,m$ and (2.2) holds. If $(\xi_1,\xi_2,\ldots,\xi_m)$ is a solution of the system of difference equations (1.2) such that either

$$(3.1)\quad \xi_1(0)>0,\ a_{1i}(0)\xi_i(0)\begin{cases}\le 0 \text{ if } n_1=0\\ \ge 0 \text{ if } n_1\ne 0\end{cases}\text{for } i=2,3,\ldots,m$$

or

$$(3.2)\quad \xi_1(0)<0,\ a_{1i}(0)\xi_i(0)\begin{cases}\ge 0 \text{ if } n_1=0,\\ \le 0 \text{ if } n_1\ne 0\end{cases}\text{for } i=2,3,\ldots,m$$

and (2.4) holds, then $\xi_1(n)\ne 0$ for $n\in N$ and the function ξ_1 changes sign r times.

Proof. Let $(\xi_1,\xi_2,\ldots,\xi_m)$ be a solution of the system (1.2) satisfying the conditions (3.1) and (2.4). Suppose that r is an even number. Denote

$$\eta_i(n):=\begin{cases}-\xi_i(n) \text{ if } n\in N_1:=\bigcup_{k=1}^{r/2}\{n_{2k-1}+1,n_{2k-1}+2,\ldots,n_{2k}\},\\ \xi_i(n) \text{ if } n\in N\setminus N_1\end{cases}$$

for $i = 1,2,..,m$. We have

$$\eta_i(n+1)=\sum_{j=1}^{m}b_{ij}(n)\eta_j(n),\ i=1,2,..,m,$$

where

$$b_{ij}(n) := \begin{cases} -a_{ij}(n) & \text{if either } n \in N_1 \text{ and } n+1 \in N \setminus N_1 \\ & \text{or } n \in N \setminus N_1 \text{ and } n+1 \in N_1, \\ a_{ij}(n) & \text{if either } n \in N_1 \text{ and } n+1 \in N_1 \\ & \text{or } n \in N \setminus N_1 \text{ and } n+1 \in N \setminus N_1 \end{cases}$$

for $i,j = 1,2,..,m$. Since

$$\left[N_1 \setminus (N_1 - 1) \right] \cup \left[(N_1 - 1) \setminus N_1 \right] =$$

$$= \bigcup_{k=1}^{r/2} \Big[\Big(\{ n_{2k-1}+1, n_{2k-1}+2, \ldots, n_{2k} \} \setminus$$

$$\setminus \{ n_{2k-1}, n_{2k-1}+1, \ldots, n_{2k}-1 \} \Big) \cup \Big(\{ n_{2k-1}, n_{2k-1}+1, \ldots, n_{2k}-1 \} \setminus$$

$$\setminus \{ n_{2k-1}+1, n_{2k-1}+2, \ldots, n_{2k} \} \Big) \Big] = \bigcup_{k=1}^{r/2} \Big(\{ n_{2k} \} + \{ n_{2k-1} \} = N_0,$$

so

$$b_{ij}(n) = \begin{cases} -a_{ij}(n) & \text{if } n \in N_0, \\ a_{ij}(n) & \text{if } n \in N \setminus N_0 \end{cases}$$

for $i,j = 1,2,..,m$. Hence

$$b_{11}(n) \geq 0, \ b_{1j}(n) \neq 0, \ b_{1j}(n+1) b_{j1}(n) \geq 0,$$

$$b_{1j}(n) b_{1i}(n) b_{ji}(n) \geq 0, \ b_{1i}(0) \eta_i(0) \geq 0$$

for $i,j = 2,3,..,m$, $n \in N$. By Lemma 1a, $\eta_1(n) > 0$ for $n \in N$. Thus $\xi_1(n) \neq 0$ for $n \in N$ and the function ξ_1 changes sign r times.

If r is an odd number, then we put

$$N_1 := \bigcup_{k=1}^{\frac{r-1}{2}} \left\{ n_{2k-1}+1, n_{2k-1}+2, \dots, n_{2k} \right\} \cup \left\{ n_r+1, n_r+2, \dots \right\}$$

and a similar argument holds.

The proof for a solution of (1.2) satisfying the conditions (3.2) and (2.4) is analogous.

4. Remarks

a) Lemma 1 is a particular case of Lemma 2 (for $N_0 = 0$).

b) The solutions which exist by Lemma 2 may be used in next parts of the paper.

c) Lemma 2 can be extended on the case when $a_{1j}(n) = 0$ for some j and n, but $\xi_j(n)$ has a determined sign. Then, instead of a sign of the expression $a_{1j}(n+1)a_{1i}(n)a_{ji}(n)$, it is necessary to assume a proper sign of the expression $a_{1j}(n+1)\xi_i(n)a_{ji}(n)$.

d) Some particular cases of Lemma 2 (without the assumption $a_{1j}(n) \neq 0$) are contained in the paper [8]. They may be used to determination of signs of the expressions $\xi_j(n)$ mentioned in the item c).

e) Obviously, solutions of the system (1.2) with the first component without zero points and changing sign a finite number times also exist, if the coefficients of this system satisfy other kinds of conditions (see Example 11).

5. **Lemma 3.** Let $a_{1m}(n) \neq 0$ for $n \in N$ and the coefficients of the system of difference equations (1.2) satisfy whatever conditions guaranteeing existence of a solution $(\xi_1, \xi_2, \dots, \xi_m)$ of this system such that $\xi_1(n) \neq 0$ for

$n = 2,3,\ldots$. Then the substitution

$$(5.1)\quad \begin{cases} x_1(n) = \xi_1(n) \sum\limits_{k=0}^{n-1} z_1(k), \\ x_i(n) = \xi_i(n) \sum\limits_{k=0}^{n-1} z_1(k) + z_i(n), \quad i = 2,3,\ldots,m, \end{cases}$$

$\left(\sum\limits_{k=0}^{-1} f(k) = 0 \right)$ transforms the system of difference equations (1.1) into the system of equations

$$(5.2)\quad z_i(n+1) = \sum_{j=1}^{m-1} c_{ij}(n) z_j(n) + g_i(n), \quad i = 1,2,\ldots,m-1,$$

where

$$(5.3)\quad \left\{ \begin{aligned} c_{11}(n) &:= \frac{1}{\xi_1(n+2)} \sum_{k=2}^{m} a_{1k}(n+1) \times \\ &\qquad \times \left[\frac{a_{km}(n)}{a_{1m}(n)} \xi_1(n+1) - \xi_k(n+1) \right], \\ c_{1j}(n) &:= \frac{1}{\xi_1(n+2)} \sum_{k=2}^{m} a_{1k}(n+1) \times \\ &\qquad \times \left[a_{kj}(n) - \frac{a_{1j}(n) a_{km}(n)}{a_{1m}(n)} \right], \\ c_{i1}(n) &:= \frac{a_{im}(n)}{a_{1m}(n)} \xi_1(n+1) - \xi_i(n+1), \\ c_{ij}(n) &:= a_{ij}(n) - \frac{a_{1j}(n) a_{im}(n)}{a_{1m}(n)}, \end{aligned} \right.$$

$$\left|\begin{aligned} g_1(n) &:= \frac{1}{\xi_1(n+2)}\left\{\sum_{k=2}^{m} a_{1k}(n+1)\left[f_k(n) - \right.\right. \\ &\left.\left. - \frac{a_{km}(n)}{a_{1m}(n)} f_1(n)\right] + f_1(n+1)\right\}, \\ g_i(n) &:= f_i(n) - \frac{a_{im}(n)}{a_{1m}(n)} f_1(n) \end{aligned}\right.$$

for $i,j = 2,3,\dots,m-1$. Moreover,

$$z_m(n) = \frac{1}{a_{1m}(n)}\left[\xi_1(n+1)z_1(n) + \right. \qquad (5.4)$$

$$\left. - \sum_{j=2}^{m-1} a_{1j}(n)z_j(n) - f_1(n)\right].$$

Proof. Let $(\xi_1,\xi_2,\dots,\xi_m)$ be a solution of the system (1.2) appearing in the assumption of the lemma. In the system (1.1) we change variables according to the formulae (5.1). We get

$$(5.5)\quad \left\{\begin{aligned} \xi_1(n+1)z_1(n) &= \sum_{j=2}^{m} a_{1j}(n)z_j(n) + f_1(n), \\ z_i(n+1) &= -\xi_i(n+1)z_1(n) + \sum_{j=2}^{m} a_{ij}(n)z_j(n) + f_i(n), \end{aligned}\right.$$

$$i = 2,3,\dots,m.$$

Hence the equation (5.4) follows. Displacing argument in the first equation of (5.5) and eliminating the variable z_m, we obtain the system

$$
(5.6)\quad \left\{
\begin{aligned}
z_1(n+1) = {} & \frac{1}{\xi_1(n+2)} \Big\{ \sum_{k=2}^{m} a_{1k}(n+1) \Big[\frac{a_{km}(n)}{a_{1m}(n)} \xi_1(n+1) + \\
& -\xi_k(n+1) \Big] z_1(n) + \sum_{j=2}^{m-1} \sum_{k=2}^{m} a_{1k}(n+1) \Big[a_{kj}(n) + \\
& - \frac{a_{1j}(n) a_{km}(n)}{a_{1m}(n)} \Big] z_j(n) + \sum_{k=2}^{m} a_{1k}(n+1) \cdot \\
& \cdot \Big[f_k(n) - \frac{a_{km}(n)}{a_{1m}(n)} f_1(n) \Big] + f_1(n+1) \Big\}, \\
z_i(n+1) = {} & \Big[\frac{a_{im}(n)}{a_{1m}(n)} \xi_1(n+1) - \xi_i(n+1) \Big] z_1(n) + \\
& + \sum_{j=2}^{m-1} \Big[a_{ij}(n) - \frac{a_{1j}(n) a_{im}(n)}{a_{1m}(n)} \Big] z_j(n) + f_i(n) + \\
& - \frac{a_{im}(n)}{a_{1m}(n)} f_1(n), \quad i = 2,3,\ldots,m-1.
\end{aligned}
\right.
$$

This is a system of the form (5.2) with the coefficients defined by the formulae (5.3).

6. **Theorem 1.** Assume that $a_{1m}(n) \neq 0$ for $n \in N$ and the coefficients of the system of difference equations (1.2) satisfy whatever conditions guaranteeing existence of a solution $(\xi_1, \xi_2, \ldots, \xi_m)$ of this system such that $\xi_1(n) \neq 0$ for $n \in N$ as well as the function ξ_1 changes sign r times. Let $(u_1, u_2, \ldots, u_m)$ be a solution of the system of difference equations (1.1). If for every solution $(v_1, v_2, \ldots, v_{m-1})$ of the system of difference equations (5.6) the function v_1 changes

sign at most p times, then the function u_1 changes sign at most p+r+1 times.

Proof. Take the solution $(u_1, u_2, \dots, u_m)$ of the system (1.1) and the solution $(\xi_1, \xi_2, \dots, \xi_m)$ of the system (1.2) appearing in the assumptions of the theorem. Let

$$\zeta_i := u_i - \frac{u_1}{\xi_1}(0)\, \xi_i, \quad i = 1,2,\dots,m.$$

Hence $(\zeta_1, \zeta_2, \dots, \zeta_m)$ is a solution of the system (1.1). Let Δ denote the forward difference operator, i.e.

$$\Delta(f)(n) := f(n+1) - f(n)$$

for $f : N \longrightarrow R$, $n \in N$. Since

$$\zeta_1(n) = \xi_1(n) \sum_{k=0}^{n-1} \Delta\left[\frac{\zeta_1}{\xi_1} \right](k),$$

so, by Lemma 3, the function $\Delta\left[\frac{\zeta_1}{\xi_1} \right]$ is the first component of a solution of the system (5.6). We have

$$u_1(n) = \xi_1(n) \left[\frac{u_1}{\xi_1}(0) + \frac{\zeta_1}{\xi_1}(n) \right].$$

Denote

$$U(n) := \frac{u_1}{\xi_1}(0) + \frac{\zeta_1}{\xi_1}(n), \quad n \in N.$$

Hence we obtain $\Delta(U) = \Delta\left[\frac{\zeta_1}{\xi_1} \right]$. Thus the function $\Delta(U)$ is the first component of a solution of the system (5.6). By the assumption, the function $\Delta(U)$ changes sign at most p times, so the function U changes sign at most p+1 times. The function u_1, as product of the functions ξ_1 and U, changes sign at most p+r+1 times.

7. C o r o l l a r y 1. Assume that $a_{1m}(n) \neq 0$ for $n \in N$ and the coefficients of the system of difference equations (1.2) satisfy whatever conditions guaranteeing existence of a solution $(\xi_1,\xi_2,\dots,\xi_m)$ of this system such that $\xi_1(n) \neq 0$ for $n \in N$ as well as the function ξ_1 is nonoscillatory. Let $(u_1,u_2,\dots,u_m)$ be a solution of the system of difference equations (1.1). If for every solution $(v_1,v_2,\dots,v_{m-1})$ of the system of difference equations (5.6) the function v_1 is either identically equal to zero or nonoscillatory, then the function u_1 has the same property.

This corollary follows from the proof of Theorem 1, immediately.

8. P r o p o s i t i o n 1. Assume that $a_{1m}(n) \neq 0$ for $n \in N$ and the coefficients of the system of difference equations (1.2) satisfy whatever conditions guaranteeing existence of a solution $(\xi_1,\xi_2,\dots,\xi_m)$ of this system such that $\xi_1(n) \neq 0$ for $n \in N$.

a) If the system of difference equations (5.6) has no solutions with the first component identically equal to zero, then the system (1.1) has the same property.

b) If $f_1 = 0$ and the system of difference equations (5.6) has no nontrivial solutions with the first component identically equal to zero, then the system (1.1) has the same property.

Proof. The assertion a) follows from the proof of Theorem 1.

Ad b). Suppose that $(u_1, u_2, \ldots, u_m)$ is a solution of the system (1.1) such that the function u_1 is identically equal to zero. We have

$$u_1(n) = \xi_1(n) \sum_{k=0}^{n-1} 0,$$

$$u_i(n) = \xi_i(n) \sum_{k=0}^{n-1} 0 + u_i(n), \; i = 2,3,\ldots,m,$$

on the set N. By Lemma 3, $(0, u_2, u_3, \ldots, u_{m-1})$ is a solution of the system (5.6) and

$$u_m = -\frac{1}{a_{1m}} \sum_{j=2}^{m-1} a_{1j} u_j. \tag{8.1}$$

By the assumption, the functions $u_2, u_3, \ldots, u_{m-1}$ are identically equal to zero and, by the formula (8.1), u_m has the same property.

9. Corollary 2. Let $m = 2$. Assume that $a_{12}(n) \neq 0$ for $n \in N$ and the function

$$F(n) := a_{12}(n+1) \left[f_2(n) - \frac{a_{22}(n)}{a_{12}(n)} f_1(n) \right] + f_1(n+1), \; n \in N,$$

changes sign p times. Let $(u_1, u_2, \ldots, u_m)$ be a solution of the system of difference equations (1.1). If the coefficients of the system of difference equations (1.2) satisfy whatever conditions guaranteeing existence of a solution (ξ_1, ξ_2) of this system such that

$$\xi_1(n) \neq 0,\ a_{12}(n+1)\xi_1(n+2)\xi_2(n+1) \le 0,$$

$$a_{12}(n)a_{12}(n+1)a_{22}(n)\xi_1(n+1)\xi_1(n+2) \ge 0$$

for $n \in N$ and the function ξ_1 changes sign r times, then the function u_1 changes sign at most $p+2r+2$ times.

Proof. Let (ξ_1, ξ_2) be the solution of the system (1.2) appearing in the assumptions of the corollary. By Lemma 3 the substitution (5.1) transforms the system (1.1) into the equation

$$z(n+1) = \frac{1}{\xi_1(n+2)}\left\{ a_{12}(n+1)\left[\frac{a_{22}(n)}{a_{12}(n)}\xi_1(n+1) + \right.\right.$$

$$\left.\left. - \xi_2(n+1) \right] z(n) + F(n) \right\}.$$

On account of the assumptions, this is an equation of the form

$$z(n+1) = b(n)z(n) + g(n),$$

where $b \ge 0$ and the function g changes sign at most $p+r$ times. Hence any solution v of this equation changes sign at most $p+r+1$ times. By Theorem 1, for every solution (u_1, u_2) of the system (1.1) the function u_1 changes sign at most $p+2r+2$ times.

10. Remark. A linear system of m first - order difference equations ($m \ge 3$) can be transformed by means of $m-2$ substitutions of the type (5.1) into a system of two first - order difference equations. Then Corollary 2 and Theorem 1 can be applied to it. In this manner we obtain properties of solutions of the system of m equations.

11. **Example.** Take the system of difference equations

$$(11.1)\qquad \begin{cases} x_1(n+1) = (3-n)x_1(n) - 2x_2(n) + x_3(n) + n, \\ x_2(n+1) = (1-n)x_1(n) + nx_2(n) + 2-2n, \\ x_3(n+1) = (1+n-n^2)x_1(n) + nx_3(n) - 1. \end{cases}$$

The linear homogeneous system corresponding to this system has the solution

$$\xi_1(n) = 1, \quad \xi_2(n) = 1, \quad \xi_3(n) = n.$$

Applying the substitution (5.1) to the system (11.1), we obtain a system of the form (5.6) the first equation of which is

$$(11.2)\qquad z_1(n+1) = z_1(n) - 4 + 5n - n^2.$$

The function

$$g_1(n) := -4 + 5n - n^2.$$

changes sign twice. Hence any solution of the equation (11.2) changes sign at most three times. By Theorem 1, the first component of any solution of the system (11.1) changes sign at most four times.

REFERENCES

[1] Z. Butlewski : O całkach rzeczywistych równań różniczkowych liniowych zwyczajnych, Wiadom. Mat. 44 (1938) 17–81.

[2] Z. Butlewski : Sur les intégrales non oscillatoires d'un système de trois équations différentielles linéaires, Ann. Polon. Math. 18 (1966) 95–106.

[3] J. Mikołajski : On a certain property of the linear system of two differentiall equations, Funct. Approximatio, 16 (in print).

[4] J. Mikołajski : On the linear systems of differential equations with a bounded number of zeros of their solutions, Comment. Math. 27.2 (in print).

[5] J. Mikołajski , Z. Ratajczak : On the nonoscillatory solutions of a system of four linear homogeneous differential equations, Fasc. Math. 13 (1981) 51–59.

[6] W. Patula : Growth and oscillation properties of second order linear difference equations, SIAM J. Math. Anal. 10.1 (1979) 55–61.

[7] W. Patula : Growth, oscillation and comparision theorems for second order linear difference equations, SIAM J. Math. Anal. 10.6 (1979) 1272–1279.

[8] J. Popenda : On the system of finite difference equations, Fasc. Math. 15 (1984) 119–126.

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ON A CHARACTERIZATION OF PROBABILITY DISTRIBUTIONS BY CONDITIONAL EXPECTATIONS

1. Introduction

There are well known results which characterize probability distributions by conditional expectations, see, for instance, Cox [1], Swartz [7] or Galambos and Kotz [3]. It has been shown there the mean residual liftime determines the distribution, namely: if X and Y are random variables whose means exist, then X and Y have the same distribution if and only if

$$E(X - c \mid X \geq c) = E(Y - c \mid Y \geq c),\ c \in R. \tag{1.1}$$

The generalization of the formula (1.1) for multivariate distributions is given by Jupp and Mardia in [5].

Many authors considered a characterization of probability distributions taking as a starting point equations describing conditional expectations in the form

$$E(h(X) \mid X \geq x) = g(x)\ ,\ x \in R, \tag{1.2}$$

where h, g are certain known functions (see, for istance, Gupta [4], Dallas [2]).

Galambos and Kotz [3] considered an example which shows that the characterization (1.2) includes distributions for which means do not exist.

The aim of this paper is to give a simple characterization of continuous distributions by conditional expectations for the cases in which means exist as well as they do not exist.

2. Characterization

Let X be a univariate random variable and F,G be the differentiable distribution and tail distribution functions of X, respectively. Assume that for $-\infty \le a < b \le +\infty$ the following conditions are satisfied

$$(2.1)\quad \begin{cases} \lim\limits_{x \to a^+} F(x) = 0, \quad \lim\limits_{x \to b^-} F(x) = 1, \\ 0 < F(x) < 1 \quad \text{for } x \in (a,b), \end{cases}$$

and

$$(2.2)\quad \begin{cases} \lim\limits_{x \to a^+} G(x) = 0, \quad \lim\limits_{x \to b^-} G(x) = 1, \\ 0 < G(x) < 1 \quad \text{for } x \in (a,b), \end{cases}$$

Theorem 1. The equation

$$(2.3)\quad E(F^{-\alpha}(X)\,|X < x) = \frac{1}{1-\alpha} F^{-\alpha}(x) \quad \text{for} \quad x \in (a,b)$$

or the equation

$$(2.4)\quad E(G^{-\alpha}(X)\,|X \ge x) = \frac{1}{1-\alpha} \cdot G^{-\alpha}(x) \quad \text{for} \quad x \in (a,b),$$

where $\alpha \in (0,1)$, determines uniquely the distribution of the random variable X.

Proof. For any random variable X with distribution function F the existence of the mean value $E(F^{-\alpha}(x))$ for all $\alpha \in (0,1)$ follows from Lemma 1.1.8 proved in [6]. We will prove first that any continuous random variable satisfies the equation (2.3). Integrating by part the left hand side of (2.3) we obtain

$$E(F^{-\alpha}(X)\,|\,X < x) = F^{-1}(x)\int_a^x F^{-\alpha}(y)F'(y)dy =$$

$$= F^{-1}(x)\Big(F^{1-\alpha}(y)\Big|_{y=a}^{y=x} + \alpha\int_a^x F^{-\alpha}(y)F'(y)dy \Big)$$

for $x \in (a,b)$. As $\alpha \in (0,1)$ and the conditions (2.1) are satisfied we have

$$E(F^{-\alpha}(X)\,|\,X < x) = \frac{1}{1-\alpha}\,F^{-\alpha}(x) \quad \text{for} \quad x \in (a,b).$$

By analogy we prove that the distribution function G satisfies (2.4).

Let $\tilde{F}$ be any differentiable distribution function satisfying (2.1) and

$$\text{(2.5)} \qquad E(\tilde{F}^{-\alpha}(X)\,|\,X < x) = \frac{1}{1-\alpha}\,\tilde{F}^{-\alpha}(x) \quad \text{for} \quad x \in (a,b).$$

We will prove that $\tilde{F} = F$. By differentiation both sides of (2.5) we get the equation

$$\tilde{F}^{-\alpha}(x)F'(x) = \frac{1}{1-\alpha}\,(-\alpha\,\tilde{F}^{-1-\alpha}(x)\tilde{F}'(x)\,F(x) + \tilde{F}^{-\alpha}(x)F'(x))$$

which leads to the following differential equation

$$\frac{d}{dx}(\ln \tilde{F}) = \frac{d}{dx}(\ln F).$$

Taking into account the conditions (2.1) we obtain $\tilde{F} = F$. The uniqueness of the tail distribution function G may be proved in the same way.

As a conclusion of Theorem 1 we obtain

Theorem 2. Continuous random variables X_1 and X_2 have the same distributions if and only if for any $\alpha \in (0,1)$ one of the following conditions is satisfied

$$E(F_1^{-\alpha}(x) - F_1^{-\alpha}(x) \,|\, X_1 < x) = E(F_2^{-\alpha}(x) - F_2^{-\alpha}(x) \,|\, X_2 < x)$$

or

$$E(G_1^{-\alpha}(x) - G_1^{-\alpha}(x) \,|\, X_1 \geq x) = E(G_2^{-\alpha}(x) - G_2^{-\alpha}(x) \,|\, X_2 \geq x)$$

for $x \in (a,b)$, where F_1, F_2 are distribution function of X_1 and X_2 and G_1, G_2 are tail distributions functions of X_1 and X_2, respectively.

3. Examples

a) Characterization of the Pareto distribution.

Let $a > 0$, $b = +\infty$. A random variable X has the Pareto distribution with parameter $\theta > 0$ if its tail distribution function G takes the form

$$G(x) = P(X \geq x) = (a^{-1}x)^{-\theta} \quad , \quad x \geq 0 .$$

According to Theorem 1 a random variable X has the Pareto distribution with parameter $\theta > 0$ if and only if for any $\alpha \in (0,1)$

$$E((a^{-1}X)^{\alpha\theta} \mid X \geq x) = \frac{1}{1-\alpha}(a^{-1}X)^{\alpha\theta} \quad ,$$

for $x \geq a > 0$. The characterization includes the case $0 < \theta < 1$. For $0 < \theta < 1$ the mean of the Pareto distribution does not exist.

b) Charakterization of the logistic distribution.

Let $a = -\infty$, $b = +\infty$. A random variable X has the logistic distribution with parameter $\theta > 0$ if its distribution function F takes the form

$$F(x) = P(X < x) = (1 + e^{-x})^{-\theta} \quad , \quad x \in R .$$

We have the following characterization: a random variable X has the logistic distribution with parameter $\theta > 0$ if and only if for any $\alpha \in (0,1)$

$$E((1 + e^{-X})^{-\alpha\theta} \mid X < x) = \frac{1}{1-\alpha}(1 + e^{-x})^{-\alpha\theta}, \quad x \in R.$$

c) Characterization of the exponential distribution.

Let $a = 0$, $b = +\infty$. A random variable X has the exponential distribution with parameter $\lambda > 0$ if the distribution and tail distribution functions of X are

$F(x) = 1 - e^{-\lambda x}$, $G(x) = e^{-\lambda x}$ for $x \geq 0$, respectively.

In this case, a random variable X has the exponential distribution with parameter $\lambda > 0$ if and only if for any $\alpha \in (0,1)$ one of the following equations

$$E((1 - e^{-\lambda X})^{-\alpha} \mid X < x) = \frac{1}{1-\alpha}(1 - e^{-\lambda x})^{-\alpha}, \quad x \geq 0$$

or

$$E(e^{\alpha\lambda X} \big| X \geq x) = \frac{1}{1-\alpha} e^{\alpha\lambda x} \quad ,x \geq 0$$

is satisfied.

d) Charakterization of the uniform distribution.

Let $a = 0$, $b = 1$. A random variable X has the uniform distribution on $(0,1)$ if and only if for any $\alpha \in (0,1)$

$$E(X^{-\alpha} | X < x) = \frac{1}{1-\alpha} x^{-\alpha} \ , \ x \in (0,1).$$

REFERENCES

[1] D.R.Cox : Renewal Theory. Methuen, London (1962) p. 128.

[2] A.C.Dallas : Characterizing the Pareto and power distributions, Ann. Inst. Statist. Math. 28 (1976) 491-497.

[3] I.Galambos, S. Kotz : Characterizations of Probability Distributions. Lecture Notes in Mathematics, 675. Springer Verlag, Berlin (1978) pp. 30-35.

[4] R.Gupta : On characterization of distribution by conditional expectations. Communications in Staticties, 4 (1975), 99-103.

[5] P.E.Jupp, K.V.Mardia : A characterization of the multivariate Pareto distribution, Ann. Statist. 10 (1982), 1021-1024.

[6] A. M. Kagan, J. V. Linnik, S. R. Rao ; Characterization problems in mathematical statistics, Wiley, New York 1973.

[7] G.B.Swartz : The mean residual liftime function. IEEE Trans Reliability R 22 (1973), 108-109.

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ON SEMI - INVARIANT SUBMANIFOLDS OF AN ALMOST PARACONTACT RIEMANNIAN MANIFOLD

Introduction

Semi-invariant submanifold of an almoct contact Riemannian manifold have been studiend in [1]. Concept of an almost paracontact Riemannian manifold which is analogous to almost contact manifold was introduced by Sato [4,5] in 1976.

In this paper we define and study semi-invariant submanifolds of an almost paracontact Riemannian manifold. We begin the paper by giving some preliminary definitions and assumptions used in the paper. Section 1 contains the definition of semi-invariant submanifold and certain results as well as conditions for the existence of such submanifolds. A number of theorems establishing necessary and sufficient conditions for the integrability of the various distributions related to such submanifolds have been proved in Section 2.

0. Preliminaries

Throughout this paper we assume the following :

A. $\overline{M}^m$ is an m-dimensional almost paracontact Riemannian manifold $\{\overline{M}^m, F, \xi, \eta, g\}$. Here F is a (1,1) tensor field, ξ is a non-zero contravariant vector field, η is a 1-form and g is a

Riemannian metric on M^m which satisfy :

(0.1) $$F^2(X) = X - \eta(X)\xi,$$

(0.2) $$F(\xi) = 0,$$

$$\eta(\xi) = 1,$$

$$\eta_o F = 0,$$

$$\mathrm{Rank}(F) = m-1,$$

$$g(FX,FY) = g(X,Y) - \eta(X)\eta(Y),$$

$$\eta(X) = g(\xi,X),$$

(0.3) $$g(FX,Y) = g(X,FY).$$

B. M^n is a n-dimensional differentable manifold isometrically immersed in M^m such that ξ is tangential to M^n.

C. TM and $TM^{\perp}$ denote the tangent bundle and the normal bundle respectively of M^n.

D. g denotes the Riemannian metric of $\overline{M}^n$ as well as the induced metric on M^n .

E. $\{\xi\}$ is the line bundle generated by the structure vector field ξ.

1. Semi-invariant submanifolds of an almost paracontact Riemannian manifold

We begin this section by defining a semi-invariant submanifold of $\overline{M}^m$ as follows :

Definition. A submanifolds M^n of an almost paracontact Riemannian manifold $\overline{M}^m$ is called a semi-invariant submanifold of $\overline{M}^m$ if the following conditions are satisfied :

(A) $TM = D \oplus D^{\perp} \oplus \{\xi\}$,

(B) the distribution D is invariant by F i.e. $F(D_x) = D_x$ for each $x \in M^n$,

(C) the distribution $D^\perp$ is anti-invariant by F i.e. $F(D_x^\perp) \subset T_x M^\perp$ for each $x \in M^n$,

where D, $D^\perp$ and $\{\xi\}$ are orthogonal distributions on M^n such that (A) holds.

It is evident that an invariant (resp. anti-invariant) submanifold is a particular case of semi-invariant submanifold when $\dim(D^\perp) = 0$ (resp. $\dim(D) = 0$). Moreover, a semi-invariant submanifold which is neither invariant nor anti-invariant is called a proper semi-invariant submanifold.

In view of (A), each $X \in TM$ can be represented by

$$X = UX + VX + \eta(X)\xi, \tag{1.1}$$

where $UX \in D$ and $VX \in D^\perp$. Thus U and V are projection morphisms of TM into D and $D^\perp$ respectively.

Remark 1.1. In the case of semi - invariant submanifold, from (1.1) we have

$$D = \{UX;\ \forall\ X \in TE\} \text{ and } D^\perp = \{VX;\ \forall\ X \in TM\},$$

and hence $U(VX) = 0$ and $V(UX) = 0$.

Next, for $N \in TM^\perp$, $X \in TM$ decomposing FN and FX into tangential and normal parts we can write

$$FN = PN + fN, \tag{1.2}$$

and

$$FX = \psi X + \omega X, \tag{1.3}$$

where PN, $\psi X \in TM$ and fN, $\omega X \in TM^\perp$.

Now we prowe a lemma :

Lemma 1.1. Let M^n be a semi-invariant submanifold of an almost paracontact Riemannian manifold $\bar{M}^m$. Then PN defined by (1.2) always belongs to $D^{\perp}$.

Proof. In the case of semi-invariant submanifold, for $X \in D$ we have from (0.3) :

$$g(PN,X) = g(FN,X) = g(N,FX) = 0,$$

and

$$g(PN,\xi) = g(FN,\xi) = g(N,F\xi) = 0,$$

Hence we get $PN \in D^{\perp}$, which proves the lemma.

Theorem 1.1. Let M^n be a hypersurface of an almost paracontact Riemannian manifold $\bar{M}^m$. If ξ and FN are tangential to M^n, then M^n is a semi-invariant submanifold of $\bar{M}^m$.

Proof. Let ξ and FN be tangential to M^n. We define the distribution $D^{\perp}$ on M^n by $D^{\perp} = \{FN\}$, where N is a vector field normal to M^n. Since FN is tangential to M^n we get $D^{\perp} \subset TM$. Also we have

$$g(FN,\xi) = g(N,F\xi) = 0.$$

Hence $D^{\perp}$ is orthogonal to $\{\xi\}$. Let D denote the complementary distribution orthogonal to $D^{\perp} \oplus \{\xi\}$ in TM. Since ξ is tangential to M^n, $\eta(N) = g(N,\xi) = 0$. Hence

$$F(FN) = F^2N = N - \eta(N)\xi = N,$$

which shows that $D^{\perp}$ is anti-invariant.

If $X \in D$, then using (0.3) we get $g(FX,N) = g(X,FN) = 0$. Let $Y \in D^{\perp}$. Since $D^{\perp}$ is anti-invariant,

$$g(FX,Y) = g(X,FY) = 0,$$

and

$$g(FX,\xi) = g(X,F\xi) = 0.$$

Hence D is invariant, which together with $D^{\perp}$ being anti-invariant completes the proof.

Theorem 1.2. Let M^n be a submanifold of an almost paracontact Riemannian manifold $\overline{M}^m$ such that the structure vector field ξ is tangential to M^n. Then a necessary and sufficient condition for M^n to be semi-invariant is

(1.4) $$\text{rank } \psi = \text{constant, on } M^n$$

and

(1.5) $$\omega \circ \psi = 0.$$

Proof Let M^n be a semi-invariant submanifold of $\overline{M}^m$. Then applying F to (1.1), using (0.2),(1.3) and equating tangential and normal parts, we get

(1.6) $$\psi X = FUX,$$

and

(1.7) $$\forall\, X \in TM\ ,\ \omega X = FVX.$$

Here (1.6) states that $\psi(TM) = F(D) = D$ (by definition of semi-invariant submanifold), which yields rank(ψ) = constant on M^n. Next, putting $X = \psi X$ in (1.7) and using (1.6) we get

$$\omega(\psi X) = FV\psi X = FV(FUX),$$

which in view of $FUX \in D$ yields

$$\omega(\psi X) = 0,$$

which is (1.5).

Conversely, suppose (1.4) and (1.5) are satisfied. We can define the distribution D by $D_x = \operatorname{Im} \psi_x$, for each $x \in M^n$. Then for each $X \in D$, $Y \in TM$, putting $X = \psi Y$ in (1.3) and using (1.5) we have

$$FX = \psi^2 Y \in \psi(TM) = D.$$

Hence D is an invariant distribution of M^n. Also since ξ is assumed to be tangential to M^n, for $X = \psi Y \in D$, (1.3), (0.2) and (0.3) yield

$$g(X,\xi) = g(\psi Y,\xi) = g(FY,\xi) = 0,$$

which shows that D is orthogonal to $\{\xi\}$.

Let $D^{\perp}$ denote the complementary distribution orthogonal to $D \oplus \{\xi\}$ in TM. In view of the definition of $D^{\perp}$ and (0.3) for $X \in D^{\perp}$, $Y \in D^{\perp}$, we have

$$g(FX,Y) = g(X,FY) = g(X,FVY) = g(X,\psi VY) = 0,$$

which implies that $D^{\perp}$ is an anti-invariant distribution of TM. Thus M^n is a semi-invariant submanifold of $\overline{M}^m$, which completes the proof.

Theorem 1.3. Let M^n be a submanifold of an almost paracontact Riemannian manifold $\overline{M}^m$ such that the structure vector field ξ is tangential to M^n. Then a necessary and sufficient condition for M^n to be a semi-invariant submanifold is

(1.8) $$\text{rank } P = \text{constant, on } M^n,$$

and

$$\psi \circ P = 0. \tag{1.9}$$

P r o o f. Let M^n be a semi-invariant submanifold of $\overline{M}^m$. Then from Lemma 1.1 we have

$$\operatorname{Im} P_x \subset D_x^\perp, \text{ for each } x \in M^n. \tag{1.10}$$

On the other hand for $X \in D^\perp$ we have $FX \in TM^\perp$ and (1.2) now yields

$$F^2X = PFX + fFX,$$

which in view of (0.1) and $\eta(X) = 0$ reduces to

$$X = PFX, \text{ and } fFX = 0; \ \forall\ X \in D^\perp.$$

The first of the above exquations gives $D_x^\perp \subset \operatorname{Im} p_x$, for each $x \in M^n$, which together with (1.10) proves $\operatorname{Im} P_x = D_x^\perp$, for each $x \in M^n$ and thus (1.8) is proved.

Next, since $PN \in D^\perp$, for $N \in TM^\perp$, from (1.3) we get

$$\psi PN = FPN - \omega PN,$$

which in view of $FPN \in TM^\perp$ yields (1.9).

Conversely, let (1.8) and (1.9) be satisfied. Then we can define a distribution $D^\perp$ by $D_x^\perp = \operatorname{Im} P_x$ for $x \in M^n$. Let $X \in D^\perp$. Hence $X = PN$ for some $N \in TM^\perp$. Then in view of (1.3) and (1.9) we have

$$g(FX, Y) = g(FPN, Y) = g(\psi PN, Y) = 0; \ \forall\ Y \in TM,$$

which implies that $D^\perp$ is an anti-invariant distribution. Also since ξ is assumed to be tangential to M^n, (1.2), (0.2) and (0.3) establish the following

$$g(X,\xi) = g(PN,\xi) = g(FN,\xi) = 0,$$

which shows that $D^\perp$ is orthogonal to $\{\xi\}$.

Let D denote the complementary distribution orthogonal to $D^\perp \oplus \{\xi\}$ in TM. Then with the use of (0.2), (0.3), and (1.2) it can be verified that for $X \in D$, $Y \in D^\perp$ and $N \in TM^\perp$

$$g(FX,Y) = g(X,FY) = 0,$$

$$g(FX,\xi) = g(X,F\xi) = 0,$$

and

$$g(FX,N) = g(X,FN) = g(X,PN) = 0,$$

which shows that D is invariant on M^n. Thus M^n is a semi-invariant submanifold of $\overline{M}^m$. Hence the proof is finished.

Theorem 1.4. Let M^n be a semi-invariant submanifold of an almost paracontact Riemannian manifold $\overline{M}^m$. Then

(1.11) $\psi X = FX, \ \forall X \in D; \ \psi X = 0, \ \forall X \in D^\perp \oplus \{\xi\},$

(1.12) $\omega X = 0, \ \forall X \in D \oplus \{\xi\}; \ \omega X = FX, \ \forall X \in D^\perp,$

(1.13) $\psi^2 X = UX, \ \forall X \in TM,$

(1.14) $PFX = X; \ fFX = 0; \ \forall X \in D^\perp.$

Proof. Applying F to (1.1), using (1.3), (0.2) and equating tangential and normal parts we get

(1.15) (i) $\psi X + FUX$ and (ii) $\omega X = FVX; \ \forall X \in TM,$

which yieds (1.11) and (1.12). Again, operating F to (1.15)(i) and using (1.3), (0.1) and $\eta(UX) = 0$, we obtain

$$\psi^2X + \omega(\psi X) = UX.$$

Equating the tangential parts in the above yields (1.13).

Finally, let $X \in D^\perp$. Then $FX \in TM^\perp$ and $\eta(X) = 0$. Now (1.2) and (0.1) yield

$$F^2X = PFX + fFX \quad \text{i.e} \quad X = PFX, \text{ and } fFX = 0:$$

which is (1.14). Hence the proof is completed.

Theorem 1.5. Let M^n be a semi-invariant submanifold of an almost paracontact Riemannian manifold $\bar{M}^m$. Then f given by (1.2) defines an $f(3,-1)$-structure [6] on the normal bundle.

Proof. Operating (1.2) by F, in view of (0.1) and (1.2) we have

$$N - \eta(N)\xi = FPN + PfN + f^2N,$$

which on using $\eta(n) = 0$, $PN \in D^\perp$ and $FPN \in TM^\perp$ yields

(1.16) $$N = FPN + f^2N.$$

Applying f to (1.16) and using (1.14), we obtain the required results.

2. Integrability of distributions $D \oplus \{\xi\}$ and $D^\perp \oplus \{\xi\}$

In this section we investigate the integrability conditions of the distributions $D \oplus \{\xi\}$ and $D^\perp \oplus \{\xi\}$ with the help of Nijenhuis tensors of F and ψ. Throughout this section M^n is the semi - invariant submanifold of $\bar{M}^m$ defined in Section 1.

The Nijenhuis tensor $[F,F]$ of F is given by

$$(2.1)\qquad [F,F]\,(X,Y) = F^2[X,Y] + [FX,FY] - F\{[X,FY] = [FX,Y]\},$$

for $X,Y \in TM$.

Now, for $X,Y \in D$, using (0.1) and (1.11) in (2.1) we have

$$[F,F]\,(X,Y) = [X,Y] - \eta([X,Y])\xi + [\psi X,\psi Y] - \psi\{[X,\psi Y] + [\psi X,Y]\}$$
$$-\omega\,\{[X,\psi Y] + [\psi X,Y]\},$$

which in view of (1.1) and (1.13) yields

$$(2.2)\qquad F,F]\,(X,Y) = [\psi,\psi]\,(X,Y) + V[X.Y] - \omega\{[X,\psi Y] + [\psi X,Y]\}.$$

Also it can be easily verified by (2.1), (0.2), (0.1) and (1.11) that for $X \in D$

$$(2.3)\qquad [F,F]\,(X,\xi) = [\psi,\psi]\,(X,\xi) + V[X.\xi] - \omega[\psi X,\xi].$$

Let superscripts T and N denote tangential and normal parts, respectively. Then (2.2) and (2.3) provide the proof of the following theorem:

Theorem 2.1. Let M^n be a semi-invariant submanifold of an almost paracontact Riemannian manifold $\overline{M}^m$. Then for $X,Y \in D$

$$(2.4)\qquad [F,F]\,(X,Y)^T = [\psi,\psi]\,(X,Y) + V[X,Y],$$

$$(2.5)\qquad [F,F]\,(X,Y)^N = -\omega\{[X,\psi Y] + [\psi X,Y]\},$$

$$(2.6)\qquad [F,F]\,(X,\xi)^T = [\psi,\psi]\,(X,\xi) + V[X,\xi],$$

and

$$(2.7)\qquad [F,F]\,(X,\xi)^N = -\omega[\psi X,\xi].$$

Theorem 2.2. Let M^n be a semi-invariant submanifold of an almost paracontact Riemannian manifold $\overline{M}^m$. Then the distribution $D \oplus \{\xi\}$ is involutive iff

(2.8) $[F,F]\,(\overline{X},\overline{Y})^T = [\psi,\psi]\,(\overline{X},\overline{Y}), \ \forall\ \overline{X},\ \overline{Y} \in D \oplus \{\xi\}$.

Proof. Let the distribution $D \oplus \{\xi\}$ be involutive. Then for $X,\ Y \in D$ and $\overline{X},\ \overline{Y} \in D \oplus \{\xi\}$ we have $V[X,Y] = 0$, $V[X,\xi] = 0$ and $V[\overline{X},\overline{Y}] = 0$. Hence (2.4) and (2.6) yield (2.8), which proves one part of the theorem.

Conversely, let (2.8) hold. For $\overline{X} = X + a\xi$, $\overline{Y} = Y + b\xi$, $X, Y \in D$ and $a\xi,\ b\xi \in \{\xi\}$; using (2.4) and (2.6) we get

$$[F,F]\,(\overline{X},\overline{Y})^T = [\psi,\psi]\,(\overline{X},\overline{Y}) + V[\overline{X},\overline{Y}],$$

which in view of (2.8) yields $V[\overline{X},\overline{Y}] = 0$. Hence the theorem is proved.

Theorem 2.3. Let M^n be a semi-invariant submanifold of an almost paracontact Riemannian manifold $\overline{M}^m$. Then the distribution $D \oplus \{\xi\}$ is involutive iff

(2.9) $$[F,F]\,(\overline{X},\overline{Y})^N = 0$$

and

$$V\{[\psi,\psi]\,(\overline{X},\overline{Y})\} = 0$$

for $\overline{X},\ \overline{Y} \in D \oplus \{\xi\}$.

Proof. Let $D \oplus \{\xi\}$ be involutive. Then using (2.5), (2.7) and (1.12) we obtain

(2.11) $$[F,F]\,(X,Y)^N = -\omega\{[X,\psi Y] + [\psi X,Y]\} = 0,$$

(2.12) $$[F,F]\,(X,\xi)^N = -\omega[\psi X,\xi] = 0.$$

for $\forall\ X, Y \in D$.

Now, $\overline{X}, \overline{Y} \in D \oplus \{\xi\}$, putting $\overline{X} = x + a\xi$, $\overline{Y} = Y + b\xi$, with $X, Y \in D$ we get

$$[F,F]\,(\overline{X},\overline{Y})^N = [F,F]\,(X,Y)^N + a[F,F]\,(\xi,Y)^N + a[F,F]\,(X,\xi)^N,$$

which by (2.11) and (2.12) yields (2.9).

Next, taking into account (1.13) and the Nijenhuis tensor of ψ, we get for $\overline{X}, \overline{Y} \in D \oplus \{\xi\}$,

$$[\psi,\psi]\,(\overline{X},\overline{Y}) = U[\overline{X},\overline{Y}] + [\psi\overline{X},\psi\overline{Y}] - \psi\{[\psi\overline{X},\overline{Y}] + [\overline{X},\psi\overline{Y}]\}.$$

Hence $V\{[\psi,\psi]\,(\overline{X},\overline{Y})\} = 0$, $\forall\ \overline{X}, \overline{Y} \in D \oplus \{\xi\}$ i.e. (2.10).

Conversely, let (2.9) and (2.10) be satisfied. Then in view of (2.5), (2.7) and (2.9) we obtain

$$\omega\{[\overline{X},\psi\overline{Y}] + [\psi\overline{X},\overline{Y}]\} = 0;\ \forall\ \overline{X},\overline{Y} \in D \oplus \{\xi\},$$

which on using (1.7) yields

$$FV\{[\overline{X},\psi\overline{Y}] + [\psi\overline{X},\overline{Y}]\} = 0.$$

Consequently, in view of F being non-singular on $D \oplus D^{\perp}$, we have,

$$V\{[\overline{X},\psi\overline{Y}] + [\psi\overline{X},\overline{Y}]\} = 0. \tag{2.13}$$

Now, putting $\overline{Y} = \xi$ and $\psi\overline{X} = Z$, for $Z \in D$ in the above equation and using (1.11), we get

$$V[Z,\xi] = 0. \tag{2.14}$$

Also putting $\overline{X} = \psi Z$ and $\overline{Y} = Y$ where $Y, Z \in D$ in (2.13) we get

$$V\{[\psi Z,\psi Y] + [Z,Y]\} = 0,\ (\text{using } \psi^2 Z = Z, \text{ for } Z \in D),$$

which in view of (1.11) becomes

$$V\{[FZ,FY] + [Z,Y]\} = 0,\ \forall\ Z,Y \in D. \tag{2.15}$$

Finally, for $X, Y \in D$ it can be verified by (2.1), (0.1) and (1.3) that

$$[F,F](X,Y) = [X,Y] - \eta([X,Y])\xi + [FX,FY] - \psi\{[FX,Y] + [X,FY]\}$$
$$- \omega\{[FX,Y] + [X,FY]\}$$

i.e

$$V\{[F,F](X,Y)^T\} = V\{[FX,FY] + [X,Y]\},$$

which in view of (2.15) reduces to

$$V\{[F,F](X,Y)^T\} = 0, \quad \forall X, Y \in D. \tag{2.16}$$

Therefore (2.4), (2.10) and (2.16) yield

$$V[X,Y] = 0, \quad \forall X, Y \in D.$$

The above equation and (2.14) provide

$$V[\bar{X},\bar{Y}] = 0, \quad \forall \bar{X}, \bar{Y} \in D \oplus \{\xi\},$$

which proves that $D \oplus \{\xi\}$ is involutive. Hence the proof is completed.

Theorem 2.4. Let M^n be a semi-invariant submanifold of an almost paracontact Riemannian manifold $\bar{M}^m$. Then the distribution $D^\perp \oplus \{\xi\}$ is involutive iff the Nijenhuis tensor of vanishes identically on $D^\perp \oplus \{\xi\}$.

Proof. For $X, Y \in D^\perp \oplus \{\xi\}$, using definition of Nijenhuis tensor of ψ, (1.11) and (1.13) we obtain

$$[\psi,\psi](X,Y) = [\psi X,\psi Y] + \psi^2[X,Y] - \psi([\psi X,Y] + [X,\psi Y])$$

i.e

$$[\psi,\psi](X,Y) = U[X,Y],$$

which provides the proof of the theorem.

REFERENCES

[1] A. Bejancu : On semi-invariant submanifold of an almost contact metric manifold, An. Sti. Univ. "Al I Cuza", Iasi 27 (1981) 17-21.

[2] D.E. Blair : Contact manifolds in Riemannian geometry, 509 (1976) Springer-Verlag.

[3] G.D. Luden, M. Okumara, K. Yano : Anti-invariant submanifolds of almost contact metric manifold, Math. Ann. 225 (1977) 253-261.

[4] I. Sato : On a structure similar to the almost contact metric structure, Tensor (N.S.) 30 (1976) 219-224.

[5] I. Sato : On a structure similar to the almost contact metric structure II, Tensor (N.S.) 31 (1977) 199-205.

[6] K.D. Singh , R.K. Vohra : A Riemannian f(3,-1) - structure manifold, PhD. Thesis, Lucknow University (INDIA) 1974.

[7] K. Yano : Differential geometry of complex and almost complex spaces, Pergamon Press 1965.

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POWER MAPS ON SPECIAL ORTHOGONAL GROUPS

1. First the power maps $P_k: G \longrightarrow G$, $P_k(X) = X^k$, for a compact and connected Lie group G were studied by H. Hopf in [3] (see also [5] and [2, vol II]).

The aim of this paper is to precise and to complete the initial proof of H. Hopf [3] as well as other results when the Lie group G is SO(n) by giving explicit computations in terms of matrices. Our approach points out the closed relations between the critical ponts of the power maps and the singularities of the Lie group SO(n). For instance, Proposition 4 shows that the regular values of P_k coincide with the regular points of SO(n) hence the first ones do not depend on k. Some examples and applications are also given.

2. The special orthogonal group SO(n) is an $\frac{n(n-1)}{2}$ - dimensional compact and connected Lie group. Its Lie algebra $\mathscr{A}_n$ is the Lie algebra of the real antisymmetric matrices of order n. The tangent space at $A \in SO(n)$ is

(1)
$$T_A(SO(n)) = \left\{ AM/M \in \mathscr{A}_n \right\}.$$

Let us consider a power map $P_k: SO(n) \longrightarrow SO(n)$, $P_k(X) = X^k$, $k \in Z - \{0\}$, $n \geq 2$.

Lemma 1. The differential $d_AP_k : T_A(SO(n)) \longrightarrow T_{A^k}(SO(n))$ of P_k at $A \in SO(n)$ is given by

$$(d_AP_k)(AM) = \sum_{i=0}^{k-1} A^{i+1}MA^{k-i-1}, \text{ if } k \geq 1, \tag{2}$$

$$(d_AP_k)(AM) = -\sum_{i=0}^{-k-1} A^{-i}MA^{k+i}, \text{ if } k < 0. \tag{3}$$

Let $\left\{ E^i_j = (\delta^i_k\delta^h_j)_{h,k} \mid i,j = 1,\dots,n \right\}$ be the canonical basis in the Lie algebra $\mathcal{M}_n(R)$ of the real matrices of order n. In the Lie algebra $\mathcal{A}_n$ we shall use the following basis $\left\{ E_h \mid h = 1,\dots,\frac{n(n-1)}{2} \right\}$, with $E_1 = E^1_2 - E^2_1$, $E_2 = E^1_3 - E^3_1, \dots, E_{n-1} = E^1_n - E^n_1$, $E_n = E^2_3 - E^3_2, \dots, E_{n(n-1)/2} = E^{n-1}_n - E^n_{n-1}$.

Throughout the paper, the basis $\left\{ AE_h \mid h = 1,\dots,\frac{n(n-1)}{2} \right\}$ of $T_A(SO(n))$ and the basis $\left\{ A^kE_h \mid h = 1,\dots,\frac{n(n-1)}{2} \right\}$ if $k > 0$ (resp. $\left\{ E_hA^k \mid h = 1,\dots,,\frac{n(n-1)}{2} \right\}$, if $k > 0$) will be referred as the canonical bases.

Lemma 2. If $A \in SO(n)$ is a critical point (respectively a critical value) for P_k and if $A' = S^{-1}AS$ with $S \in SO(n)$, then A' has the same property.

Proof. Let $k > 0$. In addition to the canonical basis in $\mathcal{A}_n$ we can consider the basis $\left\{ SE_hS^{-1} \mid h = 1,\dots,\frac{n(n-1)}{2} \right\}$. Corresponding to this basis we have in the tangent spaces $T_A(SO(n))$ and $T_{A^k}(SO(n))$ the bases

$\left\{ A(SE_hS^{-1}) \mid h = 1,\dots,\frac{n(n-1)}{2} \right\}$ and $\left\{ A^k(SE_hS^{-1}) \mid h = 1,\dots,\right.$

, $\left.\frac{n(n-1)}{2}\right\}$ respectively. Let $(\beta_h^1) \in \mathcal{M}_{n(n-1)/2}(\mathbb{R})$ be the matrix of the differential d_AP_k with respect to these bases. Then we have

$$(d_{A'}P_k)(A'E_h) = \sum_{i=0}^{k-1} A'^{\,i+1}E_hA'^{\,k-i-1} =$$

$$= S^{-1}\left[\sum_{i=0}^{k-1} A^{i+1}(SE_hS^{-1})A^{k-i-1}\right]S = S^{-1}\left[\beta_h^1A^k(SE_1S^{-1}) + \dots +\right.$$

$$\left. + \beta_h^{n(n-1)/2}A^k(SE_{n(n-1)/2}S^{-1})\right]S = \beta_h^1(A'^{\,k}E_1) + \dots +$$

$$+ \beta_h^{n(n-1)/2}(A'^{\,k}E_{n(n-1)/2}).$$

Therefore the matrix (α_h^1) of the linear map $d_{A'}P_k$ with respect to the canonical bases coincides with (β_h^1) so that $\mathrm{rank}_AP_k = \mathrm{rank}_{A'}P_k$.

If A is a critical value for P_k, then there exists a critical point X such that $X^k = A$. This implies that $A' = (SXS^{-1})^{-k}$, with $X' = SXS^{-1}$ a critical point, so that A' is a critical value for P_k.

The case $k < 0$ is similar.

Lemma 3. In the conditions of Lemma 2, if (α_h^1) and (γ_h^1) are respectively the matrices of the linear maps $d_{A'}P_k$ and d_AP_k, with respect to the canonical bases, then $\det(\alpha_h^1) = \det(\gamma_h^1)$.

Proof. As in the proof of Lemma 2, we have $(\alpha_h^1) = (\beta_h^1)$. Let T be the matrix of change of the bases $\{E_h \mid h = 1, \dots,$

$, \frac{n(n-1)}{2}\Big\}$ and $\Big\{ SE_hS^{-1} \mid h = 1,\dots, \frac{n(n-1)}{2} \Big\}$ in the space $\mathcal{A}_n$. Then, T is simultaneously the matrix of change from the basis $\Big\{ AE_h \mid h = 1,\dots,\frac{n(n-1)}{2} \Big\}$ to the basis $\Big\{ A(SE_hS^{-1}) \mid h = 1,\dots, \frac{n(n-1)}{2} \Big\}$ in $T_A(SO(n))$ and from the basis $\Big\{ A^kE_h \mid h = 1,\dots, \frac{n(n-1)}{2} \Big\}$ to the basis $\Big\{ A^k(SE_hS^{-1}) \mid h = 1,\dots, \frac{n(n-1)}{2} \Big\}$ in $T_{A^k}(SO(n))$. Hence $(\gamma_h^1) = T^{-1}(\beta_h^1)T$ and, so $\det(\gamma_h^1) = \det(\beta_h^1) = \det(\alpha_h^1)$.

Notation. For a matrix $A \in \mathcal{M}_n(\mathbb{R})$ denote by $m_2(A)$ the matrix of all minors of orders 2 of A. The precise description of $m_2(A) \in \mathcal{M}_{n(n-1)/2}(\mathbb{R})$ is the following : its columns are the minors of order 2 of the pairs of rows $(1,2)$, $(1,3),\dots,(1,n),(2,3),\dots,(2,n),\dots,(n-1,n)$ and its rows are the minors of the order 2 of columns $(1,2),(1,3),\dots,(1,n)$, $(2,3),\dots,(2,n),\dots,(n-1,n)$.

Proposition 1. With respect to the canonical bases the matrix of the differential $d_AP_k : T_A(SO(n)) \longrightarrow T_{A^k}(SO(n))$ is given by the following formula

$$d_AP_k = \operatorname{sgn} k \Big[I + m_2(A) + m_2(A)^2 + \dots + m_2(A)^{|k|-1} \Big], \tag{4}$$

where I is the unit matrix of order $n(n-1)/2$.

Proof. First, we consider the case $k > 0$. By Lemma 1 we have

$$(d_AP_k)(AE_h) = \sum_{i=0}^{k-1} A^{i+1}E_hA^{k-i-1} = A^k \Big(\sum_{i=0}^{k-1} M_h^i \Big),$$

where

$M_h^0 = I$, $M_h^1 = {}^t(A^{k-1})E_hA^{k-1}$, $M_h^2 = {}^t(A^{k-2})E_hA^{k-2}, \ldots, M_h^{k-1} = {}^tAE_hA$.

If we denote M_h^{k-1} by $M_h(A)$, then we have $M_h^{k-2} = M_h(A^2)$, $M_h^{k-3} = M_h(A^3), \ldots, M_h^1 = M_h(A^{k-1})$ so that it is enough to calculate only the matrices $M_h(A)$, $h = 1,2,\ldots,n(n-1)/2$. Now by a simple computation we obtain $d_AP_k = I + m_2(A) + \ldots + m_2(A^{k-1})$. Then we observe that $m_2(A)$ is the matrix of the derivative at $I \in \mathscr{A}_n$ of the inner automorphism $\tau_A(X) = A^{-1}XA$, with respect to the base $\left\{ E_h \mid h = 1,\ldots,n(n-1)/2 \right\}$, that is $m_2(A) = \mathrm{Ad}\,A$. Indeed, $(\mathrm{Ad}\,A)(E_h) = A^{-1}E_hA = M_h^{k-1}$. Then $m_2(A^i) = \mathrm{Ad}\,A^i = (\mathrm{Ad}\,A)^i = m_2(A)^i$. In this way we have established the formula (4) for the case $k > 0$.

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If $k < 0$, by (3) we have

$$(d_AP_k)(AE_h) = -\sum_{i=0}^{-k-1} A^{-i}E_hA^{k+i} = -\left(\sum_{i=0}^{-k-1} N_h^i\right)A^k,$$

with

$$N_h^0 = I,\ N_h^1 = {}^tAE_hA = M_h(A),\ N_h^2 = ({}^tA)^2E_hA^2 =$$

$$= M_h(A^2), \ldots, N_h^{-k-1} = ({}^tA)^{-k-1}E_hA^{-k-1} = M_h(A^{-k-1}),$$

by which we conclude the formula $d_AP_k = -\left[I + m_2(A) + \ldots + m_2(A)^{-k-1} \right]$. This finishes the proof of the proposition.

Lemma 4. If $A \in SO(n)$, then $m_2(A) \in SO(n(n-1)/2)$ and $m_2(A)$ has +1 as eigenvalue with the multiplicity at least equal to $\left[\frac{n}{2}\right]$.

Proof. Obviously $m_2({}^tA) = {}^tm_2(A)$ and because $m_2(A) = \mathrm{Ad}\,A$ we also obtain that $m_2(AB) = m_2(B)m_2(A)$. From these and

$m_2(I) = I$ we deduce that $m_2(A^{-1}) = m_2(A)^{-1}$. Thus, if ${}^tA = A^{-1}$ then $m_2({}^tA) = m_2(A^{-1})$ implies ${}^tm_2(A)=m_2(A)^{-1}$, hence $m_2(A) \in O(n(n-1)/2)$. By the definition of the matrix $m_2(A)$ and since the relation $A' = S^{-1}AS$ implies $m_2(A') = m_2(S)m_2(A)m_2(S)^{-1}$, we deduce that if $\lambda_1,\dots,\lambda_n$ are the eigenvalues of the matrix A, then the eigenvalues of $m_2(A)$ are $\left\{ \lambda_i\lambda_j \mid 1 \le i < j \le n \right\}$. Thus we obtain $\det(m_2(A)) = (\lambda_1,\dots,\lambda_n)^{n-1} = (\det A)^{n-1} = +1$ and therefore $m_2(A) \in SO(n(n-1)/2)$.

Now suppose that $\lambda_{1,2} = \cos 2\Pi\theta_1 \pm i \sin 2\Pi\theta_1$, $\lambda_{3,4} = \cos 2\Pi\theta_2 \pm i \sin 2\Pi\theta_2$, $\dots$, $\lambda_{2m-1,2m} = \cos 2\Pi\theta_m \pm i \sin 2\Pi\theta_m$, $\lambda_{2m+1} = \dots = \lambda_n = +1$, for $2m \le n$. Then, $m_2(A)$ has as eigenvalues the following products : $\lambda_1\lambda_2 = \lambda_3\lambda_4 = \dots = \lambda_{2m-1}\lambda_{2m} = +1$ and $\lambda_i\lambda_j = +1$, $2m+1 \le i < j \le n$. The number of these products is equal to $m + \binom{n-2m}{2}$.

If $n = 2p$, then $m + \binom{n-2m}{2} - \left[\frac{n}{2}\right] = 2(p-m)(p-m-1) \ge 0$. If $n = 2p+1$, then $m + \binom{n-2m}{2} - \left[\frac{n}{2}\right] = 2(p-m)^2 \ge 0$. Thus, we have $m + \binom{n-2m}{2} \ge \left[\frac{n}{2}\right]$.

Proposition 2. A matrix $A \in SO(n)$, $n \ge 2$ is a critical point for P_k, $k \in \mathbb{Z}-\{0\}$ if and only if $\operatorname{rank}(m_2(A^k)-I) < \operatorname{rank}(m_2(A)-I)$.

Proof. First we prove that A is a critical point for P_k if and only if there exists a vector $Y \in \mathbb{R}^{n(n-1)/2}$ such that $m_2(A^k)Y = Y$ and $m_2(A)Y \ne Y$. By Proposition 1 we have

$$(d_AP_k)(I-m_2(A)) = \operatorname{sgn} k(I-m_2(A)^k) \text{ and} \tag{5}$$

$$(I-m_2(A))(d_AP_k) = \operatorname{sgn} k(I-m_2(A)^k). \tag{6}$$

If we suppose that $m_2(A^k)Y = Y$ $(Y \neq 0)$, that is $m_2(A)^k Y = Y$ and $(m_2(A)-I)Y = Y' \neq 0$, then by (5) we have $(d_A P_k)Y' = 0$ and $Y' \neq 0$. It follows that $\det(d_A P_k) = 0$, which proves that A is a critical point for P_k.

Conversely, if $\det(d_A P_k) = 0$, then there exists $Y \in \mathbb{R}^{n(n-1)/2}$, $Y \neq 0$, with $(d_A P_k)Y = 0$. By (6) we obtain $m_2(A^k)Y = Y$. If we suppose $m_2(A)Y = Y$, then by (4) we obtain $(d_A P_k)Y = \operatorname{sgn} k\underbrace{(Y+\dots+Y)}_{|k|\text{-times}} = kY$ and therefore $kY = 0$, in contrast to the relations $Y \neq 0$, $k \neq 0$. Thus $m_2(A)Y \neq Y$.

Now, by Lemma 4, +1 is an eigenvalue for the matrices $m_2(A)$ and $m_2(A^k) = m_2(A)^k$. Let $V(1) = \operatorname{Ker}(m_2(A)-I)$ and $V(k;1) = \operatorname{Ker}(m_2(A^k)-I)$. Then we have $V(1) \subseteq V(k;1)$. Suppose that A is critical point for P_k. Then there exists $Y \in \mathbb{R}^{n(n-1)/2}$ such that $m_2(A^k)Y = Y$ and $m_2(A)Y \neq Y$, that is $Y \in V(k;1)$ and $m_2(A)Y \neq Y$, that is $Y \in V(k;1)$ and $Y \overline{\in} V(1)$. Thus, $\dim V(1) < \dim V(k;1)$. But $\dim V(1) = n(n-1)/2 - \operatorname{rank}(m_2(A)-I)$ and $\dim V(k;1) = n(n-1)/2 - \operatorname{rank}(m_2(A^k)-I)$ so that we obtain $\operatorname{rank}(m_2(A^k)-I) < \operatorname{rank}(m_2(A)-I)$.

Conversely, the last inequality implies the strict inclusion $V(1) \subset V(k;1)$, so that there exists $Y \in \mathbb{R}^{n(n-1)/2}$ with $m_2(A^k)\,Y = Y$ and $m_2(A)Y \neq Y$, which proves that A is a critical point for P_k.

Corollary 1. A matrix $A \in SO(n)$, $n \geq 2$ is a regular point for P_k, $k \in \mathbb{Z}-\{0\}$, if and only if the eigenvalues $\lambda_1,\dots,\lambda_n$ of A are such that in the systems $\left\{ \lambda_i\lambda_j \mid 1 \leq i < j \leq n \right\}$ and $\left\{ \lambda_i^k\lambda_j^k \mid 1 \leq i < j \leq n \right\}$ the number +1 appears with the same multiplicity.

Proof. By Proposition 2 it is clear that A is a regular point for P_k if and only if rank $(m_2(A^k)-I) = \text{rank } (m_2(A)-I)$. But since $m_2(A)$ and $m_2(A^k)$ have as eigenvalues the products $\left\{ \lambda_i\lambda_j \mid 1 \le i < j \le n \right\}$ and $\left\{ \lambda_i^k\lambda_j^k \mid 1 \le i < j \le n \right\}$, respectively, if we suppose that +1 appears with the multiplicity m_1 in the first system and m_k in the second system, then we have rank $(m_2(A)-I) = n(n-1)/2-m_1$ and rank $(m_2(A^k)-I) = n(n-1)/2-m_k$. Thus A is regular point for P_k if and only if $m_1 = m_k$.

Corollary 2. For $k > 1$ and $n \ge 3$ the unit matrix $I \in SO(n)$ is a regular point but a critical value for P_k.

Proof. The first part is obvious. For the second part, let $A = \text{diag}(1,\dots,1,D(1/k)) \in SO(n)$, where $D(\theta)$ denotes the rotation

$$\begin{bmatrix} \cos 2\Pi\theta & -\sin 2\Pi\theta \\ \sin 2\Pi\theta & \cos 2\pi\theta \end{bmatrix}$$

$\theta \in [0,1]$. We have $A^k = I$ and by Proposition 2, A is a critical point because rank $(m_2(A^k)-I) = 0 <$ rank $(m_2(A)-I)$ since $m_2(A) \ne I$.

Corollary 3. For $n = 3$ the critical points of the map P_k, $k \ne 0,1$ are all matrices $S^{-1}A_hS$ with $S \in SO(n)$ and $A_h = \text{diag}(1,D(h/k))$, $h = 1,\dots,k-1$, while I is the single critical value of P_k.

Proof. By Lemma 2 and because for any matrix $A' \in SO(3)$ there exists $S \in SO(3)$ such that $A = S^{-1}A'S$ has the form $\text{diag}(1,D(\theta))$, it is sufficient to see when this matrix is a critical point. We have

$$m_2(A)-I = \begin{bmatrix} \cos 2\Pi\theta - 1 & -\sin 2\Pi\theta & 0 \\ \sin 2\Pi\theta & \cos 2\Pi\theta - 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

and

$$m_2(A^k)-I = \begin{bmatrix} \cos 2\Pi k\theta - 1 & -\sin 2\Pi k\theta & 0 \\ \sin 2\Pi k\theta & \cos 2\Pi k\theta - 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Imposing the condition $(m_2(A^k)-I) <$ rank $(m_2(A)-I)$ one obtains as single possibility the case $\theta = \frac{h}{k}$ and k not dividing h. It is sufficient to consider $h \in \{1,2,\dots,k-1\}$. Then $P_k(S^{-1}A_hS) = S^{-1}A_h^kS = S^{-1}S = I$, which proves that I is the single critical value for P_k.

Proposition 3. For $k \in \mathbb{Z}-\{0\}$ and $A \in SO(n)$, $n \geq 2$, the Jacobian $\det(d_AP_k)$, with respect to the canonical bases, is a polynomial with real coefficients whose unknowns are the coefficients of the characteristic polynomial of A. In addition, the following inequality holds

$$k^{\left[\frac{n}{2}\right]} \det(d_AP_k) \geq 0. \tag{7}$$

Proof. If $\lambda_1,\dots,\lambda_n \in \mathbb{C}$ are the eigenvalues of the matrix A, then by Proposition 1 and Lemma 3 we have

(8) $\det(d_AP_k) = (\operatorname{sgn} k)^{n(n-1)/2} \prod_{1\leq i<j\leq n} \left[1 + \lambda_i\lambda_j + \lambda_i^2\lambda_j^2 + \right.$

$$+ \dots + \lambda_i^{|k|-1}\lambda_j^{|k|-1} \Big].$$

Thus, $\det(d_A P_k)$ is a symmetric polynomial in $\lambda_1,\dots,\lambda_n$ and by the fundamental theorem of symmetric polynomials it follows that $\det(d_A P_k)$ is a polynomial in the elementary polynomials : $\lambda_1+\dots+\lambda_n = \mathrm{Tr}\, A$, $\lambda_1\lambda_2+\dots+\lambda_{n-1}\lambda_n = \delta(A),\dots,\lambda_1,\dots,\lambda_n = \delta(A) =$ $= \det A = 1$.

Now, let $f = 1+\lambda_a\lambda_b+\lambda_a^2\lambda_b^2+\dots+\lambda_a^{|k|-1}\lambda_b^{|k|-1}$ be a factor of the product (8). If $\lambda_a = \lambda_b = 1$, then $f = |k|$. If $\lambda_a \neq 1$ then the product (8) contains also the factor $\bar{f} = 1 + \bar{\lambda}_a\bar{\lambda}_b+\bar{\lambda}_a^2\bar{\lambda}_b^2+\dots+\bar{\lambda}_a^{|k|-1}\ \bar{\lambda}_b^{|k|-1}$, such that the contribution of f in the product (8) is $|f|^2$. Finally we observe that $n(n-1)/2$ and $\left[\frac{n}{2}\right]$ have the same parity.

Corollary 4. Let A be a critical point (critical value) for P_k, $k \neq 0$. Then :

a) A^{-1} has the same property;

b) The matrices A and $A^{-1} = {}^tA$ are critical points (resp. critical values) for the power map P_{-k}.

Proof. a) The matrices $A^{-1} = {}^tA$ and A have the same characteristic polynomial.

b) $\det(d_A P_k) = (-1)^{\left[\frac{n}{2}\right]}\det(d_A P_{-k}) = (-1)^{\left[\frac{n}{2}\right]}\det(d_{A^{-1}} P_{-k})$.

Examples. 1) Let $n = 3$, $k \neq 0$, $A = \mathrm{diag}(1, D(\theta))$. Then

$$\det(d_A P_k) = k\,\frac{1-\cos 2\Pi k\theta}{1-\cos 2\Pi\theta}\,. \tag{9}$$

Since $\delta_2(A) = \operatorname{Tr} A$, it follows that $\det(d_A P_k)$ is a polynomial with real coefficients in Tr A as unknown. In particular :

$|k| = 2 \Rightarrow \det(d_A P_k) = k(1+\operatorname{Tr} A)$;

$|k| = 3 \Rightarrow \det(d_A P_k) = k(\operatorname{Tr} A)^2$;

$|k| = 4 \Rightarrow \det(d_A P_k) = k(\operatorname{Tr} A-1)^2(\operatorname{Tr} A+1)$.

2) $n = 4$, $|k| = 2 \Rightarrow \det(d_A P_k) = 2k(\operatorname{Tr} A)^2$.

3) $n = 5$, $|k| = 2 \Rightarrow \det(d_A P_k) = 2k(\operatorname{Tr} A-1)^2(1+\operatorname{Tr} A+\delta_2(A))$.

Corollary 5. For specified n and k, the following statements are necessary and sufficient conditions that a matrix $A \in SO(n)$ to be a critical point for P_k :

1) $\operatorname{Tr} A = -1$, if $n = 3$, $|k| = 2$;

2) $\operatorname{Tr} A = 0$, if $n = 3$, $|k| = 3$;

3) $\operatorname{Tr} A = \pm 1$, if $n = 3$, $|k| = 4$;

4) $\operatorname{Tr} A = 0$, if $n = 4$, $|k| = 2$;

5) $\operatorname{Tr} A = 1$ or $\operatorname{Tr} A+\delta_2(A) = -1$, if $n = 5$, $|k| = 2$.

Remark 1. For $n \geq 4$ there exist critical values for P_k which are different from the unit matrix.

Examples. 4) Let $n = 4$, $k = 2$, $A = \operatorname{diag}(D(-\frac{1}{8}), D(\frac{1}{8}))$. Then, for $X = \operatorname{diag}(D(\frac{1}{16}), D(\frac{7}{16}))$ we have $X^2 = A$ and $\operatorname{Tr} X = 0$. So X is critical point and therefore A is critical value for P_k.

5) For $n = 5$, $k = 2$, let $A = \operatorname{diag}(1, D(0), D(-\frac{1}{4}))$ and $X = \operatorname{diag}(-1,-1,1,D(-\frac{1}{8}))$. Then $X^2 = A$ and X is a critical point for P_2 because $\operatorname{Tr} X+\delta_2(X) = -1$.

Proposition 4. Let $n \geq 2$ and $|k| > 1$. Then the regular values of the power map P_k coincide with the regular points of the Lie group SO(n) and these are all matrices A ∈ ∈ SO(n) whose eigenvalues $\lambda_1,\dots,\lambda_n$ have the property that precisely $\left[\frac{n}{2}\right]$ products $\lambda_i\lambda_j$, $1 \le i < j \le n$ are equal to +1.

Proof. The above condition concerning the eigenvalues of A can also be formulated as follows : A is a regular value of P_k iff +1 is an eigenvalue of the matrix $m_2(A)$ with the multiplicity equal to $\left[\frac{n}{2}\right]$. This is true because the matrix $m_2(A)$ has as eigenvalues the products $\lambda_i\lambda_j$, $1 \le i < j \le n$. Now we can prove the proposition.

Necessity. Let A be a regular value for P_k, $|k| > 1$. Suppose that the multiplicity of the eigenvalue +1 for $m_2(A)$ is greater than $\left[\frac{n}{2}\right]$ (see Lemma 4). Then, with the same notations as in Lemma 4, we have $2m \le n-3$, such that +1 is an eigenvalue for A with the multiplicity least equal to 3. By Lemma 2 we can suppose that A is of the form $A = \mathrm{diag}(D(\theta_1),\dots,D(\theta_m),1,\dots,1,1,1)$. Because $A \neq I$, we can suppose that $\cos 2\Pi\theta_1 + i \sin 2\Pi\theta_1 \neq 1$. The matrix $X = \mathrm{diag}(D(\theta_1/k),\dots,D(\theta_m/k),1,\dots,1,D(1/k))$ is among the solutions of the equation $X^k = A$. But X is a critical point for P_k because the Jacobian $\det(d_A P_k)$ contains the following factor (see (8)) :

$$|1 + (\cos 2\Pi/k + i \sin 2\Pi/k) + \dots + (\cos 2\Pi/k + i \sin 2\Pi/k)^{|k|-1}|^2 = \left| \frac{1-(\cos 2\Pi/k + i \sin 2\Pi/k)^{|k|}}{1-(\cos 2\Pi/k + i \sin 2\Pi/k)} \right|^2 = 0.$$

Thus, A is a critical value for P_k, contradicting the hypothesis.

Sufficiency. Suppose that +1 has the order of multiplicity as eigenvalue of $m_2(A)$ equal to $\left[\frac{n}{2}\right]$.

The case $n = 2m$, $k > 0$. We can suppose that $A =$ $= \mathrm{diag}(D(\theta_1),\dots,D(\theta_m))$. Let $\lambda_{1,2} = \cos 2\Pi\theta_1 \pm i \sin 2\Pi\theta_1$, $\lambda_{3,4} =$ $= \cos 2\Pi\theta_2 \pm i \sin 2\Pi\theta_2$, $\dots, \lambda_{2m-1,2m} = \cos 2\Pi\theta_m \pm i \sin 2\Pi\theta_m$. The matrix $m_2(A)$ has as eigenvalues the products $\lambda_1\lambda_2 = \lambda_3\lambda_4 = \dots =$ $= \lambda_{2m-1}\lambda_{2m} = +1$. By hypothesis, it is necessary to have $\cos 2\Pi(\theta_h+\theta_l) \pm i \sin 2\Pi(\theta_h+\theta_l) \neq 1$ and $\cos 2\Pi(\theta_h-\theta_l) \pm i \cdot \sin 2\Pi(\theta_h-\theta_l) \neq 1$, for $h < l$. These conditions imply $\theta_h \pm \theta_l \overline{\in} \mathbb{Z}$.

Now we consider the following matrices :

$$X_{q_1\dots q_m} = \mathrm{diag}\left(D(\frac{\theta_1+q_1}{k}),\dots,D(\frac{\theta_m+q_m}{k}) \right],$$

$$q_1,\dots,q_m \in \{0,1,\dots,k-1\}.$$

Then we have $X^k_{q_1\dots q_m} = A$. Moreover $X^k_{q_1\dots q_m}$ are regular points for P_k. Indeed, by (8) :

$$\det(d_{X^k_{q_1\dots q_m}} P_k) = k^m \prod_{a+1<b} \left| 1 + \lambda_a\lambda_b + \lambda_a^2\lambda_b^2 + \right.$$

$$\left. + \dots + \lambda_a^{k-1}\lambda_b^{k-1} \right|^2,$$

where λ_a, λ_b are eigenvalues of the matrix $X^k_{q_1\dots q_m}$. If

$$\lambda_a = \cos \frac{2\Pi(\theta_h+q_h)}{k} \pm i \sin \frac{2\Pi(\theta_h+q_h)}{k} \text{ and}$$

$$\lambda_b = \cos \frac{2\Pi(\theta_l+q_l)}{k} \pm i \sin \frac{2\Pi(\theta_l+q_l)}{k} \text{ ,then}$$

$$\left| 1 + \lambda_a\lambda_b + \lambda_a^2\lambda_b^2 + \dots + \lambda_a^{k-1}\lambda_b^{k-1} \right|^2 =$$

$$\left| \frac{1 - \cos 2\Pi(\theta_h \pm \theta_1) \pm i \sin 2\Pi(\theta_h - \theta_1)}{1 - \lambda_a\lambda_b} \right|^2 \neq 0,$$

because $\theta_h \pm \theta_1 \overline{\in} \mathbb{Z}$.

We can now prove that

$$(10) \qquad P_k^{-1} = \left\{ X^k_{q_1\dots q_m} \mid q_1\dots q_m \in \{0,1,\dots,k-1\} \right\}.$$

Let X be an arbitrary solution of the equation $X^k = A$. We can write $X = S^{-1}X'S$, with $S = (s^i_j) \in SO(n)$ and $X' = \mathrm{diag}(D(\theta_1), \dots, D(\theta_m))$. Then, $X^k = A$ implies $S^{-1}X'^kS = A$ and therefore X'^k and A have the same eigenvalues. It follows that $X' \in \left\{ X^k_{q_1\dots q_m} \mid q_1\dots q_m \in \{0,1,\dots,k-1\} \right\}$ hence $X'^k = A$. Thus we obtain the condition SA = As. Let us denote $SA = (\alpha^i_j)$ and $AS = (\beta^i_j)$. By imposing the conditions $\alpha^1_3 = \beta^1_3$, $\alpha^1_4 = \beta^1_4$, $\alpha^2_3 = \beta^2_3$, $\alpha^2_4 = \beta^2_4$, we obtain :

$$\begin{cases} s^1_3(\cos 2\Pi\theta_2 - \cos 2\Pi\theta_1) - s^2_3\cos 2\Pi\theta_1 - s^1_4 \sin 2\Pi\theta_2 = 0 \\ s^1_3\sin 2\Pi\theta_2 + s^1_4(\cos 2\Pi\theta_2 - \cos 2\Pi\theta_1) - s^2_4 \sin 2\Pi\theta_1 = 0 \\ s^1_3\sin 2\Pi\theta_1 + s^2_3(\cos 2\Pi\theta_2 - \cos 2\Pi\theta_1) - s^2_4 \sin 2\Pi\theta_2 = 0 \\ s^2_3\sin 2\Pi\theta_2 + s^1_4 \sin 2\Pi\theta_1 + s^2_4(\cos 2\Pi\theta_2 - \cos 2\Pi\theta_1) = 0 \end{cases}$$

with

$$\begin{vmatrix} \cos 2\Pi\theta_2 - \cos 2\Pi\theta_1 & -\sin 2\Pi\theta_1 & -\sin 2\Pi\theta_2 & 0 \\ \sin 2\Pi\theta_2 & 0 & \cos 2\Pi\theta_2 - \cos 2\Pi\theta_1 & -\sin 2\Pi\theta_1 \\ \sin 2\Pi\theta_1 & \cos 2\Pi\theta_2 - \cos 2\Pi\theta_1 & 0 & -\sin 2\Pi\theta_2 \\ 0 & \sin 2\Pi\theta_2 & \sin 2\Pi\theta_1 & \cos 2\Pi\theta_2 - \cos 2\Pi\theta_1 \end{vmatrix} =$$

$= -4\,(\cos 2\Pi\theta_2 - \cos 2\Pi\theta_1)^2 = -8\,\sin^2\Pi(\theta_1+\theta_2)\sin^2\Pi(\theta_1-\theta_2) \neq 0$ because $\theta_1 \pm \theta_2 \overline{\in} \mathbb{Z}$. Thus we obtain $s_3^1 = s_4^1 = s_3^2 = s_4^2 = 0$.

Analogously, imposing the conditions $\alpha_1^3 = \beta_1^3$, $\alpha_1^4 = \beta_1^4$, $\alpha_2^3 = \beta_2^3$, $\alpha_2^4 = \beta_2^4$, we obtain $s_1^3 = s_2^3 = s_1^4 = s_2^4 = 0$ and we further deduce $S = \mathrm{diag}(D(\varphi_1),\ldots,(\varphi_m))$. Then we have $X = S^{-1}X'S = {}^tSX'S = X' \in \left\{ X^k_{q_1\ldots q_m} \mid q_1\ldots q_m \in \{0,1,\ldots,k-1\} \right\}$. Hence $P_k^{-1}(A)$ contains only regular points and therefore A is regular value.

The case $n = 2m+1$, $k > 0$. Let $A = \mathrm{diag}(1,D(\theta_1),\ldots,D(\theta_m))$ with the eigenvalues $\lambda_1 = 1$, $\lambda_{2,3} = \cos 2\Pi\theta_1 \pm i \cdot \sin 2\Pi\theta_1, \ldots, \lambda_{2m,2m+1} = \cos 2\Pi\theta_m \pm i \sin 2\Pi\theta_m$. The matrix $m_2(A)$ has among its eigenvalues the products $\lambda_2\lambda_3 = \ldots = \lambda_{2m}\lambda_{2m+1} = +1$. In order that the multiplicity of $+1$ as an eigenvalue of $m_2(A)$ be precisely $\left[\frac{n}{2}\right] = m$ it is necessary that $\cos 2\Pi\theta_h \pm i \sin 2\Pi\theta_h \neq 1$, $\cos 2\Pi(\theta_h+\theta_1) \pm i \sin 2\Pi(\theta_h+\theta_1) \neq 1$, $\cos 2\Pi(\theta_h-\theta_1) \pm i \sin 2\Pi(\theta_h-\theta_1) \neq 1$, which imply $\theta_h \overline{\in} \mathbb{Z}$, $\forall\, h = 1,\ldots,m$ and $\theta_h \pm \theta_1 \overline{\in} \mathbb{Z}$, $\forall\, h < l$. Consider the following matrices :

$$Y_{q_1\dots q_m} = \mathrm{diag}\left[\, 1, D\left(\frac{\theta_1+q_1}{k}\right), \dots, D\left(\frac{\theta_m+q_m}{k}\right) \right],$$

$$q_1, \dots, q_m \in \{0,1,\dots,k-1\}.$$

We have $Y_{q_1\dots q_m} = A$. Moreover each $Y_{q_1\dots q_m}$ is a regular point for P_k. Indeed, in the Jacobian $\det(d_{Y_{q_1\dots q_m}} P_k)$ the factors are of the same form as in the case $n = 2m$ excepting the factors of the form

$$\left| \frac{1 - (\cos 2\Pi\theta_h \pm i \sin 2\Pi\theta_h)}{1 - (\cos \frac{2\Pi(\theta_h+q_h)}{k} \pm i \sin \frac{2\Pi(\theta_h+q_h)}{k}} \right|^2 \neq 0.$$

Moreover, we can prove that

$$(11) \qquad P_k^{-1}(A) = \left\{ Y_{q_1\dots q_m} \mid q_1\dots q_m \in \{0,1,\dots,k-1\} \right\}.$$

For this, let Y be an arbitrary solution of the equation $X^k = A$. As above we deduce that $Y = S^{-1}Y'S$ with $Y' \in \left\{ X^k_{q_1\dots q_m} \mid q_1\dots q_m \in \{0,1,\dots,k-1\} \right\}$, $S = (s^i_j) \in SO(n)$ and $SA = AS$. If $SA = (\alpha^i_j)$, $AS = (\beta^i_j)$, then imposing the conditions $\alpha^1_2 = \beta^1_2$, $\alpha^1_3 = \beta^1_3$, we obtain

$$\begin{cases} s^1_2(\cos 2\Pi\theta_1 - 1) - s^1_3 \sin 2\Pi\theta_1 = 0 \\ s^1_2 \sin 2\Pi\theta_1 + s^1_3(\cos 2\Pi\theta_1 - 1) = 0, \text{ with} \end{cases}$$

$$\begin{vmatrix} \cos 2\Pi\theta_1 - 1 & -\sin 2\Pi\theta_1 \\ \sin 2\Pi\theta_1 & \cos 2\Pi\theta_1 - 1 \end{vmatrix} = 2(1 - \cos 2\Pi\theta_1) \neq 0,$$

since $\theta_1 \notin \mathbb{Z}$. Thus we obtain $s_2^1 = s_3^1 = 0$. Analogously, using the following pairs of relations : $\alpha_4^1 = \beta_4^1$, $\alpha_5^1 = \beta_5^1$, ... , $\alpha_{2m}^1 = \beta_{2m}^1$, $\alpha_{2m+1}^1 = \beta_{2m+1}^1$, we obtain $s_4^1 = s_5^1 = 0, \dots, s_{2m}^1 = s_{2m+1}^1 = 0$ and by relations $\alpha_j^i = \beta_1^i$, $i = 2,3,\dots,n$, we deduce that $s_1^2 = s_1^3 = \dots = s_1^{2m+1} = 0$. Then we proceed as in the case $n = 2m$ and we obtain $S = \mathrm{diag}(1, D(\varphi_1), \dots, D(\varphi_m))$. By this we get $Y = S^{-1}Y'S = Y'$ which proves (11).

The case $k < -1$ can be reduced to the case $k > 1$. In fact, $X \in P_k^{-1}(A)$ iff $X^{-1} \in P_{-k}^{-1}(A)$. Hence $P_k^{-1}(A) = \left\{ X^{-1} \mid X \in P_{-k}^{-1}(A) \right\}$. Then we can apply Corollary 4.

Now we prove that the matrices which are regular value for a power map P_k, $|k| > 1$, are regular points of the Lie group $SO(n)$. We remember that for a compact and connected Lie group G, an element $x \in G$ is a regular point iff it is contained in precisely one maximal torus [2, vol. II, p.107]. Let $T_{SO(n)}$ be the usual maximal torus of the matrices of the form

$$D(\theta_1,\dots,\theta_m) = \begin{cases} \mathrm{diag}(D(\theta_1,\dots,\theta_m)), & \text{if } n = 2m \\ \mathrm{diag}(D(\theta_1,\dots,\theta_m),1), & \text{if } n = 2m+1 \end{cases}, [4, p. 182].$$

Let A be a regular value for P_k and let $A \in T_{SO(n)}$. Another maximal torus of $SO(n)$ is of the form $S^{-1}T_{SO(n)}S$, with $S \in SO(n)$. If $A \in S^{-1}T_{SO(n)}S$ then $A = S^{-1}BS$, with $B \in T_{SO(n)}$. Because A is regular value for P_k, the matrix B has the same property. Then by the relation $SA = BS$ we deduce, in a similar way as above, that $S \in T_{SO(n)}$. Thus $S^{-1}T_{SO(n)}S = T_{SO(n)}$ and therefore the regular values of a power map P_k, $|k| > 1$, are regular points for the Lie group $SO(n)$.

Conversely, let A be a regular point of the Lie group $SO(n)$ and let $A \in S^{-1}T_{SO(n)}S$, that is $A = S^{-1}A'S$, with $A' \in$ $\in T_{SO(n)}$. Then A' is also a regular point for $SO(n)$ such that we can even suppose $A \in T_{SO(n)}$. If we assume that the condition from the statement of the proposition for the eigenvalues of A is not satisfied, then A has one of the forms :

$$A = D(\theta_1, \ldots, \underset{i}{\theta_i}, \ldots, \underset{j}{\theta_i}, \ldots, \theta_m) \text{ or}$$

$$A = D(\theta_1, \ldots, \underset{i}{\theta_i}, \ldots, -\underset{j}{\theta_i}, \ldots, \theta_m) ,$$

with $j > i$. Then, by considering the matrix $S \in SO(n)$ which changes in the matrix A the $(2i-1)$th row (column) with the $(2j-1)$th row (column) and $2i$-th row (column) with the $2j$-th row (column) respectively the $(2j-1)$th row with $2i$-th row and $2j$-th row followed by the same changes of columns as above, we obtain $A = S^{-1}AS$. Hence $A = S^{-1}T_{SO(n)}S$. But $S^{-1}T_{SO(n)}S \neq T_{SO(n)}$ which can be verified by using matrices that are regular points for $SO(n)$. This contradicts the regularity of the point A of the Lie group $SO(n)$. Hence every regular point A of the Lie group $SO(n)$ satisfies the condition for the eigenvalues of A and which, as we proved, ensures that A is a regular value for P_k, $|k| > 1$. This complete the proof of the proposition.

Corollary 6. a) If $A\ SO(n)$, $n \geq 2$ is a critical point for a power map P_k, then A^k is a singular point of the Lie group $SO(n)$.

b) If A^k is a singular point for $SO(n)$ and if A is a regular point for $SO(n)$ then A is a critical point for P_k.

Proof. a) Let $A \in SO(n)$ be a critical point for P_k. If A is a singular point for $SO(n)$ then obviously A^k is also a singular point for $SO(n)$. Suppose that A is a regular point for $SO(n)$. Then, $\text{rank}(m_2(A)-I) = n(n-1)/2 - \left[\frac{n}{2}\right]$ and, since A is a critical point for P_k, $\text{rank}(m_2(A^k)-I) < \text{rank}(m_2(A)-I)$. Therefore $\text{rank}(m_2(A^k)-I) < n(n-1)/2 - \left[\frac{n}{2}\right]$, which proves that $m_2(A^k)$ has +1 as an eigenvalue with a multiplicity grater than $\left[\frac{n}{2}\right]$. Thus A^k is a singular point of $SO(n)$.

b) Suppose now that A^k is a singular point of the Lie group $SO(n)$ and that A is a regular point of $SO(n)$. Then $\text{rank}(m_2(A^k)-I) < n(n-1)/2 - \left[\frac{n}{2}\right]$ and $\text{rank}(m_2(A)-I) < n(n-1)/2 + - \left[\frac{n}{2}\right]$. These relations imply the inequality $\text{rank}(m_2(A^k)-I) < \text{rank}(m_2(A)-I)$ and we apply Proposition 2.

Remark 2. It is possible that a singular point A of $SO(n)$ be a regular point of a power map P_k. For example $A = I$, $k \neq 0$.

Corollary 7. Let $A \in SO(n)$, n an even number and consider the matrix

$$A' = \begin{pmatrix} 1 & 0 \\ 0 & A \end{pmatrix} \in SO(n+1).$$

a) If A is a singular point for $SO(n)$ then A' is a singular point for $SO(n+1)$.

b) If A' is a regular point for $SO(n+1)$ then A is a regular point of $SO(n)$.

Proof. We apply Proposition 4 and Lemma 4.

Remark 3. It is possible that $A \in SO(n)$ be a regular point for $SO(n)$ but A' be a singular point for $SO(n+1)$. For example, $A = \mathrm{diag}(D(0), D(-1/4))$ is a regular point for $SO(4)$ but A' is a singular point for $SO(5)$.

Corollary 8. a) The unique singular point of the group $SO(3)$ is the unit matrix.

b) The singular points of the group $SO(4)$ are the matrices of the form $A_\theta = \mathrm{diag}(D(\theta), D(\theta))$, $B_\theta = \mathrm{diag}(D(\theta), -D(\theta))$ and $S^{-1}A_\theta S$, $S^{-1}B_\theta S$, with an arbitrary $S \in SO(n)$.

Proof. a) We use Corollary 3 and Proposition 4.

b) Let $\lambda_{1,2} = \cos 2\Pi\theta_1 \pm i \sin 2\Pi\theta_1$, $\lambda_{3,4} = \cos 2\Pi\theta_2 \pm i \cdot \sin 2\Pi\theta_2$ be the eigenvalues of a matrix $A \in SO(4)$. The spectrum of the matrix $m_2(A)$ is $\left\{1, \cos 2\Pi(\theta_1 \pm \theta_2) + i \cdot \sin 2\Pi(\theta_1 \pm \theta_2), \cos 2\Pi(\theta_1 \pm \theta_2) - i \sin 2\Pi(\theta_1 \pm \theta_2), 1\right\}$. In order that this set contains $+1$ with a multiplicity greater that $\left[\frac{4}{2}\right] = 2$, it is necessary and sufficient that $\theta_1 + \theta_2 \in \mathbb{Z}$ or $\theta_1 - \theta_2 \in \mathbb{Z}$, which implies the assertion of the proposition.

Corollary 9. The degree of the power map $P_k : SO(n) \longrightarrow SO(n)$, $n \geq 1$, $k \in \mathbb{Z}$ is given by the formula :

$$\deg P_k = k^{\left[\frac{n}{2}\right]}. \tag{12}$$

Proof. If A is a regular value for P_k, then $\deg P_k = \sum_{X \in P_k^{-1}(A)} \varepsilon(X)$, where $\varepsilon(X)$ is the index of P_k at X, that is $\varepsilon(X) = \mathrm{sgn}(\det(d_X P_k))$ [1, Th. 24.9.4, p. 50]. Choosing A as in the proof of Proposition 4 we see that $P_k^{-1}(A)$ has $k^{\left[\frac{n}{2}\right]}$ elements which are the all regular points for P_k. By Proposition 3, for $X \in P_k^{-1}(A)$ we have $\varepsilon(X) = \operatorname{sgn} k^{\left[\frac{n}{2}\right]}$. Thus we obtain

$$\deg P_k = k^{\left[\frac{n}{2}\right]} \operatorname{sgn} k^{\left[\frac{n}{2}\right]} = k^{\left[\frac{n}{2}\right]}.$$

Corollary 10. Let $A \in SO(n)$, $n \geq 2$ and $k \in \mathbb{Z}-\{0\}$.

a) The equation $X^k = A$ has solutions in $SO(n)$;

b) If A is regular point of the Lie group $SO(n)$ then the equation $X^k = A$ has precisely $|k|^{\left[\frac{n}{2}\right]}$ solutions in $SO(n)$.

Proof. a) Because $\deg P_k = k^{\left[\frac{n}{2}\right]} \neq 0$ it follows that the map $P_k : SO(n) \longrightarrow SO(n)$ is surjective [2, vol. I, p.241].

b) If A is an arbitrary regular point of $SO(n)$, then by Proposition 4 it is a regular value for P_k and then $\deg P_k = \sum_{X \in P_k^{-1}(A)} \varepsilon(X)$. By Proposition 3, $\varepsilon(X) = \operatorname{sgn} k^{\left[\frac{n}{2}\right]}$ and by Corollary 9 we have $\deg P_k = k^{\left[\frac{n}{2}\right]}$. Hence $k^{\frac{n}{2}} = |P_k^{-1}(A)| \operatorname{sgn} k^{\left[\frac{n}{2}\right]}$ which proves that $|P_k^{-1}(A)| = |k|^{\left[\frac{n}{2}\right]}$.

Remark 4. Corollary 10 b) is not generally true if A is a singular point for $SO(n)$. For example, if $n = 3$, $k = 2$, the equation $X^2 = I$ has as solutions ant matrix of the form

$$X = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -\cos\varphi & -\sin\varphi \\ 0 & -\sin\varphi & \cos\varphi \end{bmatrix}.$$

The values $k = \pm 1$ are exceptional values because although I is a singular point for $SO(n)$, $\eta \geq 3$, the equation $X^k = I$ has a single solution.

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REFERENCES

[1] J.Dieudonné: Éléments d'Analyse, vol. 9. Paris, 1982.

[2] W.Greub, S.Halperin, R.Vanstone: Connections, curvature, and cohomology, vol. I, II. New York and London 1975.

[3] H.Hopf: Über den Rang geschlossener Liescher Gruppen, Comment Mat. Helv., 13 (1941), 119–143.

[4] D.Husemoller: Fibre Bundles, New York 1966.

[5] H.Samelson: Topology of Lie groups, Bull. Amer. Math. Soc., 58 (1952) 2–37.

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ON NON-SINGULAR RINGS AND SEMI-PRIME RINGS

This note is essentially concerned with non-singular rings, semi-prime rings and reduced rings. Conditions are given for certain rings to be non-singular and reduced. Strongly regular rings and self-injective strongly regular rings with non - zero socle are characterized.

Throughout, A represents an associative ring with identity and A-modules are unital. Z, J denote respectively the left singular ideal and the Jacobson radical of A. A left (right) ideal of A is called reduced if it contains no non-zero nilpotent element. An ideal of A will always mean a two-sided ideal and A is called left duo if every left ideal of A is an ideal. An effective generealization of left duo rings, noted TC-rings, is introduced below. The following are among the results proved here : (1) If A is a semi-prime left TC-ring, then A/Z is a reduced ring; (2) If there exists a non-singular left ideal of A which is either maximal or a faithful left A-module, then A is left non-singular; (3) If A is a left non-singular left TC-ring, then A is reduced; (4) A semi-prime ring containing a reduced maximal left ring is reduced; (5) Let A be left non-singular such that for any proper complement left ideal C and any maximal essential left ideal M of A (if it exists) containing C, C is an ideal of M. Then A is either reduced or semi-prime with non - zero socle; (6) The following conditions are equivalent: (a) A is strongly regular; (b) A is left TC-ring whose cyclic semi - simple right modules are flat; (c) A is a left MP-injective ring containing a reduced maximal left ideal; (7) A is self - injective strongly regular with non - zero socle if, and only if, A is left TC-ring containing an injective non - singular maximal left ideal.

Semi - prime rings need not be left non - singular and the converse is not true either. But reduced rings are both semi - - prime and left (right) non - singular. Recall that A is a left CM-ring (intoduced in [8]) if, for any maximal essential left ideal M of A (if it exists), every complement left subideal is an ideal of M. Left CM-rings generalize left duo rings, semi - simple Artinial rings, left uniform rings, left PCI rings [2] and left Ore domains. Further results on left CM-rings are here developed. It is well - known that A is von Neumann regular iff every left (right) A-module is flat (cf. for example, [4], p. 262). This is also true if "flat" is replaced by "p-injective". (A left A-module M is p-injective if, for any principal left ideal P of A, every left A-homomorphism of P into M extends to one of A into M.) Flat modules need not be p-injective and the converse is not true either. However, in case A is commutative, a simple A-module is flat iff it is p-injective. Following [5], a left A-module M is called semi - simple if the intersection of all maximal left submodules of M is zero. Then A is a left V-ring iff every left A-module is semi - simple ([5], Theorem 2.1). If A is commutative, then A is V-ring iff it is von Neumann regular (I. Kaplansky). For any left A-module M, $Z(M) = \{ y \in M \setminus Ly = 0$ for some essential left ideal L of A $\}$ is the singular submodule of M and ${}_AM$ is called singular (resp. non - singular) if $Z(M) = M$ (resp. $Z(M) = 0$). A is therefore left non - singular (resp. semi - simple) iff $Z = 0$ (resp. $J = 0$).

We now introduce a non - trivial generalization of left duo rings.

Definition. A is called a left TC-ring if every non-zero complement left ideal of A contains a non-zero ideal of A.

It is easy to see that left uniform rings (in particular, left Ore domains) are left TC. But left CM-rings need not be left TC (otherwise, any semi - simple Artinian ring would be reduced (cf. Proposition 1 below)).

We start with an interesting property of semi - prime left TC-rings.

Proposition 1. If A is a semi - prime left TC-ring, then A/Z is reduced.

Proof. Suppose that A/Z is not reduced ring. Then there exists $c \in A$, $c \notin Z$ such that $c^2 \in Z$. Since $l(c)$ is not essential in $l(c^2)$, there exists a non-zero left subideal I of $l(c^2)$ such that $I \cap l(c) = 0$. The set of all essential extensions of ${}_AI$ in ${}_AA$ has, by Zorn's Lemma, a maximal member K which is a complement left ideal of A and $K \cap l(c) = 0$. By hypothesis, K contains a non-zero ideal T and since $Ic \leq l(c)$, then $TIc \leq Tl(c) \leq T \cap l(c) \leq K \cap l(c) = 0$ implies that $TI \leq$ $\leq l(c) \cap T = 0$. Now $V = T \cap I \neq 0$ (because ${}_AI$ is essential in ${}_AK$) and $V^2 \leq TI = 0$, which contradicts A semi - prime. This proves the proposition.

Recall that A is left uniform if every non - zero left ideal is essential.

Corollary 1.1. (1) If A is a prime left uniform ring, then A/Z is reduced; (2) If A is a prime left CM-ring, then either A is simple Artinian or A/Z is reduced.

Corollary 1.2. If A is a semi-prime left TC-ring, then A/I is left and right self-injective strongly regular. (Apply [2], Corollary 19.28.)

Corollary 1.3. Let A be a semi - prime left TC-ring, satisfying any one of the following conditions: (a) every simple left A-module is flat; (b) every simple right A-module is flat. Then A/Z is a strongly regular ring.

Following [2], an element c of A is called left regular if $l(c) = 0$. Any left ideal containing a left regular element is therefore a faithful left A-module.

Lemma 2. The following conditions are equivalent:

(1) A is left non - singular;

(2) There exists a non - singular left ideal of A which is either maximal or essential or a faithful left A-module;

(3) $A = N \oplus D$, where N is a non - singular left ideal and D is a singular left ideal of A.

Proof. Obviously, (1) implies (2).

Assume (2). Let I be a non – singular left ideal of A. If I is an essential left ideal, then A is also left non – singular. Now suppose that I is a maximal left ideal which is not essential. Then $A = I \oplus U$, where U is a minimal left ideal of A. If $Z \neq 0$, by ([10] Lemma 7), there exists $0 \neq z \in Z$ such that $z^2 = 0$. Since $I = Ae$, where $e = e^2 \in A$, and $U = A(1-e)$, $ze \in Z \cap I = Z(I) = 0$, the $z = z(1-e) \in U$ which implies that $U = Az$ is a projective left A–module. Therefore $l(z)$ is a direct summand of ${}_AA$ which yields $z = 0$, a contradiction! This proves that A is left non – singular if I is maximal. Finally, if ${}_AI$ is faithful, then for any $z \in Z$, $zI \subseteq Z \cap I = Z(I) = 0$ which implies that $z = 0$.

Thus $Z = 0$ in each case and (2) implies (3)

Assume (3). Let $N = Ae$, $e = e^2 \in A$ and $D = A(1-e)$. For any $z \in Z$, since $ze \in Z(N) = 0$, then $z = z(1-e) \in D$ which implies that $Z = D$. Now $1-e \in Z$ and $l(e) \cap l(1-e) = 0$ imply that $l(e) = 0$, whence $e = 1$ and $A = N$ is left non – singular. Therefore (3) implies (1).

Applying ([10], Lemma 7), we get immediately

Corollary 2.1. For any ring A, either every reduced left ideal contains no left regular element or A is left non – singular.

Proposition 3. Let A be a left TC–ring containing a non – singular maximal left ideal. Then A is reduced.

Proof. By Lemma 2, A is left non – singular. Suppose there exists $0 \neq b \in A$ such that $b^2 = 0$. Since $l(b)$ is not essential, there exists a non-zero complement left ideal K of A such that $L = l(b) \oplus K$ is an essential left ideal of A. Now K contains a non - zero ideal T and $Tb \subseteq l(b) \cap T \subseteq l(b) \cap K = 0$ which implies that $T \subseteq l(b)$, contradicting $T \cap l(b) = 0$. This proves that A must be reduced.

Proposition 4. If A is a semi - prime ring with a reduced maximal left ideal, then A is reduced.

Proof. Let M be a reduced maximal left ideal of A. If M is an essential left ideal, then A is reduced by ([10], Lemma 7). Now suppose that $A = M \oplus U$, where $M = Ae$, $e = e^2 \in A$ and $U = A(1-e)$ ia a minimal left ideal of A. Since $(1-e)Ae \subseteq M$, $((1-e)Ae)^2 = 0$ implies that $Ae \subseteq r(1-e) = eA$. Now $(1-e)A$ is a minimal right ideal (because A is semi - prime) which implies that eA is a maximal right ideal of A. Now $AeA \subseteq eA$ yields $Ae = AeA$, whence $Ae = eA$. Consequently, $ea = eae = ae$ for every $a \in A$ which proves that e is a central idempotent. If $0 \neq b \in A$ such that $b^2 = 0$, $(be)^2 = b^2e = 0$ implies $be = 0$ which yields $Ab = Ab(1-e) = U$. Therefore $U = U^2 = UAb = Ub = Ab^2 = 0$ which is impossible.

This proves that A is reduced.

Applying ([10], Lemma 7) and Lemma 2, we get

Corollary 4.1. If A is left self-injective, then either every maximal left ideal of A contains a non-zero nilpotent element or A is right self-injective strongly regular.

Semi-prime rings need not be left CM and the converse is obviously not true either.

We are now in a position to give a nice result on CM-rings.

Proposition 5. If A is a left CM - ring containing a reduced maximal left ideal, then A is reduced.

Proof. Let K be a reduced maximal left ideal. Then $_AK$ is non - singular and by Lemma 2, A is left non - singular. By ([8], Lemma 1.6 (3)), A is semi - prime and by Proposition 4, A is reduced.

Corollary 5.1. The following conditions are equivalent for a left CM-ring A: (a) A is left non - singular ; (b) A is either reduced or semi - simple Artinian.

Apply [8], Lemma 1.6 (1).

As before, write "A is ERT" if every essential right ideal of A is an ideal of A (cf. [8]). If A is ERT, then for every factor ring of A is ERT. Note that A is semi - prime, then any essential right ideal which is an ideal of A must be left essential. Also, if A is semi - prime left non - singular, then every essential left ideal is a faithful left A-module.

Proposition 6. The following conditions are equivalent:

(1) A is an ERT von Neumann regular ring;

(2) Every factor ring B of A has the following property; B has a faithful non-singular left ideal and every essential right ideal of B is an essential left ideal;

(3) A is ERT such that for every factor ring of A, every essential left ideal of B is a faithful left B-module.

Proof. If follows from what is noted above that (1) implies (2) and (3).

Assume (2). If the factor ring B of A contains a faithful non - singular left ideal, then B is left non-singular by Lemma 2. Suppose there exists an ideal T of B such that $T^2 = 0$. Then $I = T \oplus K$ is an essential right ideal of B for some complement right ideal K of B. Now $IT = 0$, (because $KT = 0$) and by hypothesis, $T \subseteq Z(B)$, the left singular ideal of B, which zero. This proves that B is semi - prime. Since every factor ring of A is semi - prime, then A is fully idempotent. If P is a prime factor ring of A, then P is an ERT prime fully idempotent ring which yields P fully right idempotent. By ([1], Theorem 3.1), B is regular and hence (2) implies (1) by ([3], Corollary 1.18).

Assume (3). Let T be an ideal of the factor ring B such that $T^2 = 0$. Then it is easily seen that $r_B(T) = L$ is an essential left ideal of B. It follows that $T \subseteq l_B(L) = 0$ which proves that B is semi - prime. Then the proof of "(2) implies (1)" shows that (3) implies (1).

A recent private communication from Professor L.C.A. van Leeuwen motivates the following result for a class of rings which generalize left CM-rings ([8], Lemma 1.6 (4) is here improved).

Proposition 7. Let A be left non-singular such that for any proper complement left ideal C of A and any maximal essential left ideal M of A (if it exists) containing C, C is an ideal of M. Then A is either reduced or semi - prime with non-zero socle.

Proof. First suppose there exists $0 \neq b \in A$ such that $(Ab)^2 = 0$. If K is relative complement of Ab in ${}_AA$, then $L = Ab \oplus K$ is an essential left ideal of A. Since $(Ab)^2 = 0$, then $L \neq A$ and if M is a maximal left ideal containing L, then $KM \subseteq K$ implies that $Kb \subseteq K \cap Ab = 0$. It follows that $Lb = 0$, whence $b \in Z = 0$. This contradiction proves that A must be semi-prime. Now suppose that A has zero socle and that there exists $0 \neq c \in A$ such that $c^2 = 0$. If V is a maximal left ideal containing $l(c)$, then V must be an essential left ideal of A and since $l(c)$ is a complement left ideal of A ([6], Lemma 1), we have $(Vc)^2 \subseteq AcVc \subseteq l(c)Vc \subseteq l(c)c = 0$ which yields $Vc = 0$, whence $V = l(c)$. Therefore Ac ($\approx A/V$) is a minimal left ideal which contradicts A with zero socle. This proves that A is either reduced or semi - prime with non-zero socle.

([7], Proposition 3 and 6) togetherwith the proof of Proposition 7 yield

Remark 1. Let A have the following properties : (a) every simple left A-module is either p-injective or projective; (b) for any proper complement left ideal C of A and any maximal essential left ideal M of A (if it exists) containing C, C is an ideal of M. Then A is fully left idempotent.

Proposition 8. The following conditions are equivalent :

(1) A is either semi-simple Artinian or strongly regular;

(2) Every factor ring of A is a semi-prime left CM-ring whose principial left ideals are complement left ideals.

Proof. Obviously, (1) implies (2).

Assume (2). For any factor ring B of A, it can be shown that B either is strongly regular or has non-zero socle. If P is a prime factor ring of A, since P is left CM, then P is simple Artinian by ([8], Theorem 2.14). Since A is fully idempotent, then A is regular by ([3], Corollary 1.18). Since A is left CM, then (2) implies (1) by Corollary 5.1.

We have seen (in Proposition 6) that if every essential left ideal of A is a faithful left A-module, then A is semi-prime. This, together with ([8], Lemma 1.6 (1)) and Proposition 5, yield

Proposition 9. The following conditions are equivalent :

(1) A is semi-simple Artinian ;

(2) A is a left CM left hereditary ring which contains either an injective maximal left ideal or an injective maximal right ideal ;

(3) Every essential left ideal of A is a faithful left annihilator ideal.

Following [11], a left A-module M is called MP-injective if, for any principial left ideal P of A, any left A-monomorphism of P into M extends to a left A-homomorphism of A into M. MP-injectivity effectively generalizes p-injectivity (any simple left (right) module over a semi-prime ring is MP-injective).

Theorem 10. The following conditions are equivalent :

(1) A is strongly regular ;

(2) A is a left non-singular left MP-injective left TC-ring ;

(3) A is a left TC-ring whose cyclic semi-simple right A-modules are flat ;

(4) Every factor ring B of A is a semi-prime left MP-injective left TC-ring ;

(5) There exists a maximal left ideal M of A such that for any $a \in M$, $r(a)$ is a MP-injective left A-module ;

(6) A is a left MP-injective ring containing a reduced maximal left ideal ;

(7) A is a right MP-injective left CM-ring containing a reduced maximal left ideal ;

(8) A is a right MP-injective left TC-ring containing a reduced maximal left ideal ;

(9) There exists a faithful left A-module F such that $l(u)$ is a MP-injective left A-module for every $u \in F$ and $l(a) =$ $= r(a)$ for every $a \in A$.

Proof. Obviously, (1) implies (2).

(2) implies (3) by ([12], Lemma 1.2) and Lemma 3.

Assume (3). Let $z \in Z$. For any $a \in A$, if $c = 1-za$, then $l(c) = 0$ and we claim that $cA = A$. Suppose not : there exists a maximal right ideal M containing cA and since A/M_A is flat, then $c = dc$ for some $d \in M$. Therefore $1 = d \in M$ which contradicts $M \neq A$. Thus c is right invertible in A which proves that $z \in J$. Now A/J_A is cyclic semi-simple and therefore flat, whence $z = vz$ for some $v \in J$. Now $w(1-v) = 1$ for some $w \in A$

which implies that $z = w(1-v)z = w(z-vz) = 0$. This proves that $Z = 0$. By Proposition 3, A is reduced and since every simple right A-module is flat, then A is strongly regular and (3) implies (4).

Assume (4). Then A is fully idempotent ring which implies that $J = 0$. By ([11], Proposition 3), $Z = 0$ and hence (4) implies (5) by Proposition 3 and ([12], Lemma 1.2).

Assume (5). Then ${}_AA$ is MP-injective. If $b \in M$ such that $b^2 = 0$, since $r(b)$ is a MP-injective left A-module, the inclusion map $Ab \longrightarrow r(b)$ yields $b = bd$ for some $d \in r(b)$. Then $1-d \in r(b)$ implies $1 \in r(b)$, whence $b = 0$. This proves that M is reduced and therefore (5) implies (6).

(6) implies (7) by ([10], Lemma 7), ([11], Proposition 3.2), ([12], Lemma 1.2), Lemma 2 and Proposition 4.

(7) implies (8) by ([12], Lemma 1.2), and Proposition 5.

(8) implies (9) by Lemma 2 and Proposition 3.

Assume (9). Suppose there exists $0 \neq b \in A$ such that $b^2 = 0$. Since ${}_AF$ is faithful, there exists $y \in F$ such that $by \neq 0$. Since ${}_Al(by)$ is MP-injective, the inclusion map $Ab \longrightarrow l(by)$ induces an element $v \in l(by)$ such that $b = bv$, whence $1-v \in r(b) = l(b) \subseteq l(by)$. Therefore $1 \in l(by)$ which contradicts $by \neq 0$. This proves that A is reduced. Since ${}_AA$ is MP-injective, then (9) implies (1) by ([12], Lemma 1.2).

Recall that any finitely generated p-injective left ideal of A is a direct summand of ${}_AA$.

Theorem 11. The following conditions are equivalent:

(1) A is strongly regular with non-zero socle ;

(2) A is a left TC-ring containing a non-singular maximal left ideal which is finitely generated p-injective ;

(3) A is a left CM-ring containing a reduced maximal left ideal which is finitely generated p-injective .

Proof . (1) implies (2) evidently.

Assume (2). Then A is reduced by Proposition 3. If M is the non-singular maximal left ideal which is finitely generated p-injective, then A = M ⊕ U, where U is a minimal left ideal of A. Since A is reduced, then M and U are generated by central idempotents. Therefore M is a maximal right ideal of A and U is a minimal right ideal. Since U_A is projective, by ([11], Lemma 1), $_A A/M$ is injective which implies that A = M ⊕ U is a left p-injective ring. Then (2) implies (3) by ([12], Lemma 1.2).

Similarly, Proposition 5 enables us to say that (3) implies (1).

The proof of Proposition 4 and Theorem 11 yield

Theorem 12. The following conditions are equivalent:

(1) A is left and right self-injective strongly regular with non-zero socle ;

(2) A is a left TC - ring containing an injective non-singular maximal left ideal ;

(3) A is a left CM-ring containing an injective reduced maximal left ideal ;

(4) A is a semi - prime ring with a reduced injective maximal left ideal.

Remark 2. The proof of ([8], Lemma 1.6(1)) shows that if A is a left CM-ring, then any non-singular left ideal containing a non-zero nilpotent element is a direct summand of ${}_AA$.

We add a last result and conclude with a few remarks.

Proposition 13. Let A be a left CM-ring satisfying any one of the following conditions : (a) is left MP-injective ; (b) every simple left A-module is flat. Then any non-singular principial left ideal of A is generated by an idempotent.

Proof. Let P = Ac be a non-singular principal left ideal.

(a) Suppose that A is left MP-injective. If P is reduced, then P is a direct summand of ${}_AA$ by ([12], Lemma 1.2). It follows from Remark 2 that, in any case, P is generated by an idempotent.

(b) Assume now that every simple left A-module is flat. If P is reduced, then P is generated by an idempotent ([9], Lemma 4). If P contains a non-zero nilpotent element, apply Remark 2.

Remark 3. If A is a left MP-injective left TC-ring, then any non-zero complement left ideal contains either a non-zero nilpotent element or a non-zero central idempotent.

Remark 4. If A is fully left idempotent ring satisfying the maximum condition on complement left ideals, then every ideal of A is generated by a central idempotent.

Remark 5. The following conditions are equivalent : (a) A is a left Ore domain ; (b) A is prime left TC-ring containing a non-zero non-singular left ideal.

The proof of Proposition 4 yields

Remark 6. If A contains a reduced maximal left ideal M, then either A is reduced or A = M ⊕ U, where U is generated by every non-zero nilpotent element of A.

REFERENCES

[1] G.Baccella : Generalized V-rings and von Neumann regular rings, Rend. Sem. Mat. Univ. Padova 72 (1984) 117-133.

[2] C.Faith : Algebra II : Ring Theory , Berlin - Heidelberg - New York. Springer (1976).

[3] K.R.Goodearl : Von Neumann regular rings, London - San Francisco - Melbourne : Pitman (1979).

[4] F.Kasch : Modules and rings, London Math. Soc. Monograph 17 (1982).

[5] G.O.Michler , O.E.Villamayor : On rings whose simple modules are injective, J.Algebra 25 (1973) 185-201.

[6] K.Chiba, H.Tominaga : On strongly regular rings, II Proc. Japan Acad. 50 (1975) 444-445.

[7] R.Yue Chi Ming : On von Neumann regular rings, II. Math. Scand. 39 (1976) 167-170.

[8] R.Yue Chi Ming : On regular rings and self - injective rings, Mh. Math. 91 (1981) 153-166.

[9] R.Yue Chi Ming : On regular rings and Artinian

rings, Riv. Mat. Univ. Parma (4)8 (1982) 443–452.

[10] R. Yue Chi Ming: On quasi–injectivity and von Neumann regularity, Mh. Math. 95 (1983) 25–32.

[11] R. Yue Chi Ming: On von Neumann regular rings, VIII. J. Korean Math. Soc. 19 (1983) 97–104.

[12] R. Yue Chi Ming: On regular ideals and P – – injectivity, Rend. Sem. Fac. Sc. Univ. Cagliari 54 (1984) 25–34.

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ON SOME INEQUALITY FOR TRACES

Let H be a Hilbert space and let A and B be symmetric positive operators on H. It is the aim of the present note to give a simple proof of the following inequality for traces

$$\mathrm{Tr}(A + B)^{\alpha} \leq \mathrm{Tr}\, A^{\alpha} + \mathrm{Tr}\, B^{\alpha}, \quad 0 \leq \alpha \leq 1.$$

Here $\mathrm{Tr} \ldots = \sum (\ldots e_i , e_i)$, where $\{ e_i \}$ is some orthonormal basis in H.

(We note that the definition of the trace is independent of the orthonormal basis and positive definite scalar product $(\cdot,\cdot)$ in an equivalence class, i.e. defining equivalent norms on H. For this and other results that will be used in the proof see e.g. [1], [2].)

Remark. It should be noticed that the inequality $\mathrm{Tr}(A+B)^{\alpha} \leq 2\mathrm{Tr}\, A^{\alpha} + 2\mathrm{Tr}\, B^{\alpha}$, $0 < \alpha < 1$ is a consequence of the norm inequality stated e.g. in ([1] Lemma 9 p.255) , ([2] Theorem 7.8, p.73).

Proof. We may (and do) assume that $\mathrm{Tr}\, A^{\alpha} < \infty$, $\mathrm{Tr}\, B^{\alpha} < \infty$ and $0 < \alpha < 1$ or else there is nothing to prove. Hence A and B are , in addition, of trace class operators.

Let $\varepsilon > 0$, I be the unit operator on H and $-1 < \beta < 0$. Then

$$0 < A + \varepsilon I \le A + B + \varepsilon I \quad \text{and} \quad 0 < (A + B + \varepsilon I)^{\beta} \le (A + \varepsilon I)^{\beta},$$

where ≤ denotes the order in the space of bounded operators on H. Therefore

$$0 \le A^{\frac{1}{2}} (A + B + \varepsilon I)^{\beta} A^{\frac{1}{2}} \le A^{\frac{1}{2}} (A + \varepsilon I)^{\beta} A^{\frac{1}{2}} = A (A + \varepsilon I)^{\beta}.$$

By replacing A by B and B by A we obtain

$$0 \le B^{\frac{1}{2}} (A + B + \varepsilon I)^{\beta} B^{\frac{1}{2}} \le B (B + \varepsilon I)^{\beta}.$$

From this and the previous inequality we get

$$0 \le A^{\frac{1}{2}} (A + B + \varepsilon I)^{\beta} A^{\frac{1}{2}} + B^{\frac{1}{2}} (A + B + \varepsilon I)^{\beta} B^{\frac{1}{2}} \le$$

$$\le A(A + \varepsilon I)^{\beta} + B(B + \varepsilon I)^{\beta}.$$

Consequently

$$\operatorname{Tr} A^{\frac{1}{2}} (A + B + \varepsilon I)^{\beta} A^{\frac{1}{2}} + \operatorname{Tr} B^{\frac{1}{2}} (A + B + \varepsilon I)^{\beta} B^{\frac{1}{2}} \le$$

$$\le \operatorname{Tr} A(A + \varepsilon I)^{\beta} + \operatorname{Tr} B(B + \varepsilon I)^{\beta}$$

and

$$\operatorname{Tr} (A + B)(A + B + \varepsilon I)^{\beta} \le \operatorname{Tr} A (A + \varepsilon I)^{\beta} + \operatorname{Tr} B (B + \varepsilon I)^{\beta}.$$

Computing the trace, by using an orthonormal basis of H consisting of eigenvectors of A + B, A and B (with eigenvalues, say $\{c_i\}$, $\{a_i\}$ and $\{b_i\}$) in the first, second and third terms of the above inequality respectively, we obtain

$$\Sigma c_i (c_i + \varepsilon)^{\beta} \le \Sigma a_i (a_i + \varepsilon)^{\beta} + \Sigma b_i (b_i + \varepsilon)^{\beta}.$$

The desired result follows now as $\varepsilon \downarrow 0$ and $\beta = \alpha - 1$.

Q.E.D.

Remark. We note that, in general, (if A and B do not commute, $0 < \alpha < 1$) the inequality $(A + B)^{\alpha} \le A^{\alpha} + B^{\alpha}$ is false. To give an example, take $\alpha = \frac{1}{2}$,

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \text{ and } B = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

BIBLIOGRAPHY

[1] N.Dunford, J.T.Schwartz: Linear operators, Part II, Interscience Publishers, New York 1963.

[2] J.Weidman: Linear operators in Hilbert spaces, Springer Verlag, New York 1980.

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MINIMAL DISSECTIONS OF BORDERED MANIFOLDS

We construct minimal (with respect to vertices) pseudodissections for any connected piecewise-linear n-manifold with boundary. As a consequence, we obtain ball coverings for bordered manifolds with nice intersection properties. Then we give a graph-theoretical description of these results.

1. Notations

Let Δ_n be the set $\{0,1,\dots,n\}$ and $N_n = \Delta_n - \{0\}$. The symbol #A means the cardinality of the set A.

All considered (compact) spaces and maps belong to the piecewise-linear (PL) category in the sense of [H] or [Z]. The prefix PL will always be omitted.

Two ball-complexes B_1, B_2 are said to be abstractly isomorphic if there exists a bijection $f: B_1 \longrightarrow B_2$ preserving the face-incidence relation. An n-pseudocomplex K is an n-dimensional principal ball-complex in which every r-ball, considered with all their faces, is abstractly isomorphic with the complex underlying an r-simplex ([HW], p.49). K is said to be a pseudodissection of the polyhedron |K|. By $S_r(K)$ and K^s we respectively denote the set of all the r-balls of K and the s-skeleton of K. We shall also call r-simplex (resp. vertex) each r-ball (resp. 0-ball) of K. An n-dimensional pseudocomplex K with exactly n+1 vertices is called a contracted complex: such a K is also said to be a contracted triangulation of |K|.

Given a simplex s in an n-pseudocomplex K, the disjoined star std(s,K) is defined to be the disjoint union of the n-simplexes of K containing s, with re-identification of the (n-1)-faces containing s and of their faces. The subcomplex $lkd(s,K) = \{\tau \in std(s,K)/\tau \cap s = \emptyset\}$ is called the disjoined link of s in K. If K is a pseudodissection of a manifold, the star st(s,K) and the link lk(s,K) of a simplex s in K are not necessarily balls or spheres; however std(s,K) and lkd(s,K) are the balls or spheres obtained by a minimal set of severings on st(s,K) and lk(s,K) respectively.

An (n+1)-coloured graph with boundary is a pair (Γ,γ), where $\Gamma = (V(\Gamma),E(\Gamma))$ is a finite non-oriented multigraph (without loops) and $\gamma: E(\Gamma) \to \Delta_n$ is an edge-coloration of Γ (i.e. $\gamma(e_1) \neq \gamma(e_2)$ for any two adjacent edges $e_1, e_2 \in E(\Gamma)$) ([CG], Def.1)).

A boundary vertex of Γ is a vertex of $V(\Gamma)$ whose degree is strictly less than n+1. If Γ has no boundary vertices, i.e. Γ is regular of degree n+1, (Γ,γ) is simply called an (n+1)-coloured graph. For every subset $B \subseteq \Delta_n, \Gamma_B$ denotes the partial graph $(V(\Gamma),\gamma^{-1}(B))$. (Γ,γ) is said to be contracted if $\Gamma_{\hat{i}} = \Gamma_{\Delta_n - \{i\}}$ is connected, for every $i \in \Delta_n$. A colour-isomorphism between two (n+1)-coloured graphs with boundary (Γ,γ) and $(\bar{\Gamma},\bar{\gamma})$ is a pair (Ψ,Φ), where Ψ is a permutation on Δ_n and $\Phi:\Gamma \to \bar{\Gamma}$ is a graph-isomoprhism such that $\Psi \circ \gamma = \bar{\gamma} \circ \Phi$.

If (Γ,γ) is an (n+1)-coloured graph (with or withour boundary), the associated n-pseudocomplex $K(\Gamma)$ is defined by the following rules:

a) take an n-simplex $\sigma^n(v)$ for each $v \in V(\Gamma)$ and label its vertices by Δ_n,

b) if $v,w \in V(\Gamma)$ are joined by an i-coloured edge, identify the (n-1)-faces of $\sigma^n(v)$ and $\sigma^n(w)$ opposite to the vertices labelled by i, so that equally labelled vertices are identified.

On the other hand, let K be an n-dimensional pseudocomplex with a vertex-coloration $\rho : V(K^1) \to \Delta_n$ which is injective on each n-simplex of K. The pair (K,ρ) is called an *n-labelled pseudocomplex*.

Obviously, each contracted triangulation gives an n-labelled pseudocomplex in a natural way.

The *(n+1)-coloured graph associated to (K,ρ)*, simply written $(\Gamma(K),\gamma_K)$, is obtained in this way:

a′) take a vertex for each n-simplex of K

b′) join two vertices by an edge e iff the associated n-simplexes of K have a common (n-1)-face $\sigma^{n-1}(e)$

c′) define $\gamma_K(e) = \Delta_n - \{\rho(v_0),\ldots,\rho(v_{n-1})\}$, v_i $(i \in \Delta_{n-1})$ being a vertex of $\sigma^{n-1}(e)$.

The graph (Γ,γ) (resp. $(\Gamma(K),\gamma_K)$) is said to *represent* $K(\Gamma)$ (resp. K) and every space homeomorphic with it.

(Γ,γ) (resp. $(\Gamma(K),\gamma_K)$) is a contracted graph iff $K(\Gamma)$ (resp. K) is a contracted complex. Further, $(\Gamma(K(\Gamma)),\gamma_{K(\Gamma)})$ is colour-isomorphic with (Γ,γ); If $|K|$ is a manifold, then $K(\Gamma(K))$ is abstractly isomorphic with K.

2. Minimal pseudodissections

The following result has been proved in [P] for the closed manifolds and extended in [CG] to the manifolds with connected boundary.

Proposition 1. Every n-dimensional connected manifold with connected boundary admits a contracted triangulation.

We give a pseudodissection theorem for every connected n-manifold with h $(h>1)$ boundary components, by means of a pseudocomplex with $n\cdot h$ vertices (n vertices on each boundary component). Since $n\cdot h$ is the minimal possible number of vertices, the result can be considered as an extension of the proposition 1 to every n-manifold, obtained by a completely different proof.

Lemma 2. Let M be a connected n-manifold with h boundary components M_i. If K_i is a contracted triangulation of M_i $(i \in \mathbb{N}_h)$, there exists a pseudodissection K of M, inducing K_i on each M_i.

Proof. Let M_i' be a copy of M_i ($i \in N_h$) and $\phi_i: M_i \to M_i \times I$ ($I = [0,1]$) the map which identifies M_i with $M_i' \times \{0\}$. Let $M^* = M \cup_{\phi_1} (M_1' \times I) \cup_{\phi_2} \ldots \cup_{\phi_h} (M_h' \times I)$ be the adjunction space homeomorphic with M.

If K_i' denotes the first barycentric subdivision of K_i, let H_1 be a simplicial complex triangulating M such that ∂H_1 is a subdivision of $\bigcup_{i=1}^{h} K_i'$. The pseudocomplex $\left(\bigcup_{i=1}^{h} K_i\right) \times I$ gives a prismation P of $\overline{M^*-M}$. Let H_2 be the pseudocomplex decomposing P obtained as follows:

1) subdivide $\bigcup_{i=1}^{h} K_i$ by ∂H_1 so that the inferior base of each prism in P is triangulated by a subcomplex of ∂H_1,

2) by recurrence on the dimension of the prisms, star each of them from the barycenter over its boundary.

Then $K^* = H_1 \cup H_2$ is a pseudodissection of M^* inducing K_i on each $M_i' \times \{1\}$. The pseudodissection K of the lemma is $f(K^*)$, where $f: M^* \to M$ is a homeomorphism.

Proposition 3. Let M be a connected n-manifold with h ($h > 1$) boundary components M_i. If K_i is a contracted triangulation of M_i ($i \in N_h$), there exists a pseudodissection H of M with no interior vertices, inducing K_i on each M_i.

Proof. Let $\eta : S_0\left(\bigcup_{i=1}^{h} K_i\right) \to \Delta_n$ be a surjective application, called a boundary coloration, such that $\eta|S_0(K_i)$ is injective for each $i \in N_h$; $\eta(v)$ is said to be the colour of the vertex v. Note that the existence of η is allowed by the condition $h > 1$ and that the number of vertices in $\bigcup_{i=1}^{h} K_i$ is just nh.

Let $P_r(M)$ ($r \in \Delta_{n+1}$) be the set of all the pseudodissections H^* of M such that:

(′) H^* induces K_i on each M_i,

(″) every n-simplex of H has at least r boundary vertices, they are labelled by different colours of a subset of Δ_n and r of them are labelled by different colours of Δ_{r-1} (set $\Delta_{-1} = \emptyset$).

Let $H_0 \in P_0(M)$ be a pseudodissection of M obtained as shown in Lemma 2. By induction, if $H_r \in P_r(M)$, we construct $H_{r+1} \in P_{r+1}(M)$. The pseudodissection $H = H_{n+1}$ will satisfy the statement, being $\partial H = \partial H_0$ and $\# S_0(H) = n \cdot h$.

Since $|H_r| = M$, there exist a finite sequence $\xi = \left\{\sigma_i^n\right\}_{i=1}^{\alpha_r}$ $(\# S_n(H_r) = \alpha_r)$ of all the n-simplexes of H_r and finite sequence $\xi' = \left\{\sigma_j^{n-1}\right\}_{j=2}^{\alpha_r}$ of (n-1)-simplexes of H_r such that, for each j, σ_j^{n-1} is a common face of σ_j^n and σ_k^n $(k < j)$.

Let τ_i^{n-r} be the (n-r)-face of σ_i^n not containing the vertices labelled by the colours of Δ_{r-1} and let j(i) be the minimal index in ξ such that $\tau_i^{n-r} < \sigma_{j(i)}^n$.

The set of the distinct elements in the family $\left\{S_n(\mathrm{std}(\tau_{j(i)}^{n-r}, H_r))\right\}_{i=1}^{\alpha_r}$ is a partition of $S_n(H_r)$.

The finite sequence $\varepsilon = \left\{\mathrm{std}(\tau_{j(i)}^{n-r}, H_r)\right\}$ and the subsequence $\varepsilon' = \left\{\sigma_{j(i)}^{n-1}\right\}$ of ξ' are increasing ordered with respect to j(i). Moreover, for each j(i), $\sigma_{j(i)}^{n-1}$ has one copy in $\partial \mathrm{std}(\tau_{j(i)}^{n-r}, H_r)$ and the other one is in $\partial \mathrm{std}(\tau_\ell^{n-r}, H_r)$, for some $\ell < j(i)$. If τ_i^{n-r} has no r-coloured vertices, then $\tau_i^{n-r} \notin \partial H_r$; so the only possible situations are the following ones:

1) $\tau_i^{n-r} \notin \partial H_r$ and τ_i^{n-r} has no r-coloured vertices,

2) there is just one r-coloured vertex of τ_i^{n-r}.

By identifying the elements of ε along suitable (n-1)-simplexes of ε', we obtain a finite sequence $\sum = \left\{B_k^n\right\}$ of n-balls such that each B_k^n contains exactly one disjoined star $\mathrm{std}(\tau_i^{n-r}, H_r)$ with τ_i^{n-r} as in case 2). Note that each n-ball B_k^n contains just one r-coloured vertex $v_k \in \partial B_k^n$. Further, $\mathrm{lk}(v_k, \partial B_k^n)$ is an (n-2)-sphere.

If $D_k^{n-1} = \overline{\partial B_k^n - \mathrm{st}(v_k, \partial B_k^n)}$, let D_k^n be the triangulated n-ball $v_k * D_k^{n-1}$; ∂D_k^n is isomorphic with ∂B_k^n and every n-simplex of D_k^n

has v_k as its vertex. Let B^n_{r+1} be the combinatorial n-ball obtained by identifying the n-balls D^n_k along the remaining (n-1)-simplexes of ε'. The pseudodissection $H_{r+1} \in P_{r+1}(M)$ is obtained from B^n_{r+1} by pairwise identification of all the (n-1)-simplexes of ∂B^n_{r+1}. ■

If M is a manifold, define $p(M) = \min\{\# S_0(K) \mid K$ pseudodissection of $M\}$.

A pseudodissection K of M is said to be minimal if $\# S_0(K) = p(M)$.

Proposition 4. Let M be a connected n-manifold.

i) If ∂M is connected (possibly empty), $p(M) = n+1$.

ii) If ∂M has h ($h > 1$) connected components, $p(M) = n \cdot h$.

Proof. i) is another version of Proposition 1. If ∂M has h ($h > 1$) connected components M_i, then $p(M) \leqslant n \cdot h$ by Prop. 3. Since $p(M_i) = n$, for every $i \in N_h$, then $p(M) \geqslant h \cdot p(M_i) = n \cdot h$. ■

Note that the pseudodissection H of Proposition 3 is minimal.

The proof of Proposition 3 can be adapted so to obtain an alternative proof of the following result of [G].

Corollary 5. Let M be a connected n-manifold with h ($h > 0$) boundary components M_i. If K_i is a contracted triangulation of M_i ($i \in N_h$), there exists a pseudodissection $\tilde{H}$ of M with only one interior vertex, inducing K_i on each M_i.

Proof. Let $\bar{\eta} : S_0\left(\bigcup_{i=1}^{h} K_i\right) \to \Delta_{n-1}$ be an application such that $\bar{\eta}|_{S_0(K_i)}$ is injective for each $i \in N_h$. With the notations of Proposition 3, let $\bar{H} \in P_n(M)$ be a pseudodissection of M inductively constructed as H_n, by using $\bar{\eta}$ instead of η. For each n-simplex $\sigma^n \in \bar{H}$, let $v(\sigma^n)$ be the unique vertex of σ^n not labelled by $\bar{\eta}$. There exist a finite sequence $\xi'' = \{\operatorname{std}(v_\alpha(\sigma^n), \bar{H})\}_{\alpha=0}^{t}$ of the disjoined stars of all the vertices $v(\sigma^n)$ in $\bar{H}$ and a finite sequence $\varepsilon'' = \{A^{n-1}_\beta\}_{\beta=1}^{t}$ of (n-1)-simplexes of $\bar{H}$ such that, for every β, we have $A^{n-1}_\beta \in \operatorname{st}(v_\beta(\sigma^n), \bar{H}) \cap \operatorname{st}(v_\gamma(\sigma^n), \bar{H})$, for some $\gamma < \beta$.

Let B be the triangulated n-ball obtained by pairwise identification of all the n-balls in ξ'' along the (n-1)-simplexes in ε''. If w is an interior point of B, let $\tilde{H}$ be the pseudodissection obtained from $D = w * \partial B$ by the identification of all the simplexes in $\partial D = \partial B$ which are copies of the same simples of $\bar{H}$.

3. Crystallizations

Let G^n be the set of all the (n+1)-coloured graphs with boundary whose boundary vertices have degree exactly n. Let $(\Gamma, \gamma) \in G^n$. If Γ has no boundary vertices, set $\partial\Gamma = \emptyset$. Otherwise, let $(\partial\Gamma, \partial\gamma)$ be the graph defined by the following rules:

a) the vertices of $\partial\Gamma$ are the boundary vertices of Γ,

b) two vertices x,y are joined in $\partial\Gamma$ by an i-coloured edge iff there is a colour j in Γ missing both in x and in y and x,y belong to the same connected component of the partial graph $\Gamma_{\{i,j\}}$. $(\partial\Gamma, \partial\gamma)$ is said to be the boundary graph of (Γ, γ). If $(\partial\Gamma, \partial\gamma)$ has h connected components, we say that (Γ, γ) has h boundary components.

Note that an (n+1)-coloured graph with boundary, regular with respect to one colour ([CG], Def.3), is a particular element of G^n: moreover, its boundary graph in the sense of ([CG], Def.4) coincides with the one just defined.

Let G^n_h $(h \geqslant 0)$ be the set of all the (n+1)-coloured graphs in G^n with h boundary components, each of which is a contracted n-coloured graph.

Let M be a connected n-manifold with h $(h \geqslant 0)$ boundary components. The following definition generalizes the analogous concepts of [F], [CG].

Definition 1. An element (Γ, γ) of G^n_h is said to be a crystallization of M iff $K(\Gamma)$ is a minimal pseudodissection of M.

Proposition 6. 1) An element (Γ, γ) of G^n_0 is a crystallization of a closed connected n-manifold M iff $|K(\Gamma_{\hat{i}})|$ is an (n-1)-sphere for each $i \in \Delta_n$.

2) An element (Γ,γ) of G_1^n is a crystallization of a connected n-manifold M with connected boundary iff $|K(\Gamma_{\hat{j}})|$ is an (n-1)-sphere for exactly one $j \in \Delta_n$ and $|K(\Gamma_{\hat{i}})|$ is an (n-1)-ball for each $i \in \Delta_n - \{j\}$.

3) An element (Γ,γ) of G_h^n $(h > 1)$ is a crystallization of a connected n-manifold M with h $(h > 1)$ boundary components iff $|K(\Gamma_{\hat{i}})|$ is the disjoint union of exactly h (n-1)-balls for each $i \in \Delta_n$.

Proof. The results for G_0^n and G_1^n have been proved in [P] and [CG] respectively. For $h > 1$, it is sufficient to observe that the connected components of $K(\Gamma_{\hat{i}})$ triangulate the disjoined links of all the vertices of $\eta^{-1}(\{i\})$, η being a boundary coloration as in Proposition 3.

Proposition 7. Let (Γ,γ) be a crystallization of a connected n-manifold M with non-empty boundary. Then each boundary component of (Γ,γ) is a crystallization of a component of ∂M.

Proof. If M_i is a connected component of ∂M, let K_i be the contracted triangulation of M_i induced by $K(\Gamma)$. If $\Gamma(K_i)$ is the associated crystallization, we prove that $\Gamma(K_i)$ is colour-isomorphic with a suitable connected component of $\partial\Gamma$.

If s_x^{n-1} is the (n-1)-simplex of K_i represented by $x \in V(\Gamma(K_i))$, let $s^n \in K(\Gamma)$ be the unique n-simplex such that $s_x^{n-1} < s^n$. If y is the boundary vertex of Γ representing s^n, define $\theta(x) = y$.

Let $\partial_i(\Gamma)$ be the connected component of $\partial\Gamma$ with $y \in V(\partial_i(\Gamma))$. The map $\theta : V(\Gamma(K_i)) \rightarrow V(\partial_i(\Gamma))$ is bijective. Note that $\Gamma(K_i)$ and $\partial_i(\Gamma)$ have the same colour set, $\Delta_n - \{c\}$ say.

If $x, \bar{x} \in V(\Gamma(K_i))$ are joined by an r-coloured edge of $\Gamma(K_i)$, let σ^{n-2} be the common (n-2)-face of s_x^{n-1}, $s_{\bar{x}}^{n-1}$. Since the n-simplexes $s_{\theta(x)}^n$, $s_{\theta(\bar{x})}^n$ belong to $std(\sigma^{n-2}, K(\Gamma))$, $\theta(x), \theta(\bar{x}) \in V(\Gamma_{\{c,r\}})$. Finally, $\theta(x)$ and $\theta(\bar{x})$ are joined by an r-coloured edge in $\partial_i(\Gamma)$.

Definition 2. A crystallization of a non-connected manifold M is defined to be a non-connected graph such that each one of its components is a crystallization, in the sense of Definition 1, of a (different) component of M.

The following result is an immediate consequence of Propositions 3, 7 and Definition 2:

Proposition 8. Let M be a connected n-manifold with h ($h > 0$) boundary components M_i ($i \in N_h$). For every crystallization (Γ', γ') of ∂M, there exists a crystallization (Γ, γ) of M such that $(\partial\Gamma, \partial\gamma)$ is colour-isomorphic with (Γ', γ').

Finally we give a graph-theoretical description of the crystallizations of all the 3-manifolds with h ($h \geq 0$) boundary components among the graphs of G^3_h. If $(\Gamma, \gamma) \in G^3_h$ and C_α is a connected component of $\Gamma_{\hat{j}}$ ($j \in \Delta_3$), let g^α_{ab} ($a, b \in \Delta_3 - \{j\}$, $a \neq b$) be the number of all the connected components of C_α bicoloured by a and b. If $C_\alpha = \Gamma_{\hat{j}}$, set $g^\alpha_{ab} = g^j_{ab}$.

Proposition 9. 1) Let (Γ, γ) be an element of G^3_0. Then (Γ, γ) is a crystallization of a closed connected 3-manifold iff $\#V(\Gamma_{\hat{j}}) = - \#E(\Gamma_{\hat{j}}) + \sum g^j_{ab} = 2$ for each $j \in \Delta_3$.

2) Let (Γ, γ) be an element of G^3_1. Then (Γ, γ) is a crystallization of a connected 3-manifold with connected boundary iff there exists exactly one colour $j \in \Delta_3$ such that $\#V(\Gamma_{\hat{j}}) - \#E(\Gamma_{\hat{j}}) + \sum g^j_{ab} = 2$ and $\#V(\Gamma_{\hat{i}}) - \#E(\Gamma_{\hat{i}}) + \sum g^i_{ab} = 1$, for each $i \in \Delta_3 - \{j\}$.

3) Let (Γ, γ) be an element of G^3_h ($h > 1$). Then (Γ, γ) is a crystallization of a connected 3-manifold with h ($h > 1$) boundary components iff, for each $i \in \Delta_3$ and for each connected component C_α ($\alpha \in N_h$) of $\Gamma_{\hat{i}}$, $\#V(C_\alpha)$, $- \#E(C_\alpha) + \sum g^\alpha_{ab} = 1$.

Proof. The three statements can be easily derived by using the following remarks. Let (Γ, γ) be a crystallization of an n-manifold. For each $B \subseteq \Delta_n$, with $\#B = h < n+1$, there is a bijection between the set of connected components of Γ_B and the set of the $(n-h)$-simplexes of $K(\Gamma)$ whose vertices are labelled by Δ_n-B. For each $j \in \Delta_3$, $\Gamma_{\hat{j}}$ is a 3-coloured graph. Then $\#V(\Gamma_{\hat{j}})$ (resp. $\#E(\Gamma_{\hat{j}})$) is the number of all the triangles (resp. edges) of $K(\Gamma_{\hat{j}})$. Finally the sum $\sum g^{j}_{ab}$ of all the bi-coloured cycles of $\Gamma_{\hat{j}}$ is just the cardinality of $S_0(K(\Gamma_{\hat{j}}))$.

4. Ball coverings of manifolds

Let M be a closed connected n-manifold and $\mathcal{B} = \{B_i \mid i \in I\}$ be a finite set of closed n-balls such that $M = \bigcup_{i \in I} B_i$. Recall the following P_α-conditions ($\alpha \in \Delta_2$) listed in [CGr]:

P_0: for every $i,j \in I$ $(i \neq j)$, $B_i \cap B_j = \partial B_i \cap \partial B_j$ has (n-1)-manifolds as connected components.

P_1: for every $i,j \in I$ $(i \neq j)$, $B_i \cap B_j = \partial B_i \cap \partial B_j$ has (n-1)-balls as connected components.

P_2: for every $J \subseteq I$, $\#J = k$, $k \leq n+1$, $\bigcap_{j \in J} B_j = \bigcap_{j \in J} (\partial B_j)$ has (n-k+1)-balls as connected components.

Definition 3. $\mathcal{B}$ is said to be a P_α-ball covering $(\alpha \in \Delta_2)$ of M if P -condition holds.

Obviously $P_2 \Rightarrow P_1 \Rightarrow P_0$.

Definition 4. Let M be an n-manifold with h $(h > 0)$ boundary components M_j $(j \in \mathbb{N}_h)$ and $\mathcal{B} = \{B_i \mid i \in I\}$ be a finite set of closed n-balls such that $M = \bigcup_{i \in I} B_i$. $\mathcal{B}$ is said to be a P_α-ball covering $(\alpha \in \Delta_2)$ of M if $\mathcal{B}$ satisfies the P_α-condition and $\mathcal{B}_j = \{B_i \cap M_j \mid i \in I\}$ is a P_α-ball covering of the closed (n-1)-manifold M_j, for every $j \in \mathbb{N}_h$. Note that a P_0-ball (resp. P_1-ball) covering is a ball covering (resp. strong ball covering) in the sense of [KT], [IY] (resp. [FG]).

As an immediate application of Proposition 3 it follows:

Proposition 10. Every connected n-manifold M with h $(h > 1)$ boundary components M_j $(j \in \mathbb{N}_h)$ admits a P_2-ball covering $\mathcal{B}$, such that:

(') $\#\mathcal{B} = n \cdot h$,

(") $\#\mathcal{B}_j = n$, for every $j \in \mathbb{N}_h$.

Proof. Let K_j be a contracted triangulation of M_j ($j \in \mathbb{N}_h$) and let H be a pseudodissection of M with no interior vertices which induces K_j on each M_j. If H' is the first barycentric subdivision of H, let $B_v = |st(v,H')|$ be the polyhedron underlying the star in H' of a vertex $v \in S_0(H)$. Then $\mathcal{B} = \{B_v \mid v \in S_0(H)\}$ is a P_2-ball covering of M satisfying (′) and (″).

Proposition 10 improves the analogous result in ([FG], Proposition 3 and Remark 6), obtained by making use of a pseudodissection of M with ($n \cdot h+1$) vertices (see Corollary 5).

REFERENCES

[CG] A. Caviochioli, C. Gagliardi: Crystallizations of PL-manifolds with connected boundary, Boll. Un. Mat. Ital. 17-B (1980), 902-917.

[CGr] A. Cavicchioli, L. Grasselli: Minimal atlases of manifolds, Cahiers de Topologie et Geometrie Diff. Categoriques, 26-4 (1985).

[F] M. Ferri: Una rappresentazione delle n-varietà topologiche triangolabili mediante grafi (n+1)-colorati, Boll. Un. Mat. Ital. 13-B (1976), 250-260.

[FG] M. Ferri, C. Gagliardi: Strong ball coverings of manifolds, Atti Sem. Mat. Fis. Univ. Modena 28 (1979), 289-293.

[G] C. Gagliardi: Cobordant crystallizations, Discrete Math. 45 (1983), 61-73.

[H] J.F.P. Hudson: Piecewise linear topology, W.A.Benjamin Inc. New York, Amsterdam, 1969.

[KT] K. Kobayashi, Y. Tsukui: The ball coverings of manifolds, J. Math. Soc. Japan 28 (1976), 133-143.

[IY] H. Ikeda, M. Yamashita: Closed 4-manifolds covered by three 4-balls, Yokohama Math. J. 26 (1978), 109-117.

[HW] P.J. Hilton, S. Wylie: An introduction to algebraic topology-Homology theory, Cambridge Univ. Press, 1960.

[P] M. Pezzana: Sulla struttura topologica delle varietà compatte, Atti Sem. Mat. Fis. Univ. Modena 23 (1974), 269-277.

[Z] E.C. Zeeman: Seminar on combinatorial topology, Inst. Hautes Etudes Sci. Publ. Math., Paris, 1963.

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ОБОБЩЕННЫЕ ГАРМОНИЧЕСКИЕ СЕКУЩИЕ В СПЕЦИАЛЬНЫХ ВИДАХ РАССЛОЕННЫХ РИМАНОВЫХ ПРОСТРАНСТВАХ

Пусть E^4-4-мерное риманово многообразие расслоиваемое проекцией $p:E^4 \to N^2$ на 2-мерные слои M^2, причем N^2-2-мерная база.

Будем принимать, что индексы $a,b = 1,2$; $\lambda, \mu = 3,4$; $i,k = 1,2,3,4$.

Выберем некоторое поле $\left\{e_a, e_\lambda\right\}$ адаптированных реперов на E^4. Пусть $\left\{\omega^a, \omega^\lambda\right\}$ соответствующее полю $\left\{e_a, e_\lambda\right\}$ поле дуальных кореперов таких, что формы ω^a равны нулю на слоях M^2, а уравнение $\omega^\lambda = 0$ определяет подпространство пространства E^4 ортогональное к базе N^2. Формы римановой связности в этом корепере обозначим через ω^i_k. Они удовлетворяют структурному уравнению $d\omega^i = \omega^k \wedge \omega^i_k$. Отсюда

$$(1)\qquad \begin{cases} d\omega^a = \omega^b \wedge \omega^a_b + \omega^\lambda \wedge \omega^a_\lambda, \\ d\omega^\lambda = \omega^a \wedge \omega^\lambda_a + \omega^\mu \wedge \omega^\lambda_\mu, \end{cases}$$

причем

$$(2)\qquad \begin{cases} \omega^a_\lambda = H^a_{\lambda b}\omega^b + H^a_{\lambda\mu}\omega^\mu, \\ \omega^\lambda_a = H^\lambda_{ab}\omega^b + H^\lambda_{a\mu}\omega^\mu. \end{cases}$$

Компоненты $H^a_{\lambda b}$, H^λ_{ab} и $H^a_{\lambda\mu}$, $H^\lambda_{a\mu}$ образуют тензоры. Их геометрический смысл выяснен в [3].

Так как система $\omega^a = 0$ вполне интегрируема, то $H^a_{[\lambda\mu]}=0$.

Применяя к структурным уравнениям (1) соотношения (2) и учитывая симметричность тензора $H^a_{\lambda\mu}$ по нижним индексам они примут вид

$$\left\{\begin{aligned}
d\omega^1 &= \omega^2\wedge\omega^1_2 - H^1_{31}\omega^1\wedge\omega^3 - H^1_{32}\omega^2\wedge\omega^3 - H^1_{41}\omega^1\wedge\omega^4 - H^1_{42}\omega^2\wedge\omega^4,\\
d\omega^2 &= \omega^1\wedge\omega^2_1 - H^1_{32}\omega^1\wedge\omega^3 - H^2_{32}\omega^2\wedge\omega^3 - H^1_{42}\omega^1\wedge\omega^4 - H^2_{42}\omega^2\wedge\omega^4,\\
d\omega^3 &= \omega^4\wedge\omega^3_4 + H^3\omega^1\wedge\omega^2 + H^3_{13}\omega^1\wedge\omega^3 + H^3_{14}\omega^1\wedge\omega^4 + H^3_{23}\omega^2\wedge\omega^3 +\\
&\quad + H^3_{24}\omega^2\wedge\omega^4,\\
d\omega^4 &= \omega^3\wedge\omega^4_3 + H^4\omega^1\wedge\omega^2 + H^3_{14}\omega^1\wedge\omega^3 + H^4_{14}\omega^1\wedge\omega^4 + H^3_{24}\omega^2\wedge\omega^3 +\\
&\quad + H^4_{24}\omega^2\wedge\omega^4,
\end{aligned}\right.\tag{3}$$

где $H^3 = 2H^3_{12}$, $H^4 = 2H^4_{12}$.

В дальнейших рассуждениях для удобства будем пользоваться другой связностью. Пусть дано преобразование формы связности вида

$$\omega^2_1 \to \omega^2_1 + \lambda_1\omega^1 + \lambda_2\omega^2 + \lambda_3\omega^3 + \lambda_4\omega^4,$$

$$\omega^4_3 \to \omega^4_3 + \mu_1\omega^1 + \mu_2\omega^2 + \mu_3\omega^3 + \mu_4\omega^4$$

с коэффициентами λ_i, μ_k. Поэтому $H^1_{32} = H^2_{31}$, $H^1_{42} = H^2_{41}$, $H^3_{14} = H^4_{13}$, $H^3_{24} = H^4_{23}$. Тогда структурные уравнения (3) будут

$$\left\{\begin{aligned}
d\omega^1 &= \omega^2\wedge\omega^1_2 - H^1_{31}\omega^1\wedge\omega^3 - H^1_{32}\omega^2\wedge\omega^3 - H^1_{41}\omega^1\wedge\omega^4 - H^1_{42}\omega^2\wedge\omega^4,\\
d\omega^2 &= \omega^1\wedge\omega^2_1 - H^1_{32}\omega^1\wedge\omega^3 - H^2_{32}\omega^2\wedge\omega^3 - H^1_{42}\omega^1\wedge\omega^4 - H^2_{42}\omega^2\wedge\omega^4,\\
d\omega^3 &= \omega^4\wedge\omega^3_4 + H^3\omega^1\wedge\omega^2 + H^3_{13}\omega^1\wedge\omega^3 + H^3_{14}\omega^1\wedge\omega^4 + H^3_{23}\omega^2\wedge\omega^3 + H^3_{24}\omega^2\wedge\omega^4,\\
d\omega^4 &= \omega^3\wedge\omega^4_3 + H^4\omega^1\wedge\omega^2 + H^3_{14}\omega^1\wedge\omega^3 + H^4_{14}\omega^1\wedge\omega^4 + H^3_{24}\omega^2\wedge\omega^3 + H^4_{24}\omega^2\wedge\omega^4.
\end{aligned}\right.\tag{4}$$

можно рассматривать [4] системы самостоятельных тензоров с компонентами

$$\{H^1_{31}+H^2_{32},\ H^1_{41}+H^2_{42}\},\ \{H^1_{31}-H^2_{32},\ H^1_{32},\ H^1_{41}-H^2_{42},\ H^1_{42}\},$$

$$\{H^3_{13}+H^4_{14},\ H^3_{23}+H^4_{24}\},\ \{H^3_{13}-H^4_{14},\ H^3_{14},\ H^3_{23}-H^4_{24},\ H^3_{24}\},\ \{H^3,\ H^4\}.$$

В дальнейшем будем заниматься случаями, когда некоторые из них не разны нулю, а остальные нули, именно

$-\{H^3, H^4\} \neq 0;$

$-\{H^3_{13} + H^4_{14},\ H^3_{23} + H^4_{24}\} \neq 0;$

$-\{H^3_{13} + H^4_{14},\ H^3_{23} + H^4_{24}\} \neq 0$ и $\{H^3, H^4\} \neq 0.$

Структурные уравнения (4) примут соответственно вид:

$$(5)\quad \begin{cases} d\omega^1 = \omega^2 \wedge \omega^1_2, \\ d\omega^2 = \omega^1 \wedge \omega^2_1, \\ d\omega^3 = \omega^4 \wedge \omega^3_4 + H^3\omega^1 \wedge \omega^2, \\ d\omega^4 = \omega^3 \wedge \omega^4_3 + H^4\omega^1 \wedge \omega^2; \end{cases}$$

$$(6)\quad \begin{cases} d\omega^1 = \omega^2 \wedge \omega^1_2, \\ d\omega^2 = \omega^1 \wedge \omega^2_1, \\ d\omega^3 = \omega^4 \wedge \omega^3_4 - A_1\omega^1 \wedge \omega^3 - A_2\omega^2 \wedge \omega^3, \\ d\omega^4 = \omega^3 \wedge \omega^4_3 - A_1\omega^1 \wedge \omega^4 - A_2\omega^2 \wedge \omega^4; \end{cases}$$

$$(7)\quad \begin{cases} d\omega^1 = \omega^2 \wedge \omega^1_2, \\ d\omega^2 = \omega^1 \wedge \omega^2_1, \\ d\omega^3 = \omega^4 \wedge \omega^3_4 + H^3\omega^1 \wedge \omega^2 - A_1\omega^1 \wedge \omega^3 - A_2\omega^2 \wedge \omega^3, \\ d\omega^4 = \omega^3 \wedge \omega^4_3 + H^4\omega^1 \wedge \omega^2 - A_1\omega^1 \wedge \omega^4 - A_2\omega^2 \wedge \omega^4, \end{cases}$$

причем $H^3_{13} = -A_1$, $H^3_{23} = -A_2$.

Рассмотрим в пространстве N^2 ограниченную область S с гладкой границей ∂S. Пусть F – пространство над S, всевозможных гладких поверхностей на которых

$$(8) \qquad \omega^3 = \xi^3_1\omega^1 + \xi^3_2\omega^2, \quad \omega^4 = \xi^4_1\omega^1 + \xi^4_2\omega^2$$

с коэффициентами ξ^λ_a, то есть сечений над S. Рассмотрим функционал [3] $I = \int_S \Omega$ где $\Omega = \frac{1}{2}\sum_{\substack{a=1,2\\ \lambda=3,4}} (\xi^\lambda_a)^2\omega^1\wedge\ldots\wedge\omega^n$.

Поверхности на которых имеется (8) являющиеся стационарными для функционала I называем обобщенными гармоническими секущими.

Рассмотрим пространство определенное структурными уравнениями (5). Из теоремы 1 работы [3] следует

Теорема 1. Поверхность, на которой имеется (8) является стационарной для функционала I тогда и только тогда, когда $\xi^3_1, \xi^3_2, \xi^4_1, \xi^4_2$ удовлетворяют уравнениям

$$(9) \qquad \begin{cases} \Delta\xi^3_1 + \frac{1}{2}H^3\omega^2 = \xi^3_{11}\omega^1 + \xi^3_{12}\omega^2, \\ \Delta\xi^3_2 - \frac{1}{2}H^3\omega^1 = \xi^3_{12}\omega^1 + \xi^3_{22}\omega^2, \\ \Delta\xi^4_1 + \frac{1}{2}H^4\omega^2 = \xi^4_{11}\omega^1 + \xi^4_{12}\omega^2, \\ \Delta\xi^4_2 - \frac{1}{2}H^4\omega^1 = \xi^4_{12}\omega^1 + \xi^4_{22}\omega^2, \end{cases}$$

причем $\xi^3_{11} + \xi^3_{22} = 0$, $\xi^4_{11} + \xi^4_{22} = 0$.

В этой теореме и в дальнейшем $\Delta\xi^\lambda_a = d\xi^\lambda_a - \xi^\lambda_b\omega^b_a + \xi^\mu_a\omega^\lambda_\mu$.

Обозначая

$$(10) \qquad \xi^3_{11} = \eta^3, \quad \xi^3_{12} = \zeta^3, \quad \xi^4_{11} = \eta^4, \quad \xi^4_{12} = \zeta^4$$

можем уравнения (9) записать в виде

$$\Delta\xi^3_1 + \frac{1}{2} H^3\omega^2 + \eta^3\omega^1 + \zeta^3\omega^2,$$

$$\Delta\xi^3_2 - \frac{1}{2} H^3\omega^1 = \zeta^3\omega^1 - \eta^3\omega^2,$$

$$\Delta\xi^4_1 + \frac{1}{2} H^4\omega^2 = \eta^4\omega^1 + \zeta^4\omega^2,$$

$$\Delta\xi^4_2 - \frac{1}{2} H^4\omega^1 = \zeta^4\omega^1 - \eta^4\omega^2.$$

Дифференцируя их с учетом уравнений возникающих в результате дифференцирования структурных уравнений (5), то есть

$$d\omega^4_3 = -H\omega^1\wedge\omega^2, \quad d\omega^2_1 = K\omega^1\wedge\omega^2,$$

$$dH^3 + H^4\omega^3_4 = H^3_1\omega^1 + H^3_2\omega^2 + H\omega^4,$$

$$dH^4 + H^3\omega^4_3 = H^4_1\omega^1 + H^4_2\omega^2 - H\omega^3$$

приходим к формулам

$$\Delta\eta^3\wedge\omega^1 + \Delta\zeta^3\wedge\omega^2 + \left(\xi^3_2 K - \frac{3}{2}\xi^4_1 H - \frac{1}{2}H^3_1\right)\omega^1\wedge\omega^2 = 0,$$

$$\Delta\zeta^3\wedge\omega^1 - \Delta\eta^3\wedge\omega^2 + \left(-\xi^3_1 K - \frac{3}{2}\xi^4_2 H - \frac{1}{2}H^3_2\right)\omega^1\wedge\omega^2 = 0,$$

$$\Delta\eta^4\wedge\omega^1 + \Delta\zeta^4\wedge\omega^2 + \left(\xi^4_2 K + \frac{3}{2}\xi^3_1 H - \frac{1}{2}H^4_1\right)\omega^1\wedge\omega^2 = 0,$$

$$\Delta\zeta^4\wedge\omega^1 - \Delta\eta^4\wedge\omega^2 + \left(-\xi^4_1 K + \frac{3}{2}\xi^3_2 H - \frac{1}{2}H^4_2\right)\omega^1\wedge\omega^2 = 0,$$

причем

$$(11)\quad \begin{cases} \Delta\eta^3 = d\eta^3 - 2\zeta^3\omega^2_1 - \eta^4\omega^4_3, & \Delta\zeta^3 = d\zeta^3 + 2\eta^3\omega^3_1 - \zeta^4\omega^4_3, \\ \Delta\eta^4 = d\eta^4 - 2\zeta^4\omega^2_1 + \eta^3\omega^4_3, & \Delta\zeta^4 = d\zeta^4 + 2\eta^4\omega^2_1 + \zeta^3\omega^4_3. \end{cases}$$

Так как система уравнений

$$
(12)\qquad \begin{cases}
d\omega^1 = \omega^2 \wedge \omega^1_2, \ d\omega^2 = \omega^1 \wedge \omega^2_1, \\
d\omega^3 = \omega^4 \wedge \omega^3_4 + H^3 \omega^1 \wedge \omega^2, \\
d\omega^4 = \omega^3 \wedge \omega^4_3 + H^4 \omega^1 \wedge \omega^2, \\
d\omega^2_1 = 0, \ d\omega^4_3 = 0, \\
dH^3 + H^4 \omega^3_4 = 0, \ dH^4 + H^3 \omega^4_3 = 0
\end{cases}
$$

вполне интегрируема, то из вышесказанного следует

Теорема 2. Существует пространство определенное системой уравнений (12) и отыскание в нем обобщенных гармонических секущих на которых имеется (8) сводится к решению линейной системы уравнений

$$\Delta\eta^3 \wedge \omega^1 + \Delta\zeta^3 \wedge \omega^2 = 0, \quad \Delta\zeta^3 \wedge \omega^1 - \Delta\eta^3 \wedge \omega^2 = 0,$$

$$\Delta\eta^4 \wedge \omega^1 + \Delta\zeta^4 \wedge \omega^2 = 0, \quad \Delta\zeta^4 \wedge \omega^1 - \Delta\eta^4 \wedge \omega^2 = 0.$$

Справедлива тоже следующая [4]

Теорема 3. Существуют пространства определенные системой уравнений

$$d\omega^1 = \omega^2 \wedge \omega^1_2, \quad d\omega^2 = \omega^1 \wedge \omega^2_1,$$

$$d\omega^3 = \omega^4 \wedge \omega^3_4 + H^3 \omega^1 \wedge \omega^2,$$

$$d\omega^4 = \omega^3 \wedge \omega^4_3 + H^4 \omega^1 \wedge \omega^2,$$

$$d\omega^2_1 = K\omega^1 \wedge \omega^2, \quad d\omega^4_3 = -\frac{2}{3} K\omega^1 \wedge \omega^2,$$

$$dH^3 + H^4\omega^3_4 = H^3_1\omega^1 + H^3_2\omega^2 + \frac{2}{3} K\omega^4, \ dH^4 + H^3\omega^4_3 = -H^3_2\omega^1 + H^3_1\omega^2 - \frac{2}{3}K\omega^3,$$

$K = \mathrm{const}$

и отыскание в них обобщенных гармонических секущих на которых

$$\omega^3 = \left(Y - \frac{H^3_2}{4K}\right)\omega^1 + \left(X + \frac{H^3_1}{4K}\right)\omega^2, \quad \omega^4 = \left(X - \frac{H^3_1}{4K}\right)\omega^1 + \left(-Y - \frac{H^3_2}{4K}\right)\omega^2$$

сводится к решения линейной системы уравнений

$$\Delta Y\wedge\omega^1 + \Delta X\wedge\omega^2 = -H^3\omega^1\wedge\omega^2,$$

$$\Delta Y\wedge\omega^2 - \Delta X\wedge\omega^1 = H^4\omega^1\wedge\omega^2,$$

где $\Delta Y = dY + X(\omega^3_4 - \omega^2_1)$, $\Delta X = dX - Y(\omega^3_4 - \omega^2_1)$.

Сейчас рассмотрим случай, когда структурные уравнения пространства имеют вид (6). Из теоремы 1 работы [3] следует

Теорема 4. Поверхность, на которой имеется (8) является стационарной для функционала I тогда и только тогда, когда ξ^3_1, ξ^3_2, ξ^4_1, ξ^4_2 удовлетворяют системе уравнений

$$(13)\quad \begin{cases} \Delta\xi^3_1 - \xi^3_1A_1\omega^1 - \xi^3_2A_1\omega^2 = \xi^3_{11}\omega^1 + \xi^3_{12}\omega^2, \\ \Delta\xi^3_2 - \xi^3_1A_2\omega^1 - \xi^3_2A_2\omega^2 = \xi^3_{12}\omega^1 + \xi^3_{22}\omega^2, \\ \Delta\xi^4_1 - \xi^4_1A_1\omega^1 - \xi^4_2A_1\omega^2 = \xi^4_{11}\omega^1 + \xi^4_{12}\omega^2, \\ \Delta\xi^4_2 - \xi^4_1A_2\omega^1 - \xi^4_2A_2\omega^2 = \xi^4_{12}\omega^1 + \xi^4_{22}\omega^2, \end{cases}$$

причем $\xi^3_{11} + \xi^3_{22} = 0$, $\xi^4_{11} + \xi^4_{22} = 0$.

Вводя обозначения (10), уравнения (13) примут вид

$$\Delta\xi^3_1 - \xi^3_1A_1\omega^1 - \xi^3_2A_1\omega^2 = \eta^3\omega^1 + \zeta^3\omega^2,$$

$$\Delta\xi^3_2 - \xi^3_1A_2\omega^1 - \xi^3_2A_2\omega^2 = \zeta^3\omega^1 - \eta^3\omega^2,$$

$$\Delta\xi^4_1 - \xi^4_1A_1\omega^1 - \xi^4_2A_1\omega^2 = \eta^4\omega^1 + \zeta^4\omega^2,$$

$$\Delta\xi^4_2 - \xi^4_1A_2\omega^1 - \xi^4_2A_2\omega^2 = \zeta^4\omega^1 - \eta^4\omega^2.$$

Дифференцируя их с учетом уравнений возникающих в результате дифференцирования структурных уравнений (6), то есть уравнений

$$d\omega^4_3 = 0, \quad d\omega^2_1 = K\omega^1 \wedge \omega^2,$$

$$dA_1 - A_2\omega^2_1 = A_{11}\omega^1 + A_{12}\omega^2,$$

$$dA_2 - A_1\omega^1_2 = A_{12}\omega^1 + A_{22}\omega^2$$

и применяя обозначения (11) приходим к формулам

$$\Delta\eta^3 \wedge \omega^1 + \Delta\zeta^3 \wedge \omega^2 + (\xi^3_1(A_1A_2 - A_{12}) + \xi^3_2(K - (A_1)^2 + A_{11}))\omega^1 \wedge \omega^2 = 0,$$

$$\Delta\zeta^3 \wedge \omega^1 - \Delta\eta^3 \wedge \omega^2 + (\xi^3_1(-K + (A_2)^2 - A_{22}) + \xi^3_2(-A_1A_2 + A_{12}))\omega^1 \wedge \omega^2 = 0,$$

$$\Delta\eta^4 \wedge \omega^1 + \Delta\zeta^4 \wedge \omega^2 + (\xi^4_1(A_1A_2 - A_{12}) + \xi^4_2(K - (A_1)^2 + A_{11}))\omega^1 \wedge \omega^2 = 0,$$

$$\Delta\zeta^4 \wedge \omega^1 - \Delta\eta^4 \wedge \omega^2 + (\xi^4_1(-K + (A_2)^2 - A_{22}) + \xi^4_2(-A_1A_2 + A_{12}))\omega^1 \wedge \omega^2 = 0.$$

Так как система уравнений

$$(14) \quad \left\{ \begin{aligned} & d\omega^1 = \omega^2 \wedge \omega^1_2, \quad d\omega^2 = \omega^1 \wedge \omega^2_1, \\ & d\omega^3 = \omega^4 \wedge \omega^3_4 - A_1\omega^1 \wedge \omega^3 - A_2\omega^2 \wedge \omega^3, \\ & d\omega^4 = \omega^3 \wedge \omega^4_3 - A_1\omega^1 \wedge \omega^4 - A_2\omega^2 \wedge \omega^4, \\ & d\omega^2_1 = K\omega^1 \wedge \omega^2, \quad d\omega^4_3 = 0, \\ & dA_1 - A_2\omega^2_1 = (-K + (A_1)^2)\omega^1 + A_2A_2\omega^2, \\ & dA_2 - A_1\omega^1_2 = A_1A_2\omega^1 + (-K + (A_1)^2)\omega^2, \\ & K = \text{const} \end{aligned} \right.$$

вполне интегрируема, то имеет место

Теорема 5. Существуют пространства определенные системой уравнений (14) и отыскание в них обобщенных гармонических секущих на которых имеется (8) сводится к решению линейной системы уравнений

$$\Delta\eta^3\wedge\omega^1 + \Delta\zeta^3\wedge\omega^2 = 0,\quad \Delta\zeta^3\wedge\omega^1 - \Delta\eta^3\wedge\omega^2 = 0,$$

$$\Delta\eta^4\wedge\omega^1 + \Delta\zeta^4\wedge\omega^2 = 0,\quad \Delta\zeta^4\wedge\omega^1 - \Delta\eta^4\wedge\omega^2 = 0.$$

Сейчас рассмотрим пространство определенное системой уравнений (7). Из теоремы 1 работы [3] следует

Теорема 6. Поверхность на которой имеется (8) является стационарной для функционала I тогда и только тогда, когда ξ^3_1, ξ^3_2, ξ^4_1, ξ^4_2 удовлетворяют системе уравнений

$$\Delta\xi^3_1 + \frac{1}{2} H^3\omega^2 - \xi^3_1 A_1\omega^1 - \xi^3_2 A_1\omega^2 = \xi^3_{11}\omega^1 + \xi^3_{12}\omega^2,$$

$$\Delta\xi^3_2 - \frac{1}{2} H^3\omega^1 - \xi^3_1 A_2\omega^1 - \xi^3_2 A_2\omega^2 = \xi^3_{12}\omega^1 + \xi^3_{22}\omega^2,$$

$$\Delta\xi^4_1 + \frac{1}{2} H^4\omega^2 - \xi^4_1 A_1\omega^1 - \xi^4_2 A_1\omega^2 = \xi^4_{11}\omega^1 + \xi^4_{12}\omega^2,$$

$$\Delta\xi^4_2 - \frac{1}{2} H^4\omega^1 - \xi^4_1 A_2\omega^1 - \xi^4_2 A_2\omega^2 = \xi^4_{12}\omega^1 + \xi^4_{22}\omega^2,$$

причем $\xi^3_{11} + \xi^3_{22} = 0,\ \ \xi^4_{11} + \xi^4_{22} = 0.$

Аналогично как в прежних случаях, приходим к [4]

Теорема 7. Существуют пространства определенные системой уравнений

$$d\omega^1 = \omega^2\wedge\omega^1_2,\quad d\omega^2 = \omega^1\wedge\omega^2_1,$$

$$d\omega^3 = \omega^4\wedge\omega^3_4 + H^3\omega^1\wedge\omega^2 - A_1\omega^1\wedge\omega^3 - A_2\omega^2\wedge\omega^3,$$

$$d\omega^4 = \omega^3\wedge\omega^4_3 + H^4\omega^1\wedge\omega^2 - A_1\omega^1\wedge\omega^4 - A_2\omega^2\wedge\omega^4,$$

$$d\omega^2_1 = K\omega^1\wedge\omega^2,\quad d\omega^4_3 = 0,$$

$$dH^3 + H^4\omega^3_4 = 2H^3A_1\omega^1 + 2H^3A_2\omega^2,$$

$$dH^4 + H^3\omega_3^4 = 2H^4A_1\omega^1 + 2H^4A_2\omega^2,$$

$$dA_1 - A_2\omega_1^2 = (-K + (A_1)^2)\omega^1 + A_1A_2\omega^2,$$

$$dA_2 - A_1\omega_2^1 = A_1A_2\omega^1 + (-K + (A_2)^2)\omega^2, \quad K = \mathrm{const}$$

и отыскание в них обобщенных гармонических секущих на которых имеется (8), сводится к решению линейной системы уравнений

$$\Delta\eta^3 \wedge \omega^1 + \Delta\zeta^3 \wedge \omega^2 = 0, \quad \Delta\zeta^3 \wedge \omega^1 - \Delta\eta^3 \wedge \omega^2 = 0,$$

$$\Delta\eta^4 \wedge \omega^1 + \Delta\zeta^4 \wedge \omega^2 = 0, \quad \Delta\zeta^4 \wedge \omega^1 - \Delta\eta^4 \wedge \omega^2 = 0.$$

Сейчас рассмотрим пространства, являющиеся обобщениями пространства определенных системой уравнений (5), а именно пространства со структурными уравнениями

$$d\omega^1 = \omega^b \wedge \omega_b^a,$$

$$d\omega^\lambda = \omega^\mu \wedge \omega_\mu^\lambda + 2H^\lambda_{[ab]}\omega^a \wedge \omega^b,$$

$$a,b,c = 1,\dots,n; \quad \lambda, \mu = n+1, n+2.$$

Будем в них искать обобщенные гармонические секущие на которых

(15) $\omega^{n+1} = \xi_a^{n+1}\omega^a, \quad \omega^{n+2} = \xi_a^{n+2}\omega^a$.

Форма Ω имеет сейчас вид $\Omega = \frac{1}{2} \sum\limits_{\substack{a=1,\dots,n \\ \lambda=n+1,n+2}} (\xi_a^\lambda)^2 \omega^1 \wedge \dots \wedge \omega^n.$

Из теоремы 1 работы [3] следует

Теорема 8. Поверхность на которой имеется (15), является стационарной для функционала I тогда и только тогда, когда ξ_a^λ удовлетворяют уравнениям $\Delta\xi_a^\lambda + H^\lambda_{[ab]}\omega^b = \xi^\lambda_{ab}\omega^b$ причем

$$\Delta\xi_a^\lambda = d\xi_a^\lambda + \xi_a^\mu\omega_\mu^\lambda - \xi_b^\lambda\omega_a^b, \quad \xi_{ab}^\lambda = \xi_{ba}^\lambda, \quad \sum_a \xi_{aa}^\lambda = 0.$$

Аналогично как в 4-мерном случае приходим к

Теорема 9. Существует пространство определенное системой уравнений

$$d\omega^a = \omega^b \wedge \omega^a_b,$$

$$d\omega^\lambda = \omega^\mu \wedge \omega^\lambda_\mu + 2H^\lambda_{[ab]}\,\omega^a \wedge \omega^b,$$

$$d\omega^a_b = \omega^c_b \wedge \omega^a_c, \quad d\omega^\lambda_\mu = 0,$$

$$dH^\lambda_{[ab]} + H^\mu_{[ab]}\,\omega^\lambda_\mu - H^\lambda_{[cb]}\,\omega^c_a - H^\lambda_{[ac]}\,\omega^c_b = 0$$

и отыскание в нем обобщенных гармонических секущих на которых имеется (15), сводится к решению линейной системы уравнений $\Delta\xi^\lambda_{ab} \wedge \omega^b = 0$, причем $\xi^\lambda_{ab} = \xi^\lambda_{ba}$, $\sum_a \xi^\lambda_{aa} = 0$.

ЛИТЕРАТУРА

[1] А.И. Плужников: О гармонических отображениях римановых поверхностей и расслоенных многообразий, Mat. Sb. 113 (1980) 339-347.

[2] I. Eells, L. Lemaire: A report on harmonic maps, Bull. London Math. Soc. 10 (1978) 1-68.

[3] H. Kleczewska: Обобщенные гармонические секущие в расслоенном римановом пространстве, Demonstratio Math. 19 (1986) 785-790.

[4] H. Kleczewska: Расслоенные римановые пространства и обобщенные гармонические секущие в них, Doctoral Thesis, Univ. Łomonosova, Moskwa 1985.

[5] L. Lemaire: Applications harmoniques de varietes a bord, C.R. Acad. Sci. Paris A, 279 (1974) 925-927.

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О ФУНДАМЕНТАЛЬНОМ РЕШЕНИИ ПАРАБОЛИЧЕСКОГО УРАВНЕНИЯ С МАЛЫМИ ПАРАМЕТРАМИ

В этой работе изучаем асимптотику решения $p_{\delta_1\delta_2}(t,x,y)$ задачи

$$(1)\quad \begin{cases} \frac{\partial p}{\partial t} = \frac{1}{2}\Delta_M p - \frac{1}{\delta_1}\left[B(x), \nabla_x\right]p - \frac{1}{\delta_2}q(x)p, \quad p = p(t,x,y), \\ p(0,x,y) = \delta_y(x), \end{cases}$$

с параметрами $\delta_1 = \delta_1(t) \to 0$, $\delta_2 = \delta_2(t) \to 0$ при $t \to 0$ такими, что $\frac{t}{\delta_1} \to k_1$, $\frac{t^2}{\delta_2} \to k_2$, $k_1, k_2 > 0$. Здесь $B(x)$ - вектор-функция и $q(x)$ - скалярная функция достаточной гладкости определены на ν-мерном римановом многообразии M, с краем или без края. Символ $[\cdot,\cdot]$ обозначает скалярное произведение векторов. На многообразии M задана метрическая форма

$$ds^2 = \sum_{i,j=1}^{\nu} g_{ij}(x)dx^i dx^j,$$

которая определяет оператор Лапласа-Бельтрами

$$\Delta_M = \left(\det\{g_{ij}\}\right)^{-\frac{1}{2}} \sum_{i,j=1}^{\nu} \frac{\partial}{\partial x^i}\left[g^{ij}\left(\det\{g_{ij}\}\right)^{\frac{1}{2}} \frac{\partial}{\partial x^j}\right].$$

Символ ∇_x - обозначает оператор набла по отношению к x. Пусть (см. [1]) на множестве непрерывных кривых $\varphi(s)$ на M, $s \in [0,1]$, $\varphi(0) = x$, $\varphi(1) = y$, определен функционал

$$I(\varphi) = \frac{1}{2} \int_0^1 \left| \dot{\varphi}(\tau) + k_1 B(\varphi(\tau)) \right|^2 d\tau + k_2 \int_0^1 q(\varphi(\tau)) d\tau,$$

где $|\cdot|$ обозначает длину вектора согласно метрике ds^2 в касательном пространстве к M.

Теорема. Пусть:

а) точки $x, y \in M$ расположены столь близко, что существует регулярная система координат в некоторой карте U содержащей x и y,

б) существует единственная экстремаль $\hat{\varphi}$ функционала $I(\varphi)$ класса не меньше C^3 с условиями $\varphi(0) = x$, $\varphi(1) = y$, $I(\hat{\varphi}) = \inf I(\varphi) = V(x,y)$, причем точки x, y не сопряжены вдоль $\hat{\varphi}$,

в) функции $B(x)$, $q(x)$, $x \in M$, класса не меньше C^3,

г) $\frac{t}{\delta_1(t)} \to k_1$, $\frac{t^2}{\delta_2(t)} \to k_2$ при $t \to 0$, $\delta_1, \delta_2 \to 0$, $k_1, k_2 > 0$.

Тогда задача (1) имеет решение

$$p_{\delta_1 \delta_2}(t,x,y) \underset{t \to 0}{\sim} \frac{\Phi(x,y)}{(2\pi t)^{\nu/2}} \exp\left[-\frac{V(x,y)}{t} - C_1(x,y) k_1 - C_2(x,y) k_1^2 - C_3(x,y) k_2 \right],$$

где, согласно результатам в [4],

$$\Phi(x,y) = \rho^{\frac{\nu-1}{2}}(x,y) \left[\det \int_0^{\rho(x,y)} \bar{g}^{-1}(\tau, 0) d\tau \right]^{-\frac{1}{2}},$$

$\rho(x,y)$ - длина дуги экстремали $\hat{\varphi}$ соединяющей точки x, y, $\bar{g}^{-1}(\tau_0, u)$ - обратная матрица метрического тензора $\bar{g}_{ij}(\tau_0, u)$, $i,j = 1,2,\ldots,\nu-1$, в полугеодезической системе координат (τ, u) в слое $\tau = \tau_0$.

Доказательство: Для простоты доказательства ограничимся случаем, когда $M = R^{\nu}$; тогда $\Delta_M = \Delta_x$ – оператор Лапласа в R^{ν} и согласно условиям теоремы наша задача имеет вид

$$(2)\quad \begin{cases} \dfrac{\partial \bar{p}}{\partial s} = \dfrac{1}{2}\,\Delta_x \bar{p} - \dfrac{k_1}{t}\left[B,\nabla_x\right]\bar{p} - \dfrac{k_2}{t^2}\,q\bar{p}, \quad \bar{p} = \bar{p}(s,x,y),\ s \leqslant t, \\[2ex] \bar{p}(0,x,y) = \delta_y(x). \end{cases}$$

Пусть D_y – некоторая область точки y. Тогда, положив

$$\int_{D_y} \bar{p}_{\delta_1\delta_2}(s,x,y)\,dy = \mu_s(x,D_y),$$

по формуле Каца-Фейнмана (см. [2]), получим

$$(3)\qquad \mu_s(x,D_y) = M_x^X \exp\left[-\frac{k_2}{t^2}\int_0^s q(X_u)\,du\right]\chi_{D_y}(X_s),$$

где X_s – процесс с инфинитезимальным оператором

$$A = \frac{1}{2}\,\Delta_x - \frac{k_1}{t}\left[B,\nabla_x\right]$$

и соответствующим дифференциальным уравнением

$$dX_s = dw_s - \frac{k_1}{t}\,B(X_s)\,ds,$$

w_s – винеровский процесс.

Здесь удобно воспользоваться процессом

$$Y_s = w_s + \hat{\varphi}\left(\frac{s}{t}\right)$$

таким, что $Y_0 = x$. Очевидно, что

$$dY_s = dw_s + \frac{1}{t}\,\dot{\hat{\varphi}}\left(\frac{s}{t}\right)ds,$$

где $\dot{\hat{\varphi}}\left(\frac{s}{t}\right) = \dfrac{d}{d\tau}\,\hat{\varphi}(\tau)$, $\tau = \dfrac{s}{t}$.

Согласно теореме Камерони-Мартина-Гирсанова (см. [3]) в формуле (3) произведем замену меры процесса X_s на меру процесса Y_s.

Тогда

$$\mu_t(x, D_y) = M_x^Y \exp\left[-\frac{k_2}{t^2}\int_0^t q(Y_s)\,ds\right]\chi_{D_y}(Y_t)\,\frac{dP^X}{dP^Y}(Y_\circ) =$$

$$= M_x^Y \exp\left[-\frac{k_2}{t^2}\int_0^t q(Y_s)\,ds\right]\chi_{D_0}(w_t)\,\frac{dP^X}{dP^Y}(Y_\circ),$$

где

$$(4)\qquad \frac{dP^X}{dP^Y}(Y_\circ) = \exp\left\{-\frac{1}{t}\int_0^t\left[\dot{\hat\varphi}\left(\frac{s}{t}\right) + k_1 B(Y_s),\ dw_s\right] - \right.$$

$$\left. - \frac{1}{2t^2}\int_0^t\left|\dot{\hat\varphi}\left(\frac{s}{t}\right) + k_1 B(Y_s)\right|^2 ds\right\},$$

D_0 - область начала координат, которая является сдвигом области D_y на вектор y.

Очевидно, что (см. [4]) решение задачи (2)

$$(5)\qquad \bar p_{\delta_1\delta_2}(t,x,y) = \lim_{D_y \to y}\frac{\mu_t(x, D_y)}{\operatorname{mes} D_y} =$$

$$= \frac{1}{(2\pi t)^{\frac{\nu}{2}}} M_0^{\hat Y} \exp\left[-\frac{k_2}{t^2}\int_0^t q(\hat Y_s)\,ds\right]\frac{dP^X}{dP^{\hat Y}}(\hat Y_\circ),$$

где $\hat Y_s = \hat w_s + \hat\varphi\left(\frac{s}{t}\right)$, $\hat w_s$ - условный процесс w_s такой, что $w_0 = 0$, $w_t = 0$.

Теперь займемся преобразованием формулы (4) с подстановкой $Y_s = \hat Y_s$ и преобразованием интеграла в экспоненте формулы (5). Для этого нам понадобятся разложения в ряд Тейлора по степеням $\hat w_s$ следующих функций:

(6а) $$B\left(\hat{w}_s + \hat{\varphi}\left(\frac{s}{t}\right)\right) = B\left(\hat{\varphi}\left(\frac{s}{t}\right)\right) + \left[\hat{w}_s, \nabla_x\right] B\left(\hat{\varphi}\left(\frac{s}{t}\right)\right) + O(|\hat{w}_s|^2),$$

(6б) $$\left|\dot{\hat{\varphi}}\left(\frac{s}{t}\right) + k_1 B\left(\hat{w}_s + \hat{\varphi}\left(\frac{s}{t}\right)\right)\right|^2 = \left|\dot{\hat{\varphi}}\left(\frac{s}{t}\right) + k_1 B\left(\hat{\varphi}\left(\frac{s}{t}\right)\right)\right|^2 +$$
$$+ \left[\hat{w}_s, \nabla_x\right] \left|\dot{\hat{\varphi}}\left(\frac{s}{t}\right) + k_1 B\left(\hat{\varphi}\left(\frac{s}{t}\right)\right)\right|^2 + \frac{1}{2}\left[\hat{w}_s, \nabla_x\right]^2 \left|\dot{\hat{\varphi}}\left(\frac{s}{t}\right) +\right.$$
$$\left. + k_1 B\left(\hat{\varphi}\left(\frac{s}{t}\right)\right)\right|^2 + O(|\hat{w}_s|^3),$$

(6в) $$q\left(\hat{w}_s + \hat{\varphi}\left(\frac{s}{t}\right)\right) = q\left(\hat{\varphi}\left(\frac{s}{t}\right)\right) + \left[\hat{w}_s, \nabla_x\right] q\left(\hat{\varphi}\left(\frac{s}{t}\right)\right) +$$
$$+ \frac{1}{2}\left[\hat{w}_s, \nabla_x\right]^2 q\left(\hat{\varphi}\left(\frac{s}{t}\right)\right) + O(|\hat{w}_s|^3).$$

Учитывая (6а) и интегрируя по частям в первом интеграле экспоненты формулы (4), получим

(7а) $$-\frac{1}{t}\int_0^t \left[\left(\dot{\hat{\varphi}}\left(\frac{s}{t}\right) + k_1 B\left(\hat{w}_s + \hat{\varphi}\left(\frac{s}{t}\right)\right)\right), d\hat{w}_s\right] =$$
$$= \frac{1}{t}\int_0^t \left[\frac{d}{ds}\left(\dot{\hat{\varphi}}\left(\frac{s}{t}\right) + k_1 B\left(\hat{\varphi}\left(\frac{s}{t}\right)\right)\right), \hat{w}_s\right] ds -$$
$$- \frac{k_1}{t}\int_0^t \left[\left[\hat{w}_s, \nabla_x\right] B\left(\hat{\varphi}\left(\frac{s}{t}\right)\right), d\hat{w}_s\right] - \frac{k_1}{t}\int_0^t O(|\hat{w}_s|^2)\, d\hat{w}_s \underset{t\to 0}{\sim}$$
$$\underset{t\to 0}{\sim} \frac{1}{\sqrt{t}}\int_0^1 \left[\frac{d}{d\tau}\left(\dot{\hat{\varphi}}(\tau) + k_1 B(\hat{\varphi}(\tau))\right), \hat{w}_\tau\right] d\tau - k_1 C_1'(x,y).$$

Здесь заметим, что первое слагаемое результата мы получили применяя подстановку $\frac{s}{t} = \tau$ и пользуясь свойством автомодельности винеровского процесса. Второе слагаемое при той же подстановке имеет результат, не зависящий от t, который обозначаем $k_1 C_1'(x,y)$. Третье слагаемое согласно неравенству Чебышева при $t \to 0$ стремится к нулю с вероятностью 1. Так как

$$M\left|\frac{1}{t}\int_0^t O(|\hat{w}_s|^2)\,d\hat{w}_s\right|^2 = \frac{1}{t^2}\int_0^t M(O(|\hat{w}_s|^2))^2\,ds \leqslant$$

$$\leqslant \frac{1}{t^2}\int_0^t Cs^2\,ds = \frac{1}{t^2}\,C^1t^3, \quad C^1 > 0.$$

Далее, учитывая (6б) и применяя подстановку $\frac{s}{t} = \tau$, второй интеграл в экспоненте формулы (4) имеет вид

$$(7б) \qquad -\frac{1}{2t^2}\int_0^t \left|\hat{\varphi}\left(\frac{s}{t}\right) + k_1B\left(\hat{w}_s + \hat{\varphi}\left(\frac{s}{t}\right)\right)\right|^2 ds =$$

$$= -\frac{1}{2t}\int_0^1 \left|\dot{\hat{\varphi}}(\tau)+k_1B(\hat{\varphi}(\tau))\right|^2 d\tau - \frac{1}{2t^2}\int_0^t \left[\hat{w}_s, \nabla_x\right]\left|\dot{\hat{\varphi}}\left(\frac{s}{t}\right)+k_1B\left(\hat{\varphi}\left(\frac{s}{t}\right)\right)\right|^2 ds -$$

$$-\frac{1}{4t^2}\int_0^t \left[\hat{w}_s, \nabla_x\right]^2\left|\dot{\hat{\varphi}}\left(\frac{s}{t}\right)+k_1B\left(\hat{\varphi}\left(\frac{s}{t}\right)\right)\right|^2 ds - \frac{1}{2t^2}\int_0^t O(|\hat{w}_s|^3)\,ds \underset{t\to 0}{\sim}$$

$$\underset{t\to 0}{\sim} -\frac{1}{2t}\int_0^1 \left|\dot{\hat{\varphi}}(\tau)+k_1B(\hat{\varphi}(\tau))\right|^2 d\tau - \frac{1}{2\sqrt{t}}\int_0^1 \left[\hat{w}_\tau, \nabla_x\right]\left|\dot{\hat{\varphi}}(\tau)+k_1B(\hat{\varphi}(\tau)\right|^2 d\tau -$$

$$- k_1C_1''(x,y) - k_1^2C_2(x,y),$$

где

$$k_1C_1''(x,y) + k_1^2C_2(x,y) = \frac{1}{4t^2}\int_0^t \left[\hat{w}_s, \nabla_x\right]^2\left|\dot{\hat{\varphi}}\left(\frac{s}{t}\right) + k_1B\left(\hat{\varphi}\left(\frac{s}{t}\right)\right)\right|^2 ds.$$

Здесь согласно неравенству Чебышева

$$\frac{k_2}{2t^2}\int_0^t O(|\hat{w}_s|^3)\,ds \to 0$$

при $t \to 0$ с вероятностью 1. Так как

$$M\left|\frac{1}{t^2}\int_0^t O(|\hat{w}_s|^3)ds\right|^2 \leq M\left|\frac{1}{t^2}\int_0^t Cs^{\frac{3}{2}}\,ds\right|^2 = C^1 t, \quad C^1 > 0.$$

И, наконец, учитывая (6в) интеграл формулы (5) преобразуется следующим образом

$$(7в)\quad -\frac{k_2}{t^2}\int_0^t q\left(\hat{w}_s+\hat{\varphi}\left(\frac{s}{t}\right)\right)ds = -\frac{k_2}{t^2}\int_0^t q\left(\hat{\varphi}\left(\frac{s}{t}\right)\right)ds - \frac{k_2}{t^2}\int_0^t\left[\hat{w}_s, \nabla_x\right]q(\hat{\varphi})ds -$$

$$-\frac{k_2}{t^2}\int_0^t\left[\hat{w}_s, \nabla_x\right]^2 q\left(\hat{\varphi}\left(\frac{s}{t}\right)\right)ds - \frac{k_2}{t^2}\int_0^t O(|\hat{w}_s|^3)ds \underset{t\to 0}{\sim}$$

$$\underset{t\to 0}{\sim} -\frac{k_2}{t}\int_0^1 q(\hat{\varphi}(\tau))d\tau - \frac{k_2}{\sqrt{t}}\int_0^1\left[w_\tau, \nabla_x\right]q(\hat{\varphi}(\tau))d\tau - C_3(x,y)k_2.$$

Здесь

$$C_3(x,y) = \frac{1}{t^2}\int_0^t\left[\hat{w}_s, \nabla_x\right]^2 q\left(\hat{\varphi}\left(\frac{s}{t}\right)\right)ds,$$

$$\frac{k_2}{t^2}\int_0^t O(|\hat{w}_s|^3)ds \to 0 \quad \text{при} \quad t \to 0$$

с вероятностью 1.

Теперь, собирая результаты (5), (7а), (7б), (7в) и учитывая формулу Эйлера для экстремали $\hat{\varphi}$ функционала $I(\varphi)$ в виде

$$\frac{d}{d\tau}\left(\dot{\hat{\varphi}}(\tau)+k_1B(\hat{\varphi}(\tau))\right) = \frac{1}{2}\nabla_x\left|\dot{\hat{\varphi}}(\tau)+k_1B(\hat{\varphi}(\tau))\right|^2 + k_2\nabla_x\, q(\hat{\varphi}(\tau)),$$

$$x = \hat{\varphi}(\tau),$$

получим решение задачи (2) в виде

$$\bar{p}_{\delta_1\delta_2}(t,x,y) \underset{t\to 0}{\sim} \frac{1}{(2\pi t)^{\frac{\nu}{2}}} \exp\Big[-\frac{V(x,y)}{t} - C_1(x,y)k_1 - C_2(x,y)k_1^2 - C_3(x,y)k_2\Big],$$

где

$$C_1(x,y) = C_1'(x,y) + C_1''(x,y).$$

В общем случае для оператора Δ_M и соответствующих уравнений

$$dX_s = d\xi_s - \frac{k_1}{t} B(X_s)ds,$$

$$Y_s = \xi_s + \hat{\varphi}\Big(\frac{s}{t}\Big), \quad Y_0 = x,$$

формула (5) имеет вид

$$p_{\delta_1\delta_2}(t,x,y) = \frac{\Phi(x,y)}{(2\pi t)^{\frac{\nu}{2}}} M_0^{\hat{Y}} \exp\Big[-\frac{k_2}{t}\int_0^t q(\hat{Y}_s)ds\Big]\frac{dP^x}{dP^{\hat{Y}}}(\hat{Y}.),$$

где $\hat{Y}_s = \hat{\xi}_s + \hat{\varphi}\big(\frac{s}{t}\big)$, $\hat{\xi}_s$ - условный гауссовский процесс ξ_s такой, что $\xi_0 = 0$, $\xi_t = 0$. В дальнейших выкладках функции $C_1(x,y)$, $C_2(x,y)$, $C_3(x,y)$ вычисляются аналогичным образом как в случае R^ν с оператором Δ_x только с той разницей, что на место процесса $\hat{w}_s$ появляется процесс $\hat{\xi}_s$.

Автор выражает признательность С.А.Молчанову за ценные беседы, которые стимулировали появление этой работы.

ЛИТЕРАТУРА

[1] А.Д. Вентцель, М.И. Фрейдлин: О малых случайных возмущениях динамических систем, Uspiehi Mat. Nauk 24 (1970) 3-55.

[2] М. Рид, Б. Саймон: Методы современной математической физики. М., т.2, 1978.

[3] Г. Маккин : Стохастические интегралы. Москва, 1972.

[4] С.А. Молчанов : Диффузионные процессы и риманова геометрия, Uspiehi Mat. Nauk 30 (1975) 3-59.

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Ewa Kubicka, Tadeusz Poreda

ON THE PARAMETRIC REPRESENTATION OF STARLIKE MAPS OF THE UNIT BALL IN C^N INTO C^N

In this paper we give the parametric representation of a starlike map of the unit ball B^n into C^n. Besides, we prove the existence and the uniqueness of the solution of the equation $Df(z)h(z) = f(z)$ (see [6], Th.1), where $h \in M$, among locally biholomorphic maps satisfying the conditions $f(0) = 0$, $Df(0) = I$. The final part of the paper includes an estimation of the norm of the starlike map.

Let C^n stand for the space of n complex variables $z = (z_1, z_2, \ldots, z_n)$, $z_j \in C$, $j = 1,2,\ldots,n$. The scalar product and the norm will be denoted by $\langle z,w \rangle = \sum_{j=1}^{n} z_j \overline{w}_j$ and $\|z\| = \sqrt{\langle z,z \rangle}$, respectively. The open ball $\{z; \|z-a\| < r\}$ is denoted by $B^n(a;r)$, the unit ball $B^n(0;1)$, for short, by B^n.

The letter I will always represent the identity map on C^n.

The class of holomorphic maps

$$f(z) = (f_1(z), \ldots, f_n(z)), \qquad z \in \Omega,$$

of a domain Ω (contained in C^n) into C^n is denoted by $H(\Omega)$. A function $f \in H(\Omega)$ is called locally biholomorphic in Ω if its derivative

$$Df(z) = \left(\frac{\partial f_j(z)}{\partial z_k}\right), \qquad 1 \leqslant j, \quad k \leqslant n,$$

is non-singular at each point $z \in \Omega$. Gurganus in his paper ([2] Th. 3) gave a necessary and sufficient condition for a map f to be starlike.

A map $v(z) \in H(B^n)$ is called a Schwarz function if $\|v(z)\| \leqslant \|z\|$ for all $z \in B^n$ or, which is equivalent (by Schwarz's lemma), $v(0) = 0$ and $\|v(z)\| \leqslant 1$ for all $z \in B^n$.

Let us now define the following class of functions

$$M = \{h \in H(B^n);\ h(0) = 0,\ \ \mathrm{Re} < h(z), z > \ > 0,\ \ z \in B^n,\ \ z \neq 0,\ \ Dh(0) = I\}.$$

Gurganus [2] analyses the problem

$$\frac{\partial v}{\partial t} = -h(v), \quad v(0) = z,$$

where $h \in M$, $z \in B^n$. From Lemma 2 of [2] it follows that the solution of this problem is determined for $t \geqslant 0$ and any $z \in B^n$. Consequently, such a solution v is a function of two variables (z,t), with $z \in B^n$, $t \geqslant 0$. In the sequel, the problem will be written down in the form

$$\frac{\partial v(z,t)}{\partial t} = -h(v(z,t)), \quad v(z,0) = z, \quad z \in B^n, \quad t \geqslant 0.$$

Lemma 1. Let $h \in M$. Then the problem

$$(1) \qquad \frac{\partial v(z,t)}{\partial t} = -h(v(z,t)), \quad v(z,0) = z, \quad z \in B^n, \quad t \geqslant 0,$$

possesses exactly one solution $v = v(z,t)$ determined for $z \in B^n$ and all $t \geqslant 0$; for a fixed $t_0 \geqslant 0$ the function $v(z,t_0)$ is a univalent Schwarz function on B^n.

The proof of Lemma 1 runs similarly as that of Theorem 2.1 of [4].

The following lemma follows directly from Th. IX 5′ of [3].

Lemma 2. The solution $v = v(z,t)$, $z \in B^n$, $t \geqslant 0$, of problem (1) is a function infinitely many times differentiable with respect to the group of variables $(z,t) \in B^n \times [0,\infty)$.

Remark 1. If $v = v(z,t)$, $z \in B^n$, $t \geqslant 0$, is the solution of problem (1), then for $z \in B^n$, $t \geqslant 0$, the following inequalities take place

$$\frac{\|v(z,t)\|}{(1-\|v(z,t)\|)^2} \leqslant e^{-t} \frac{\|z\|}{(1-\|z\|)^2},$$

$$\frac{\|v(z,t)\|}{(1+\|v(z,t)\|)^2} \geqslant e^{-t} \frac{\|z\|}{(1+\|z\|)^2}.$$

These inequalities follow at once from Lemma 2.2 of [4].

Theorem 1. If $v = v(z,t)$, $z \in B^n$, $t \geqslant 0$, is the solution of problem (1), then the function v satisfies the differential equation

$$\frac{\partial v(z,t)}{\partial z} \cdot h(z) = -\frac{\partial v(z,t)}{\partial t} \quad \text{for } z \in B^n, \quad t \geqslant 0.$$

Proof. To begin with, let us introduce the following notation

$$H(z,t) = \left(\frac{\partial v(z,t)}{\partial z}\right)^{-1} \cdot \left(-\frac{\partial v(z,t)}{\partial t}\right), \quad z \in B^n, \quad t \geqslant 0.$$

The above definition of the function H is correct, because, by Lemma 1, $v(z,t)$ is a univalent Schwarz function for any $t_0 \geqslant 0$, and thus, $\frac{\partial v(z,t)}{\partial z}$ is a non-singular matrix (see Th.15.1.8 of [5]).

Since v satisfies problem (1), therefore, in virtue of Lemma 2, we obtain the following equalities

$$(2) \quad \begin{cases} \dfrac{\partial^2 v(z,t)}{\partial t^2} = -Dh(v(z,t)) \dfrac{\partial v(z,t)}{\partial t}, \\[2ex] \dfrac{\partial^2 v(z,t)}{\partial z \partial t} = -Dh(v(z,t)) \dfrac{\partial v(z,t)}{\partial z}, \end{cases}$$

for $z \in B^n$, $t \geqslant 0$.

It follows from the definition of the function H that it is differentiable with respect to the parameter t, thus

$$(3)\qquad -\frac{\partial^2 v(z,t)}{\partial t^2} = \frac{\partial^2 v(z,t)}{\partial t\partial z}\cdot H(z,t) + \frac{\partial v(z,t)}{\partial z}\cdot\frac{\partial H(z,t)}{\partial t}$$

for $z\in B^n$, $t\geqslant 0$.

By Lemma 2 and Lemma VII 5 of [3], the equality

$$\frac{\partial^2 v}{\partial t\partial z} = \frac{\partial^2 v}{\partial z\partial t}$$

takes place. This and equalities (2) and (3) imply that

$$Dh(v(z,t))\cdot\frac{\partial v(z,t)}{\partial t} =$$

$$= -Dh(v(z,t))\cdot\frac{\partial v(z,t)}{\partial z}\cdot H(z,t) + \frac{\partial v(z,t)}{\partial z}\cdot\frac{\partial H(z,t)}{\partial t}$$

for $z\in B^n$, $t\geqslant 0$.

From this equality and the fact that v is the solution of (1) we get

$$\frac{\partial v(z,t)}{\partial z}\cdot\frac{\partial H(z,t)}{\partial t} = 0,\quad z\in B^n,\ t\geqslant 0.$$

Since $\frac{\partial v(z,t)}{\partial z}$ is a non-singular matrix, therefore, $\frac{\partial H(z,t)}{\partial t} = 0$ for $z\in B^n$, $t\geqslant 0$, that is $H(z,t) = g(z)$ for $z\in B^n$, $t\geqslant 0$, where g is some analytic function in B^n. So,

$$\frac{\partial v(z,t)}{\partial z}\cdot g(z) = -\frac{\partial v(z,t)}{\partial t}\quad\text{for}\quad z\in B^n,\ t\geqslant 0.$$

From the facts that $v(z,0) = z$ and $\frac{\partial v(z,0)}{\partial z} = -h(z)$ it follows that $g(z) = h(z)$ for $z\in B^n$. Consequently, v satisfies the equation

$$\frac{\partial v(z,t)}{\partial z}\cdot h(z) = -\frac{\partial v(z,t)}{\partial t}.$$

Theorem 2. If $v = v(z,t)$, $z \in B^n$, $t \geqslant 0$, is the solution of (1) with $h \in M$, then there exists

$$\lim_{t \to \infty} e^t v(z,t) = f(z), \quad z \in B^n, \tag{4}$$

and this limit is a holomorphic function in B^n.

Proof. Denote $u(z,t) = e^t v(z,t)$ for $z \in B^n$ and $t \geqslant 0$, where v is the solution of (1). Thus, u is the solution of the problem

$$\frac{\partial u(z,t)}{\partial t} = u(z,t) - e^t h(e^{-t} u(z,t)), \quad u(z,0) = z.$$

Further, let us introduce the notation $G(z) = h(z) - z$ for $z \in B^n$. The function G is analytic in B^n, and $DG(0) = 0$. Taking into account the above notation, we see that u is the solution of the problem

$$\frac{\partial u(z,t)}{\partial t} = -e^t G(e^{-t} u(z,t)), \quad u(z,0) = z, \quad z \in B^n, \quad t \geqslant 0. \tag{5}$$

Integrating both sides of the above equality in the interval $[t_1, t_2]$, $0 < t_1 < t_2$, we get

$$u(z,t_2) - u(z,t_1) = -\int_{t_1}^{t_2} e^{\tau} G(e^{-\tau} u(z,\tau))\, d\tau. \tag{6}$$

Since the function G is holomorphic in B^n, there exists a ball $B^n(0,\varrho)$, $0 < \varrho < 1$, such that, for $z \in B^n(0,\varrho)$,

$$G(z) = \sum_{n=2}^{\infty} \frac{1}{n!} D^{(n)} G(0)(z^n),$$

and

$$L_{\varrho} = \sup_{\|z\| < \varrho} \sum_{n=2}^{\infty} \left\| \frac{1}{n!} D^{(n)} G(0)(z^n) \right\|$$

is a finite quantity.

On account of Remark 1, for each $r \in (0,1)$, there exists some $\tau_0 \geqslant 0$ such that $\frac{u(z,\tau)}{\tau} \in B^n(0,\rho)$ for $\tau > \tau_0$ and $\|z\| \leqslant r$. Making use of the elementary inequality

$$\sum_{n=2}^{\infty} \left\| \frac{1}{n!} D^{(n)} G(0) \left(\left(\frac{u(z,\tau)}{\tau} \right)^n \right) \right\| \leqslant L_\rho ,$$

we obtain

$$(7) \qquad \|G(e^{-\tau} u(z,\tau))\| \leqslant \sum_{n=2}^{\infty} \left\| \frac{1}{n!} D^{(n)} G(0) (e^{-\tau} u(z,\tau))^n \right\| \leqslant$$

$$\leqslant \sum_{n=2}^{\infty} (e^{-\tau})^n \tau^n \left\| \frac{1}{n!} D^{(n)} G(0) \left(\frac{u(z,\tau)}{\tau} \right)^n \right\| \leqslant \frac{e^{-2\tau}}{\tau^{-2}} L_\rho$$

for $\tau > \tau_0$ and $\|z\| \leqslant r$.

From (6) we get

$$\|u(z,t_2) - u(z,t_1)\| \leqslant \int_{t_1}^{t_2} \frac{e^{-\tau}}{\tau^{-2}} L_\rho \, d\tau$$

for $z \in B^n(0,r)$, $t_1, t_2 > \tau_0$. From the integrability of the function $g(\tau) = \frac{e^{-\tau}}{\tau^{-2}}$ in the interval $[0,\infty)$ it follows that, for each $\varepsilon > 0$ and $r \in (0,1)$, there is τ_1 such that

$$\sup_{\|z\| < r} \|u(z,t_2) - u(z,t_1)\| \leqslant \varepsilon \quad \text{for} \quad t_1, t_2 > \tau_1 .$$

Consequently, from the Weierstrass theorem (see [1]) it follows that $\lim_{t \to +\infty} u(z,t)$ exists for $z \in B^n$ and is a function holomorphic in B^n.

Corollary 1. Let v be the solution of (1). Then

$$\lim_{t \to +\infty} \left(-e^t \frac{\partial v(z,t)}{\partial t} \right) = \lim_{t \to +\infty} e^t v(z,t), \qquad z \in B^n .$$

P r o o f . Analogously as in the proof of Theorem 2, let $u(z,t) = e^{t}v(z,t)$, $z \in B^n$, $t \geqslant 0$. Considering the fact that $\frac{\partial v(z,t)}{\partial t} = -e^{-t}u(z,t) + e^{-t}\frac{\partial u(z,t)}{\partial t}$, it suffices to show that

$$\lim_{t \to +\infty} \frac{\partial u(z,t)}{\partial t} = 0 \quad \text{for} \quad z \in B^n.$$

From (5) and (7) it follows that

$$\left\| \frac{\partial u(z,t)}{\partial t} \right\| \leqslant \frac{e^{-t}}{t^{-2}} L_{\rho}, \qquad t > \tau_0.$$

Hence it follows directly that

$$\lim_{t \to +\infty} \left\| \frac{\partial u(z,t)}{\partial t} \right\| = 0, \qquad z \in B^n,$$

which implies the corollary.

T h e o r e m 3. Let $h \in M$, and let $v = v(z,t)$ for $z \in B^n$, $t \geqslant 0$ be the solution of (1). Then

$$\lim_{t \to +\infty} e^{t}v(z,t) = f(z) \quad \text{for} \quad z \in B^n$$

is a starlike map of the ball B^n into $\mathbf{C}^n$ such that $Df(0) = I$.

P r o o f . Fix $z_0 \in B^n$ and $t_0 \geqslant 0$. Let $\tilde{z}_0 = v(z_0, t_0)$ and $u(\tilde{z}_0, t) = v(z_0, t+t_0)$ for $t \geqslant 0$. So, we have

$$\frac{\partial u(\tilde{z}_0, t)}{\partial t} = -h(u(\tilde{z}_0, t)) \quad \text{and} \quad u(\tilde{z}_0, 0) = \tilde{z}_0,$$

since v is the solution of (1). Hence

$$f(\tilde{z}_0) = \lim_{t \to +\infty} e^{t}u(\tilde{z}_0, t) = \lim_{t \to +\infty} e^{t}v(z_0, t+t_0) =$$

$$= \lim_{s \to \infty} e^{-t_0} e^{s}v(z_0, s) = e^{-t_0} f(z_0).$$

Consequently,

$$(8)\qquad f(v(z,t)) = e^{-t}f(z) \quad \text{for} \quad z \in B^n, \quad t \geqslant 0,$$

which proves that $f(B^n)$ is a starlike set.

We shall now demonstrate that $Df(0) = I$. Differentiating the equation in (1) with respect to the variable z, by Lemma 2, we get

$$\frac{\partial}{\partial t}\left(\frac{\partial v(0,t)}{\partial z}\right) = -\frac{\partial v(0,t)}{\partial z}, \quad t \geqslant 0.$$

Since $v(z,0) = z$ for $z \in B^n$, so $\dfrac{\partial v(0,0)}{\partial z} = I$.

Taking into account this condition we get $\dfrac{\partial v(0,t)}{\partial z} = e^{-t}I$. Making use of Weierstrass theorem (see [1]) we obtain that $Df(0) = I$.

It remains to show that the map f is univalent. Suppose to the contrary that there exist $z_1, z_2 \in B^n$ such that $z_1 \neq z_2$ and $f(z_1) = f(z_2)$. Hence and from (8) it follows that $f(v(z_1,t)) = f(v(z_2,t))$ for $t \geqslant 0$. Since $Df(0) = I$, we state that, by Th. 1.3.7 of [5], there is an $r \in (0,1)$ such that f is univalent in $B^n(0,r)$. Consequently, there exists some $\tau_0 > 0$ such that $v(z_1,t) = v(z_2,t)$ for $t > \tau_0$. From the uniqueness of the solution of (1) it follows that $v(z_1,t) = v(z_2,t)$ for $t \geqslant 0$, and thus $z_1 = z_2$, which contradicts the supposition. Consequently, f is a map univalent in B^n.

Theorem 4. Let v be the solution of (1). Then the function f defined by the formula

$$f(z) = \lim_{t \to +\infty} e^t v(z,t), \quad z \in B^n,$$

satisfies the differential equation $Df(z)\,h(z) = f(z)$ for $z \in B^n$. Moreover, f is the only locally biholomorphic solution of this equation, satisfying the conditions $f(0) = 0$, $Df(0) = I$.

Proof. By Theorem 1, the function v satisfies the equation

$$\frac{\partial v(z,t)}{\partial z} h(z) = -\frac{\partial v(z,t)}{\partial t}, \quad z \in B^n, \ t \geq 0.$$

From the definition of the function f and from Corollary 1 it follows that

$$\lim_{t \to +\infty} \left(-e^t \frac{\partial v(z,t)}{\partial t}\right) = f(z), \quad z \in B^n.$$

Consequently, the equality

$$\lim_{t \to +\infty} \left(e^t \frac{\partial v(z,t)}{\partial z} h(z)\right) = \lim_{t \to +\infty} \left(-e^t \frac{\partial v(z,t)}{\partial t}\right)$$

implies, by the Weierstrass theorem (see [1]), that the function f satisfies the equation $Df(z)h(z) = f(z)$ for $z \in B^n$. From Theorem 3 we have that $f(0) = 0$ and $Df(0) = I$.

We shall now show that the solution of the problem

$$Df(z)h(z) = f(z), \quad f(0) = 0, \quad Df(0) = I,$$

where h is a fixed function belonging to M, is unique in $H(B^n)$. Suppose to the contrary that there exist two locally biholomorphic maps $f_1, f_2 \in H(B^n)$ satisfying the above problem. By Theorem 3 of [2], f_1 and f_2 are starlike maps of B^n into C^n. It is not difficult to show that the functions v_i, i=1,2,

$$v_i(z,t) = f_i^{-1}(e^{-t} f_i(z)), \quad t \geq 0, \ z \in B^n,$$

satisfy (1). In consequence, from the uniqueness of the solution of (1) we get that $v_1 = v_2$. Since

$$f_i(z) = \lim_{t \to +\infty} e^t v_i(z,t), \quad z \in B^n, \quad i=1,2,$$

therefore $f_1 = f_2$, q.e.d.

Let G_0 denote the class of starlike maps $f: B^n \to C^n$ such that $Df(0) = I$. For the definition of starlike maps see [6].

Corollary 2. The map $f \in G_0$ if and only if there exists a function $h \in M$ such that

$$f(z) = \lim_{t \to +\infty} e^{t} v(z,t),$$

where v is the solution of (1).

This corollary follows immediately from Theorems 3 and 4.

Corollary 3. If $f \in G_0$, then

$$\frac{\|z\|}{(1+\|z\|)^2} \leqslant \|f(z)\| \leqslant \frac{\|z\|}{(1-\|z\|)^2} \quad \text{for} \quad z \in B^n. \tag{9}$$

This is a consequence of Remark 1 and Corollary 2.

It is not difficult to notice that the map $f \in G_0$ defined by the formula

$$f(z_1, z_2, \ldots, z_n) = \left(\frac{z_1}{(1-z_1)^2}, \frac{z_2}{(1-z_2)^2}, \ldots, \frac{z_n}{(1-z_n)^2}\right)$$

realizes the equalities in inequalities (9) at the points $\tilde{z} = (r,0,\ldots,0)$ and $\tilde{\tilde{z}} = (-r,0,\ldots,0)$, respectively, where $r \in (0,1)$.

REFERENCES

[1] J. Dieudonné: Foundations of modern analysis. New York and London 1960 (Russian translation).

[2] K.R. Gurganus: Φ-like holomorphic functions in C^n and Banach spaces, Trans. Amer. Math. Soc. 205 (1975) 389-406.

[3] K. Maurin: Analiza. Vol. 1. Warszawa 1971.

[4] J.A. Pfaltzgraff: Subordination chains and univalence of holomorphic mappings in C^n, Math. Ann. 210 (1974), 55-68.

[5] W. Rudin: Function theory in the unit ball of C^n. New York, Heidelberg, Berlin 1980.

[6] T.J. Suffridge: Starlike and convex maps in Banach spaces, Pacific J. Math. 46 (1973), 575-589.

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Leszek J. Ciach

ON THE CONJUGATES OF SOME OPERATOR SPACES, II

This paper is a continuation of [1]. Here, continuous linear functionals on the non-commutative Marcinkiewicz spaces $\mathcal{M}_\psi(\mathcal{A})$ are discussed. In the commutative case, for $\psi(\lambda) = \lambda^\alpha$, $0<\alpha<1$, this problem is studied in detail in [3].

1. Preliminaries

Throughout this paper, the notation and terminology will be the same as in [1]. We shall now give a few facts which will be used mainly in part III of this paper.

Proposition 1.1. Suppose that $a \in \mathcal{I}_m(\mathcal{A})$, $|a| = \int_0^\infty \lambda de_\lambda$. Then

$$\alpha - m(e_{\{a(\alpha)\}}) \leqslant m(e^{\perp}_{a(\alpha)}) \leqslant \alpha$$

for all $0<\alpha<m(1)$.

Proof. The inequality $m(e^{\perp}_{a(\alpha)}) \leqslant \alpha$, $\alpha > 0$ $(a \in \mathcal{L})$, is obvious. To prove the first inequality in the case $a(\alpha) > 0$, it suffices only to observe that

$$m(e^{\perp}_{a(\alpha)}) + m(e_{\{a(\alpha)\}}) = m(e_{\langle a(\alpha),\infty)}) = \lim_n m(e^{\perp}_{a(\alpha)-1/n})$$

for $\alpha \in (0,\infty)$ and, by the definition of $a(\alpha)$, $m(e_{a(\alpha)-1/n}) \geqslant \alpha$.

If $a(\alpha) = 0$, then $m(e_0^{\perp}) < \infty$ and $m(1) = m(e_0^{\perp}) + m(e_{\{0\}})$. Hence

$$\alpha - m(e_{\{0\}}) = \alpha - m(1) + m(e_0^{\perp}) \leqslant m(e_0^{\perp}).$$

For an arbitrary $a \in \mathcal{L}$, we define two non-increasing functions $a^R(\alpha)$, $a^L(\alpha)$ on $(0, \infty)$ by

$$a^R(\alpha) = \inf\{\|p|a|p\| : m(p^{\perp}) \leqslant \alpha\} =$$

$$= \inf\{\sup\{(p|a|p\xi, \xi) : \|\xi\| = 1, \xi \in D(p|a|p)\} : m(p^{\perp}) \leqslant \alpha\},$$

$$a^L(\alpha) = \begin{cases} \sup\{\inf\{(p|a|p\xi, \xi) : \|\xi\| = 1, \xi \in D(p|a|p)\} : m(p) \geqslant \alpha\}, & \alpha \leqslant m(1) \\ 0 & \alpha > m(1). \end{cases}$$

Proposition 1.2. For any $a \in \mathcal{L}_m$, $|a| = \int_0^{\infty} \lambda \, de_{\lambda}$.

(i) $a^R(\alpha) = a(\alpha)$, $\alpha > 0$,

(ii) the set $\{\alpha : a^L(\alpha) \neq a(\alpha)\}$ is countable.

Proof. (i). Let $p = e_{a(\alpha)}$. Then $m(e_{a(\alpha)}^{\perp}) \leqslant \alpha$ and $\|e_{a(\alpha)}|a|e_{a(\alpha)}\| \leqslant a(\alpha)$. In other words, $a^R(\alpha) \leqslant a(\alpha)$.

Suppose now that $\|p_{\lambda}|a|\ p_{\lambda}\| < \lambda < a(\alpha)$ for some $p_{\lambda} \in \mathrm{Proj}(\mathcal{A})$, $m(p_{\lambda}^{\perp}) \leqslant \alpha$. Then $p_{\lambda} \wedge e_{\lambda}^{\perp} = 0$, and so, $e_{\lambda}^{\perp} = e_{\lambda}^{\perp} - p_{\lambda} \wedge e_{\lambda}^{\perp} \sim e_{\lambda}^{\perp} \vee p_{\lambda} - p_{\lambda} \leqslant p_{\lambda}^{\perp}$, that is, $m(e_{\lambda}^{\perp}) \leqslant m(p_{\lambda}^{\perp}) \leqslant \alpha$. In consequence, $a(\alpha) \leqslant \lambda$, which leads to a contradiction. Thus, $a^R(\alpha) \geqslant a(\alpha)$ which, together with the previous inequality, gives $a^R(\alpha) = a(\alpha)$.

(ii) For each $\lambda > 0$, we have

$$\operatorname{mes}(\{\alpha : a^L(\alpha) > \lambda\}) = \sup\{\alpha : a^L(\alpha) > \lambda\} =$$

$$= \sup\{m(p) : p \in \mathrm{Proj}(\mathcal{A}), \inf_{\|\xi\| = 1} (p|a|p\xi, \xi) > \lambda\} = m(e_{\lambda}^{\perp})$$

.

because $p \wedge e_\lambda = 0$ and, in consequence, $p = p - p \wedge e_\lambda \sim p \vee e_\lambda - e_\lambda \leq e_\lambda^\perp$, i.e. $m(p) \leq m(e_\lambda^\perp)$. On the other hand, $m(e_\lambda^\perp) = \sup_n (e^\perp_{\lambda+1/n})$ and

$$\inf\left\{\left(e^\perp_{\lambda+1/n}\ |a|\ e^\perp_{\lambda+1/n}\ \xi, \xi\right) : \|\xi\| = 1,\ \xi \in D\left(|a|\ e^\perp_{\lambda+1/n}\right)\right\}$$

$$\geq \lambda + 1/n > \lambda .$$

To finish the proof, it is enough to notice that, by the definition of $a(\alpha)$,

$$\operatorname{mes}\left(\{\alpha : a(\alpha) > \lambda\}\right) = m(e_\lambda^\perp).$$

Assume that the algebra $\mathcal{A}$ does not possess any non-zero minimal projections.

Definition 1.3. A continuous monotone family (c.m.f.) on $\mathcal{A}$ is a family $\{p_\alpha\}_{0 \leq \alpha \leq m(1)}$ of projections from $\mathcal{A}$ such that: $p_{m(1)} = 1$ and

(i) $\alpha_1 < \alpha_2$ implies $p_{\alpha_1} \leq p_{\alpha_2}$,
(ii) $m(p_\alpha) = \alpha$ for any $0 \leq \alpha \leq m(1)$.

For any $a \in \mathcal{S}_m$, $|a| = \int_0^\infty \lambda\, de_\lambda$ and $0 \leq \alpha \leq m(1)$, let $p^a_\alpha \in \operatorname{Proj}(\mathcal{A})$ satisfy the relations

(i) $p^a_0 = 0$, $p^a_{m(1)} = 1$,
(ii) $m(p^a_\alpha) = \alpha$, $e^\perp_{a(\alpha)} \leq p^a_\alpha \leq e^\perp_{a(\alpha)} + e_{\{a(\alpha)\}}$, $0 < \alpha < m(1)$.

Note that $\{p^a_\alpha\}$ is a monotone family.

Lemma 1.4. ([2], Lemma 6.2). For any $a \in \mathcal{S}_m$ and $\alpha > 0$,

$$m(|a|p^a_\alpha) = \int_0^\alpha a(\beta)\,d\beta = \sup\left\{m(|a|p) : p \in \operatorname{Proj}(\mathcal{A}),\ m(p) = \alpha\right\}.$$

Proposition 1.5. For any $a \in \mathcal{S}_m$, $|a| = \int_0^\infty \lambda\, de_\lambda$,

(i) $a^L(\alpha) = \inf\{(p^a_\alpha|a|p^a_\alpha\xi, \xi) = (|a|p^a_\alpha\xi, \xi) : \|\xi\| = 1, \xi \in D(|a|p^a_\alpha)\} \geq a(\alpha)$,

(ii) $a^L(\alpha)$ is left continuous,

(iii) $a^R(\alpha) = \| |a| p^{a\perp}_\alpha \| = \sup\{(p^{a\perp}_\alpha |a| p^{a\perp}_\alpha \xi, \xi) : \|\xi\| = 1\}$.

Proof. (i). $a^L(\alpha) \geq \inf\{(p^a_\alpha|a|p^a_\alpha\xi, \xi) : \|\xi\| = 1\}$ since $m(p^a_\alpha) = \alpha$. Suppose that $a^L(\alpha) > \lambda > \inf\{(p^a_\alpha|a|p^a_\alpha\xi, \xi) : \|\xi\| = 1\} \geq a(\alpha)$ for some $\alpha > 0$, and let $\inf\{(q_\lambda|a|q_\lambda\xi, \xi) : \xi \in D(q_\lambda|a|q_\lambda), \|\xi\| = 1\} > \lambda$ for some $q_\lambda \in \mathrm{Proj}(\mathcal{A})$, $m(q_\lambda) \geq \alpha$.

Then $e_\lambda \wedge q_\lambda = 0$, and consequently, $q_\lambda = q_\lambda - e_\lambda \wedge q_\lambda \sim e_\lambda \vee q_\lambda - e_\lambda \leq e^\perp_\lambda$, which implies $m(e^\perp_\lambda) \geq \alpha$.

On the other hand, $\alpha \geq m(e^\perp_{a(\alpha)}) \geq m(e^\perp_\lambda)$. Consequently, $\alpha = m(e^\perp_\lambda) = m(e^\perp_{a(\alpha)})$, $e^\perp_\lambda = e^\perp_{a(\alpha)} = p^a_\alpha$. But

$$\inf\{(p^a_\alpha|a|p^a_\alpha\xi, \xi) : \|\xi\| = 1\} \geq \lambda$$

yields a contradiction.

(ii) Assume that $\alpha_n \uparrow \alpha$ and $a^L(\alpha_n) \downarrow \lambda_0 > \lambda > a^L(\alpha)$. As in the proof of (i), $m(e^\perp_\lambda) \geq \alpha_n$, $n = 1, 2, \ldots$, that is, $m(e^\perp_\lambda) \geq \alpha$ and, in consequence, $e^\perp_\lambda = e^\perp_{a(\alpha)} = p^a_\alpha$. Hence and from (i)

$$a^L(\alpha) = \inf\{(p^a_\alpha|a|p^a_\alpha\xi, \xi) : \|\xi\| = 1\} \geq \lambda,$$

a contradiction.

(iii) $p^{a\perp}_\alpha \leq e_{a(\alpha)}$ implies $\| |a| p^{a\perp}_\alpha \| \leq a(\alpha) = a^R(\alpha)$. Suppose that $\| |a| p^{a\perp}_\alpha \| < \lambda < a^R(\alpha)$. Then $p^{a\perp}_\alpha \leq e_\lambda$ and, in consequence, $a^R(\alpha) = a(\alpha) \leq \lambda$, yielding a contradiction.

2. The decomposition $\mathcal{M}_\psi(\mathcal{A})^* = \mathcal{L}^1_{m\psi}(\mathcal{A}) \oplus S_0 \oplus S_\infty$

In this section we suppose that one of conditions (i)-(ii) is satisfied:

(i) $\lim_{\alpha\to 0} \alpha/\psi(\alpha) = 0$,

(ii) $\inf\{m(p) : 0 \neq p \in \mathrm{Proj}(\mathcal{A})\} = \gamma > 0$.

Proposition 2.1. $\mathcal{M}_\psi(\mathcal{A})^* = \mathcal{L}^1_{m\psi}(\mathcal{A})$ if and only if $\bar{\mathcal{F}}_m = \mathcal{M}_\psi(\mathcal{A})$ (the closure in $\mathcal{M}_\psi(\mathcal{A})$).

Proof. Sufficiency. On the ground of Theorem 4.1 [1]. Necessity (to prove the necessity, we do not have to assume that (i) or (ii) holds). Suppose that, $0 \neq a_0 \in \mathcal{M}_\psi - \bar{\mathcal{F}}_m$ and $h(a_0) = h_b(a_0) = 1$, $0 = h(a) = h_b(a) = m(ba)$, $a \in \bar{\mathcal{F}}_m$ for some $b \in \mathcal{L}^1_{m\psi} \subset \mathcal{J}$ and $h \in \mathcal{M}^*_\psi$. Let $b = u|b|$, $|b^*| = \int_0^\infty \lambda \, de_\lambda$ denote, respectively, the polar decomposition of b and the spectral decomposition $|b^*| = u|b|u^*$. Then $u^*(e_n - e_{1/n}) \in \mathcal{F}_m$ and

$$0 = h(u^*(e_n - e_{1/n})) = m(|b^*|(e_n - e_{1/n})), \quad n=1,2,\ldots,$$

i.e. $|b^*| = 0$, which implies $b = 0$, $h = h_b \equiv 0$. But $h(a_0) = 1$, yielding a contradiction.

By Theorem 4.1 [1], for any $h \in \mathcal{M}_\psi(\mathcal{A})^*$, $h|\bar{\mathcal{F}}_m$ (the closure in $\mathcal{M}_\psi(\mathcal{A})$) $= h_b = m(b\cdot)$ where $b \in \mathcal{L}^1_{m\psi}(\mathcal{A})$, $\|h|\bar{\mathcal{F}}_m\| = \|b\|_{1,m\psi}$. It is clear that each $h \in \mathcal{M}^*_\psi$ uniquely determines such a $b \in \mathcal{L}^1_{m\psi}$ (cf. the proof of Prop. 2.1 (necessity)). Let $h_S = h - h_b$. $h_S \in S$ where S is the space of "singular" functionals on $\mathcal{M}_\psi$ which vanish on $\bar{\mathcal{F}}_m$. We have shown that

$$\mathcal{M}_\psi(\mathcal{A})^* = \mathcal{L}^1_m(\mathcal{A}) \oplus S.$$

Let

$S_0 = \{h \in \mathcal{M}^*_\psi : h$ annihilates all bounded operators$\}$,

$S_\infty = \{h \in \mathcal{M}^*_\psi : h$ annihilates all operators with m-finite supports$\}$.

It is evident that

$S_0 \subset S$, $S_\infty \subset S$, $S_0 \cap S_\infty = \{0\}$ (since, for any $a \in \mathcal{M}_\psi \subset \mathcal{L}$, $a = a_1 + a_2$ where a_1 is bounded, a_2 has the m-finite support). We shall prove that $S = S_0 \oplus S_\infty$.

Definition 2.2. For any $h_S \in S$, $a \in \mathcal{M}_\psi(\mathcal{A})$, we define

(1) $h_0(a) = h_S(ap^\perp)$,

(2) $h_\infty(a) = h_S(ap) = h_S(a) - h_0(a)$,

where p is a projection from $\mathcal{A}$ such that $m(p^\perp) < \infty$, $\|ap\| < \infty$.

To show that $S = S_0 \oplus S_\infty$, we have to prove only that $h_0(h_\infty) \in S_0(S_\infty)$.

Proposition 2.3. For an arbitrary $h_S \in S$, h_0 and h_∞ are linear continuous functionals on $\mathcal{M}_\psi(\mathcal{A})$, and $h_0(h_\infty) \in S_0(S_\infty)$.

Proof. We shall first show that the definitions of h_0 and h are independent of the choice of p. Let $m(p_i^\perp) < \infty$, $\|ap_i\| < \infty$, $i = 1,2$. Then $m((p_1 \wedge p_2)^\perp) \leqslant m(p_1^\perp) + m(p_2^\perp)$, $\|a(p_1 \wedge p_2)\| \leqslant \min\{\|ap_1\|, \|ap_2\|\}$ and

$$h_S(a(p_1^\perp \vee p_2^\perp)) - h_S(ap_i^\perp) = h_S(a(p_1^\perp \vee p_2^\perp - p_i^\perp)) = 0$$

because

$$\|a(p_1^\perp \vee p_2^\perp - p_i^\perp)\| \leqslant \|ap_i\| < \infty, \quad m(p_1^\perp \vee p_2^\perp - p_i^\perp) < \infty, \quad i = 1,2.$$

In consequence,

$$h_S^\perp(ap_1^\perp) = h_S(ap_2^\perp) = h_S(a(p_1^\perp \vee p_2^\perp)).$$

To show that h_0 is a linear continuous functional on $\mathcal{M}_\psi$, one has only to observe that

$$|h_0(a)| = |h_S(ap^\perp)| \leqslant \|h_S\| M_\psi(ap^\perp) \leqslant \|h_S\| M_\psi(a)$$

and

$$h_0(a+b) = h_S((a+b)r^\perp) = h_S(ar^\perp) + h_S(br^\perp) = h_0(a) + h_0(b),$$

where

$\|ap\| < \infty$, $\|bq\| < \infty$, $m(p^{\perp}) < \infty$, $m(q^{\perp}) < \infty$, $r = p \wedge q$.

For h_{∞}, take proof is almost the same. It is evident that $h_0(h_{\infty}) \in S_0(S_{\infty})$.

Corollary 2.4. $\mathcal{B}(\mathcal{H})^* = \mathcal{L}^1_m(\mathcal{B}(\mathcal{H})) \oplus S_{\infty}$ where $S_{\infty} = h \in \mathcal{B}(\mathcal{H})^*$: h annihilates all compact operators.

Remark 2.5. If condition (ii) is satisfied then $\mathcal{M}_{\psi}(\mathcal{A}) \subset \mathcal{A}$ and, in consequence, S_0 has no non-trivial functionals.

Assume that the following condition is satisfied:

(iii) $\lim\limits_{\alpha \to \infty} \alpha/\psi(\alpha) = \sup\{\alpha/\psi(\alpha) : \alpha > 0\} = \infty$.

In this case, $\mathcal{M}_{\psi} \subset \mathcal{I}_m(\mathcal{A})$ and, in consequence,

$$h_0(a) = h_S(ae^{\perp}_{\lambda}), \quad h_{\infty}(a) = h_S(ae_{\lambda}), \quad \lambda > 0,$$

where $|a| = \int\limits_0^{\infty} \lambda \, de_{\lambda}$ is the spectral decomposition of $|a|$.

Proposition 2.6. Assume that condition (i) is satisfied. Then $h_0 \in S_0$ if and only if h_0 is continuous with respect to the seminorm N_0.

Proof. It is easy to see that $h_0 \in S_0$ if $|h_0(a)| \leqslant$ const $N_0(a)$, $a \in \mathcal{M}_{\psi}(\mathcal{A})$. We shall prove the necessity. Let $|a| = \int\limits_0^{\infty} \lambda \, de_{\lambda}$. Then

$$|h_0(a)| = |h_0(ae^{\perp}_{\lambda})| \leqslant \|h_0\| M_{\psi}(ae^{\perp}_{\lambda}) =$$

$$= \|h_0\| \sup\left\{1/\psi(\alpha) \int\limits_0^{\alpha} a(\beta) \, d\beta : 0 < \alpha \leqslant m(e^{\perp}_{\lambda})\right\}$$

for an arbitrary $\lambda > 0$. Hence, since $m(e^{\perp}_{\lambda}) \to 0$, $\lambda \to \infty$,

$$|h_0(a)| \leqslant \|h_0\| \limsup_{\alpha \to 0} \left\{1/\psi(\alpha) \int\limits_0^{\alpha} a(\beta) \, d\beta\right\} = \|h_0\| N_0(a).$$

For h_{∞}, we have the following

Proposition 2.7. Suppose that condition (iii) is satisfied. Then $h_\infty \in S_\infty$ if and only if h_∞ is continuous with respect to the seminorm N_∞.

Proof. It is easy to check that $|h_\infty(a)| \leqslant$ $\leqslant \text{const } N_\infty(a)$, $a \in \mathcal{M}_\psi$, implies $h_\infty \in S_\infty$.

Necessity. First, we may suppose that $a = a^* \in \mathcal{M}_\psi$, $|a| = \int\limits_0^\infty \lambda \, de_\lambda$. Then

$$h_\infty(a) = h_\infty(ae_\lambda + e_\lambda^\perp), \ \lambda > 0$$

and

$$(ae_\lambda + \lambda e_\lambda^\perp)(\alpha) = \begin{cases} a(\alpha), & \alpha \geqslant m(e_\lambda^\perp) \\ \lambda, & 0 < \alpha < m(e_\lambda^\perp). \end{cases}$$

Hence

$$\sup\left\{1/\psi(\alpha) \int\limits_0^\alpha (ae_\lambda + \lambda e_\lambda^\perp)(\beta)\,d\beta \ : \ \alpha > 0\right\} \leqslant$$

$$\sup\left\{1/\psi(\alpha) \int\limits_0^\alpha a(\beta)\,d\beta \ : \ \alpha \geqslant m(e_\lambda^\perp)\right\}$$

since

$$\frac{m(e_\lambda^\perp)}{\psi(m(e_\lambda^\perp)}\lambda \leqslant \frac{1}{\psi(m(e_\lambda^\perp))} \int\limits_0^{m(e_\lambda^\perp)} a(\beta)\,d\beta.$$

Consequently

$$|h_\infty(a)| \leqslant \|h_\infty\| \, M_\psi(ae_\lambda + \lambda e_\lambda^\perp) \leqslant$$

$$\|h_\infty\| \limsup_{\alpha \to \infty} \frac{1}{\psi(\alpha)} \int\limits_0^\alpha a(\beta)\, d\beta = N_\infty(a)\|h_\infty\|$$

if $m(e_\lambda^\perp) \to \infty$ as $\lambda \to 0^+$ (in the case $m(e_{(0,\infty)}) < \infty$, $|h_\infty(a)| = 0 = N_\infty(a)$).

For an arbitrary a, we have

$$|h_\infty(a)| \leqslant |h_\infty(\mathrm{Re}\,a)| + |h_\infty(\mathrm{Im}\,a)| \leqslant \|h_\infty\|(N_\infty(\mathrm{Re}\,a)+N_\infty(\mathrm{Im}\,a)) \leqslant$$
$$\leqslant \mathrm{const}\,\|h_\infty\|\,N_\infty(a)$$

since the seminorms N_∞ and $N_\infty(\mathrm{Re}\,\cdot) + N_\infty(\mathrm{Im}\,\cdot)$ are equivalent.

3. The representation of $h \in \mathcal{M}_\psi(\mathcal{A})^*$ as a bounded regular Borel measure

In this section we suppose that $\mathcal{A}$ has no atoms.

Let $\mathcal{U}$ be the set of all partial isometries from $\mathcal{A}$, $\overline{\mathrm{Proj}(\mathcal{A})} = \{p \in \mathrm{Proj}(\mathcal{A}) : m(p) < \infty\}$, and let $\Omega = \overline{\mathrm{Proj}(\mathcal{A})} \times \mathcal{U}$. Ω is a metric space with the metric

$$d((p,u),\,(q,v)) = \rho_m(p-q) + \|u-v\|.$$

As usual, the space of all bounded continuous complex-valued functions on Ω equipped with the supremum norm is denoted by $C(\Omega)$.

Proposition 3.1. For each $a \in \mathcal{M}_\psi(\mathcal{A})$, the formula

$$a(p,u) = \frac{1}{\psi(m(p))}\, m(uap),\quad (p,u) \in \Omega,$$

defines a function in $C(\Omega)$. Moreover,

$$M_\psi(a) = \sup\left\{|a(p,u)| \;:\; (p,u) \in \Omega\right\}.$$

Proof. Indeed, let $\rho_m(p-p_n) \to 0$, $\|u-u_n\| \to 0$ as $n \to \infty$, and let $a = a_1 + a_2$, $a_1 \in \mathcal{A}$, $a_2 \in \mathcal{L}^1_m$. Then

$$|m(ua_1p) - m(u_na_1p_n)| =$$
$$= |m(ua_1p) - m(ua_1p_n) + m(ua_1p_n) - m(u_na_1p_n)| \leqslant$$
$$|m(ua_1(p-p_n))| + |m((u-u_n)\,a_1p_n)| \leqslant$$
$$\|a_1\|\,\|p-p_n\|_1 + \|u-u_n\|\,\|a_1\|\,m(p_n) \xrightarrow[n]{} 0,$$

and

$$\left|m(ua_2p) - m(u_na_2p_n)\right| \leq \left|m(ua_2(p-p_n))\right| + \left|m((u-u_n)a_2p_n)\right| \leq$$

$$\left\|a_2(p-p_n)\right\|_1 + \left\|u-u_n\right\| \left\|a_2\right\|_1 \xrightarrow[n]{} 0.$$

In both above inequalities, their first right-hand side terms tend to zero in virtue of the Lebesgue dominated convergence theorem (cf. e.g. [2], th.4.1).

Consequently, $a(p_n,u_n) \xrightarrow[n]{} a(p,u)$ because the function $\psi(\lambda)$ is continuous. In addition,

$$\sup_{\Omega} |a(p,u)| \leq \sup\left\{\frac{1}{\psi(m(p))}\, m(|ap|) \; : \; p \in \overline{\mathrm{Proj}(\mathcal{A})}\right\} =$$

$$= ([2], \text{ Lemma } 6.4)\left\{\sup \frac{1}{\psi(m(p))}\, m(|a|\,p) \; : \; p \in \overline{\mathrm{Proj}(\mathcal{A})}\right\} =$$

$$= ([2], \text{ Lemma } 6.3) \sup\left\{1/\psi(\lambda) \int_0^{\lambda} a(\alpha)\, d\alpha \; : \; \lambda > 0\right\} = M_{\psi}(a).$$

Let now $a = v|a|$ be the polar decomposition of a. Then $a(p,v) = 1/\psi(m(p))\, m(|a|\,p)$, and consequently,

$$\sup_{\Omega} |a(p,u)| \geq \sup\left\{\frac{1}{\psi(m(p))}\, m(|a|\,p) \; : \; p \in \overline{\mathrm{Proj}(\mathcal{A})}\right\} = M_{\psi}(a).$$

Finally, $\sup_{\Omega} |a(p,u)| = M_{\psi}(a)$.

Corollary 3.2. The map $a \rightarrow a(p,u)(\mathcal{M}_{\psi}(\mathcal{A}) \rightarrow C(\Omega))$ is a linear isometry onto a closed subspace of $C(\Omega)$.

Corollary 3.3. (cf. [3]). Each $h \in \mathcal{M}_{\psi}(\mathcal{A})^*$ may be represented as a bounded regular Borel measure μ on $\beta\Omega$:

$$h(a) = \int_{\beta\Omega} a(p,u)\, d\mu(p,u), \quad a \in \mathcal{M}_{\psi},$$

where, as usual, $\beta\Omega$ is the Stone-Cech compactification of Ω. This representation is the starting point for a characterization of S_0 and S_{∞} in the case $\psi(\lambda) = \lambda^{1-1/\delta}$, $1 < \delta < \infty$. The proof of the theorem below will be a topic of the next part of this paper (cf. [3], th.1).

Theorem 3.3. Let $h \in S_0$ (S_∞ respectively) be such that $h = h^*$ ($h^*(b) = \overline{h(b^*)}$) and, for some $a \in \mathcal{M}_\psi(\mathcal{A})$, $h(a) = \|h\| M_\psi(a)$. Then there exists a positive finitely additive set function $\nu \in L^\infty((0,m(1)))^*$ such that

(1) $\|\nu\| = \nu((0,m(1))) = \|h\|$,

(2) $$h(b) = \int_{(0,m(1))} b(p^a_\alpha, u)\, d\nu(\alpha), \quad b \in \mathcal{M}_\psi,$$

(3) $$\int_{(0,m(1))} f(\alpha)\, d\nu(\alpha) \leqslant \text{const} \lim_{n\to\infty} \operatorname*{ess\,sup}_{t\in(0,1/n)} |f(t)|$$

$$\Big(\text{const} \lim_{n\to\infty} \operatorname*{ess\,sup}_{t\in(n,\infty)} |f(t)|, \quad f \in L^\infty((0,m(1)))\Big),$$

where

$$a = u|a|, \quad u = e_{\langle 0,\infty)} - e_{(-\infty,0)}, \quad a = \int_{-\infty}^{\infty} \lambda\, de_\lambda,$$

$$\psi(\lambda) = \lambda^{1-1/\delta}, \quad 1 < \delta < \infty .$$

REFERENCES

[1] L.J. Ciach: On the conjugates of some operator spaces, I, Demonstratio Math. 18 (2) (1985) 537-554.

[2] L.J. Ciach: Subadditive measure on projectors of a von Neumann algebra, Doctoral dissertation, Univ. Łódź, 1983.

[3] M. Cwikel: The dual of weak L^p, Ann. Inst. Fourier, Grenoble, 25, 2 (1975) 81-126.

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T h e o r e m 3. Let m ∈ [illegible] (E_∞ respectively) be [illegible] such that [illegible] for some [illegible], $g(a) = \inf$ [illegible]. Then there exists a positive [illegible] the function [illegible] such that

(1) [illegible]

(2) [illegible]

(3) [illegible] ≤ const [illegible]

where

[illegible]

REFERENCES

[1] L.J. C i a c h : On the conjugates of some operator spaces, I, Demonstratio Math. 18 (2) (1985) [illegible].

[2] L.J. C i a c h : Subadditive measure on projections of a von Neumann algebra, Doctoral dissertation, Univ. Łódź, 1983.

[3] M. C w i k e l : The dual of weak L^p, Ann. Inst. Fourier, Grenoble, 25, 2 (1975) 81-126.

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ON GROUP-LIKE BCI-ALGEBRAS

1. By a BCI-algebra we mean a non-empty set G together with a binary multiplication denoted by juxtaposition and some distinguished element 0 such that the following axioms are satisfied:

(1) $((xy)(xz))(zy) = 0$,

(2) $xy = yx = 0$ implies $x = y$,

(3) $x0 = x$.

This axioms system is independent (cf. [3]). As a simple consequence of the above axioms system (cf. [3], [6]) we obtain:

(4) $(x(xy))y = 0$,

(5) $xx = 0$,

(6) $(xy)z = (xz)y$.

A BCI-algebra is called a BCK-algebra if it satisfies the identity

(7) $0x = 0$.

The class of all BCK-algebras is a proper subquasivariety of the quasivariety of all BCI-algebras. This subquasivariety is uniquely determined by the independent axioms system: (1), (2), (3) and (7) (cf. [3]).

On the other hand, one can prove that the quasivariety of all BCI-algebras contains some classes of BCI-algebras which form a variety. For example: the class of all medial BCI-algebras (cf. [2]), i.e. BCI-algebras with the additional identity

(8) $(xy)(zu) = (xz)(yu)$,

and the class of all BCI-algebras with the Iséki's condition (S) (cf. [1]) are varieties which contain only one-elements BCK-algebras.

In this paper we describe the variety of all BCI-algebras which are quasigroups.

The main result of this paper was presented during the Conference on Universal Algebra and its Applications, Opole (Jarłontówek), May 23-25, 1985.

2. A BCI-algebra $(G,\cdot,0)$ is called a BCI-quasigroup iff it is a quasigroup, i.e. iff for every $a,b \in G$ there exist uniquely determined elements $x,y \in G$ such that $ax = b$ and $ya = b$.

It is clear that a BCK-algebra which is a quasigroup has only one element.

We start from some simple examples.

Examples. It is not difficult to verify that the following algebras are BCI-quasigroups:

1. The set of integers modulo 3 with respect to the operation defined as $x * y = x+2y+1$ and 1 as the distinguished element.
2. A Boolean group.
3. The centralizer Z_0 of 0 in a BCI-algebra $(G,\cdot,0)$, i.e. the set $Z_0 = \{x \in G : 0x = x0\} = \{x \in G : 0x = x\}$. It has only one element or is a Boolean group (cf. [3]).
4. The family of all subsets of a given set with the symmetric difference of sets as an operation.
5. If $(G,+,0)$ is a commutative group, then $(G,*,0)$, where $x * y = x-y$, is a BCI-quasigroup. This example shows that there exist BCI-quasigroups of all orders.
6. In [4] it will be proved that the mapping $\varphi(x) = 0x$ is a BCI-endomorphism such that BCI-algebras $\varphi(G)$ and $G/\mathrm{Ker}\varphi$ are isomorphic BCI-quasigroups.
7. From our Corollary 1 it follows that every medial BCI-algebra is a BCI-quasigroup.

In the sequel we give some characterizations of BCI-quasigroups. These characterizations are based on the following lemma.

Lemma 1. In BCI-algebras the following identities are equivalent:

(9) $(xy)(xz) = zy$,

(10) $x(xz) = z$,

(11) $0(xz) = zx$.

Proof. Putting $y = 0$ in (9) and using (3) we obtain (10). From (10), (6) and (5) we get $zx = (x(xz))x = (xx)(xz) = 0(xz)$, which gives (11). Since from (11) there follows $0(0z) = z$, hence (11) implies also (9). Indeed, by (6) and (5) we get $(xy)(xz) = (x(xz))y = (0((xz)x))y = (0((xx)z))y = (0(0z))y = zy$, which completes the proof.

Lemma 2. For BCI-algebras the following conditions are equivalent:

1) $(G,\cdot,0)$ is a BCI-quasigroup,

2) for any $a \in G$ the equation $ax = 0$ has a unique solution $x \in G$,

3) for any $a \in G$ the equation $ya = 0$ has a unique solution $y \in G$.

Proof. In a BCI-quasigroup the equation $ax = 0$ has a unique solution for every $a \in G$. If the equation $ax = 0$ has a unique solution, then the equation $ya = 0$ also has a unique solution because $aa = ya = 0 = yy$ implies $y = a$. If only $y = a$ satisfies the equation $ya = 0$, then (1) implies (9). Thus (10) and (11) hold and $ax = b$ for $x = ab$. If also $az = b$ for some $z \in G$, then $0 = bb = (az)(ax) = (a(ax))z = xz$, which implies $x = z$. Hence the equation $ax = b$ has a unique solution for all $a,b \in G$.

Similarly for $y = b(0a)$ we have $ya = (b(0a))a = (ba)(0a) = 0((0a)(ba)) = 0((0(ba))a) = a(0(ba)) = a(ab) = b$, i.e. the equation $ya = b$ has a solution for every $a,b \in G$. If also $za = b$, then $zz = 0 = (za)(ya) = (z(ya))a = (0((ya)z))a = (0((yz)a))a = (a(yz))a = 0(yz) = zy$, which gives $y = z$.

Thus the solution of the equation $ya = b$ also exists and is uniquely determined. This proves that $(G,\circ,0)$ is a quasigroup.

Lemma 3. A BCI-algebra is a quasigroup iff it satisfies (10).

Proof. Obviously any BCI-quasigroup satisfies (10). Conversely, if (10) holds, then for every $a \in G$ the equation $ax = 0$ has only one solution $x = a$. Indeed, $aa = 0$ by (5) and $ax = 0$ gives $ax = 0 = (aa)(ax) = (a(ax))a = xa$, which implies by (2) $x = a$.

As a simple consequence of the above result we obtain

Corollary 1. A BCI-algebra is a quasigroup iff it is medial.

Proof. Every medial BCI-algebra satisfies (10) because $x(xy) = (x0)(xy) = (xx)(0y) = 0(0y) = (yy)(0y) =$ $= (y0)(yy) = y0 = y$. Thus this BCI-algebra is a quasigroup.

On the other hand, if it is a quasigroup, then (11) holds. Hence $(xy)(zu) = (x(zu))y = (0((zu)x))y = (0((zx)u))y =$ $= (u(zx))y = (uy)(zx) = (0(yu))(zx) = (0(zx))(yu) = (xz)(yu)$, which gives (8).□

We say that a BCI-algebra $(G,\cdot,0)$ satisfies the Iséki condition (S) or, as we will alternatively say, has a sum + (cf. [8]), if there exists a largest element, denoted by $a+b$, in the set $\{x \in G : xa \leqslant b\}$, for any $a,b \in G$, where $\leqslant$ is a partial order in G defined by the formula: $x \leqslant y$ iff $xy = 0$. Observe that from Lemma 2 follows that a BCI-algebra is a quasigroup iff $\leqslant$ is a trivial order.

In [1] it is proved that a BCI-algebra $(G,\cdot,0)$ has a sum + and $(G,+)$ is a group iff $(G,\cdot,0)$ satisfies (10). Applying this result to Lemma 3 and Corollary 1 we obtain

Corollary 2. A BCI-algebra $(G,\cdot,0)$ has a sum + such that $(G,+)$ is a group if it is a BCI-quasigroup, or equivalently, if it is medial.

As an immediate consequence of the above result and Theorem 1 from [1], we obtain

Theorem 1. The class of all BCI-quasigroups is equationally definable by (6) and (10). These identities are independent.

3. Let $(G,\cdot,0)$ be a BCI-algebra and let $x+y = x(0y)$. A groupoid $(G,+)$ is called an associated groupoid of $(G,\cdot,0)$.

Theorem 2. An associated groupoid has a neutral element iff it is a commutative group.

Proof. Let e be a neutral element of $(G,+)$. Thus from the definition $0 = e+0 = e(00) = e$, which shows that $e = 0$. Since $x = 0+x = 0(0x)$ for every $x \in G$, then

$$x+y = x(0y) = (0(0x))(0y) = (0(0y))(0x) = y(0x) = y+x,$$

which gives the commutativity of $(G,+)$. Hence

$$(x+y)+z = (y+x)+z = (y(0x))(0z) =$$
$$= (y(0z))(0x) = (y+z)+x = x+(y+z)$$

and

$$0x+x = x+0x = x(0(0x)) = x(0+x) = xx = 0.$$

This ends the first part of our proof. The second part is obvious.

Corollary 3. A BCI-algebra is a BCI-quasigroup iff its associated groupoid has a neutral element, or equivalently, iff it satisfies the identity

(12) $0(0x) = x$.

Proof. The condition (12) is a consequence of (10) which is satisfied in every BCI-quasigroup (Lemma 3). Hence $0+x = x = x+0$ for all $x \in G$. Conversely, if (12) holds, then $(G,+)$ has a neutral element and it is a group (Theorem 2). Thus $x-y = x+(-y) = x+0y = x(0(0y)) = xy$, i.e. $(G,\cdot,0)$ is a BCI-quasigroup.

Corollary 4. A BCI-algebra $(G,\cdot,0)$ is a BCI-quasigroup iff $xy = x-y$, where $(G,+)$ is an associated group of $(G,\cdot,0)$.

This result can be restated in the form

Corollary 5. A BCI-algebra $(G,\cdot,0)$ is a BCI-quasigroup iff there exists a commutative group $(G,+,0)$ such that $xy = x-y$.

As a simple consequence of the above considerations we obtain the main result of this section. We give this result in the language of definitionally equivalent classes (cf. [7] or [5] p.351).

Theorem 3. The variety of all BCI-quasigroups and the variety of all abelian groups are definitionally equivalent.

Lemma 3 shows that the variety of all BCI-quasigroups is selected from the quasivariety of all BCI-algebras by (10). This condition is equivalent to (8) (Corollary 1), (9), (11) (Lemma 1) and (12) (Corollary 3). Obviously (8) implies

$$(13)\quad (xy)(zx) = (xz)(yx).$$

Putting in this identity $x = y$ and $z = 0$ we obtain (12). Thus (8) and (13) are equivalent. Similarly putting $y = z = 0$ in

$$(14)\quad ((xy)z)((xu)y) = uz$$

we obtain (10). But in every BCI-quasigroup we have

$$((xy)z)((xu)y) = ((xy)z)((xy)u) = ((xy)(xy))(zu) = 0(zu) = uz,$$

which gives the following characterization of BCI-quasigroups.

Theorem 4. The variety of all BCI-quasigroups is selected from the quasivariety of all BCI-algebras by everyone of identities (8)-(14).

4. It is easy to show that a non-empty subset of a BCI-algebra is a subalgebra iff it is closed with respect to BCI-operation. Thus a non-empty subset of a BCI-quasigroup is a subquasigroup iff it is a subgroup of its associated group.

A non-empty subset A of a BCI-algebra is called an ideal if $0 \in A$ and if $x, yx \in A$ implies $y \in A$. If $(G,\cdot,0)$ is a BCI-quasigroup and $(G,+)$ is its associated group, then a non-empty subset A of G is an ideal iff $0 \in A$ and if $x, y-x \in A$ implies $y \in A$. Obviously any subalgebra of a BCI-quasigroup is an ideal. The converse is not true. For example, the set N of all non-negative integers is an ideal of a BCI-quasigroup determined by the additive group of all integers, but N is not a sub-BCI-quasigroup.

Theorem 5. A finite subset of a BCI-quasigroup is an ideal iff it is closed with respect to BCI-operation.

Proof. From the above remarks it follows that any subset which is closed with respect to BCI-operation is an ideal. Conversely, if A is a finite ideal of a BCI-quasigroup $(G,\cdot,0)$, then $0 \in A$. Moreover, if $x_1 \in A$, then $x_1+x_1 \in A$, because $(x_1+x_1)-x_1 = x_1 \in A$. Since $3x_1-x_1 = 2x_1 \in A$, then also $3x_1 \in A$ and so on. Thus A contains a cyclic subgroup $C(x_1)$ of $(G,+)$ generated by $x_1 \in A$. If $x_2 \in A - C(x_1)$, then also $C(x_2) \subset A$ and $(nx_1+mx_2)-mx_2 \in C(x_2) \subset A$, which shows that $C(x_1) + C(x_2) \subset A$. In the next step we select an element $x_3 \in A - (C(x_1) + C(x_2))$ et cetera. In this way we obtain the sequence $x_1,x_2,x_3,\dots$ of elements of A. But A is finite, hence this sequence is finite and $A = C(x_1) + C(x_2) + \dots + C(x_n)$ for some natural n, which shows that A is a subgroup of $(G,+)$. Thus A is closed with respect to BCI-operation.

Corollary 6. A finite subset of a BCI-quasigroup is an ideal iff it is a subgroup of its associated group, i.e. iff it is a sub-BCI-algebra.

Direct computations show that a relation ρ is a congruence relation of a BCI-quasigroup iff it is a congruence relation of its associated group. Hence ρ is a congruence of a BCI-quasigroup $(G,\cdot,0)$ iff there exists a sub-BCI-quasigroup A (i.e. a subgroup of $(G,+)$) such that $x \rho y$ iff $xy \in A$ (i.e. $x-y \in A$).

Similarly it is not difficult to verify that ψ is a homomorphism of BCI-quasigroups iff it is a homomorphism of their associated groups. Therefore a BCI-quasigroup is free in the variety of all BCI-quasigroups iff its associated group is free in the variety of all abelian groups. Obviously a free BCI-quasigroup is a direct product (cf. [2]) of its subalgebras. In the other words, a BCI-quasigroup is free iff its associated group is a direct product of infinite cyclic groups.

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Added in proof. Similar results as in our Corollaries 3, 4 and 5 have been, independently, obtained by L. Tiande and X. Changchang in "p-radicals in BCI-algebras" Math. Japonica 30 (1985), 511-517.

REFERENCES

[1] W.A. Dudek: On some BCI-algebras with the condition (S), Math. Japonica 31 (1986), 25-29.

[2] W.A. Dudek: On medial BCI-algebras, Prace Nauk. WSP, Częstochowa, Matematyka 1 (1987), 25-33.

[3] W.A. Dudek: On the axioms system for BCI-algebras, Prace Nauk. WSP, Częstochowa, Matematyka 2 (1988), (in print).

[4] W.A. Dudek, J. Tomys: On decompositions of BCH-algebras, (to appear).

[5] G. Grätzer: Universal algebra, second edition, Springer-Verlag 1979.

[6] K. Iséki: On BCI-algebras, Math. Sem. Notes 8 (1980), 125-130.

[7] B. Jónsson: Congruence varieties, Algebra Universalis 10 (1980), 355-394.

[8] H. Yutani: An axiom system for a BCK-algebra with the condition (S), Math. Sem. Notes 7 (1979), 427-432.

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ON THE PROPERTY W FOR THE GROUP UT(n, F)

Throughout the paper $UT(n,F)$ is the group of unitriangular matrices of degree n over a field F of characteristic $p>0$ i.e. $UT(n,F)$ is the subgroup of the linear group $GL(n,F)$ consisting of upper triangular matrices with diagonal entries equal to 1. It is well known that $UT(n,F)$ is a p-group and that the group is regular for $n \leqslant p$ (see [4]).

The aim of this paper is to study the W-property for the group $UT(n,F)$. Recall [1] that a p-group G has the property W (is a W-group) if for every natural number n such that

$K_n = \{x \in G : o(x) = p^n\} \neq \emptyset$, K_nK_n is a subgroup of G.

In ([2]) it was proved that each regular p-group is a W-group. Hence if $n \leqslant p$ then $UT(n,F)$ is a W-group. In this paper we obtain that it is also true for $n \leqslant p^2$. We obtain this as a consequence of the following theorem.

Theorem 1. If $\exp UT(n,F) = p^m$, $F \neq GF(2)$ then for $\frac{m}{2} < r \leqslant m$ we have $K_rK_r = UT(n,F)$.

For the proof we need some technical lemmas.

Lemma 1. If $A = (a_{ij}) \in UT(n,F)$ and $A^m = (b_{ij})$ then

$$
(1) \qquad b_{ij} = \begin{cases} \binom{m}{1} a_{i,j} + \binom{m}{2} \sum\limits_{i<k_1<j} a_{i,k_1} a_{k_1,j} + \ldots + \\ + \binom{m}{j-i} \sum\limits_{i<k_1<\ldots<k_{j-i-1}<j} a_{i,k_1} \cdots a_{k_{j-i-1},j} & \text{for } i<j \\ 1 & \text{for } i=j \\ 0 & \text{for } i>j. \end{cases}
$$

This can be proved by an easy induction.

Using (1) and the fact that $p \Big| \binom{p^m}{k}$ for $k = 1, \ldots, p^m-1$ we obtain the following:

Lemma 2. If $A = (a_{ij}) \in UT(n,F)$ and $A^{p^m} = (b_{ij})$ then

$$
(2) \qquad b_{ij} = \begin{cases} \sum\limits_{i<k_1<\ldots<k_{p^m-1}<j} a_{i,k_1} \cdots a_{k_{p^m-1},j} & \text{for } j-i \geqslant p^m \\ 1 & \text{for } i=j \\ 0 & \text{otherwise.} \end{cases}
$$

Lemma 3. $\exp UT(n,F) = p^m$ if and only if $p^{m-1}+1 \leqslant n \leqslant p^m$.

Proof. Suppose first that $\exp UT(n,F) = p^m$. Then for every $A \in UT(n,F)$ we have $A^{p^m} = E$ and there exists a matrix X such that $X^{p^{m-1}} \neq E$.

By Lemma 2 $j-i < p^m$ for $i,j = 1, \ldots, n$ and thus $n-1 < p^m$. If $n < p^{m-1}+1$ then by (2) $A^{p^{m-1}} = E$ for each matrix A from $UT(n,F)$, so $\exp UT(n,F) \leqslant p^{m-1}$, a contradiction. Hence $p^{m-1}+1 \leqslant n$.

Now suppose that $p^{m-1}+1 \leqslant n \leqslant p^m$. As an obvious consequence of Lemma 2 we obtain that $\exp UT(n,F) \leqslant p^m$. Now let $A = (a_{ij}) \in UT(n,F)$ where

$$a_{ij} = \begin{cases} 1 & \text{for } i = j \text{ or } j = i+1 \\ 0 & \text{otherwise.} \end{cases}$$

By Lemma 2 $A^{p^{m-1}} = (b_{ij})$ with $b_{1,p^{m-1}+1} = a_{12} \cdots a_{p^{m-1},p^{m-1}+1} = 1$.

Thus $A^{p^{m-1}} \neq E$. It implies that $o(A) = p^m$ and hence $\exp UT(n,F) = p^m$. The Lemma is proved.

For given natural numbers t,s let $M_{t\times s}(F)$ denote the set of all $t\times s$ matrices over F. Now let $r \leq n$ be fixed natural numbers and let $n = (k-1)r + l$, where k,l are natural numbers and $0 < l \leq r \leq n$. Every matrix of $M_{n\times n}(F)$ can be represented in the form $X = (X_{ij})$ where

$$(3) \qquad X_{ij} \in \begin{cases} M_{r\times r}(F) & \text{for } 1 \leq i \leq k-1, \ 1 \leq j \leq k-1, \\ M_{r\times l}(F) & \text{for } i = k, \ 1 \leq j \leq k-1, \\ M_{l\times r}(F) & \text{for } 1 \leq i \leq k-1, \ j = k, \\ M_{l\times l}(F) & \text{for } i = j = k. \end{cases}$$

The multiplication of elements of $M_{r\times l}(F)$ by elements of $M_{r\times r}(F)$ from the left and by the elements of $M_{l\times l}(F)$ from the right is well defined. So we can use identical formulas for multiplication of elements of $M_{n\times n}(F)$ as in the case where $X_{ij} \in F$ (see [3]).

Now we define two subsets of elements of $UT(n,F)$.

$A_r := \{A = (A_{ij}) : A_{ij}$ are as in (3) and $A_{ii} = E$ for $i = 2,\dots,k\}$,

$B_r := \{B = (B_{ij}) : B_{ij}$ are as in (3) and $B_{ij} = 0$ for $i \neq j\}$,

where E denotes an identity element of a suitable group of matrices.

Now we prove some properties of these sets.

Lemma 4. If $A = (A_{ij}) \in A_r$ and $A_{11} = E$, then for $p^s \geq k$ $o(A) \leq p^s$.

Proof. Let $A = (A_{ij})$ be an element of A_r satisfying the assumptions of the Lemma. From (2) we have $A^{p^s} = (c_{tu})$, where $c_{tu} \in F$ and

$$(4)\qquad c_{tu} = \begin{cases} \sum\limits_{t<t_1<\ldots<t_{p^s-1}<u} a_{t,t_1}\ldots a_{t_{p^s-1},u} & \text{for } u-t \geqslant p^s, \\ 1 & \text{for } u = t, \\ 0 & \text{otherwise.} \end{cases}$$

We show that all the products $a_{t,t_1}\ldots a_{t_{p^s-1},u}$ occurring in the above sum are equal to zero ($a_{t_i,t_j} \in F$ are entries of A). Suppose by way of contradiction that $a_{t,t_1}a_{t_1,t_2}\ldots$ $\ldots a_{t_{p^s-1},u} \neq 0$ and $t = t_0 < t_1 < \ldots < t_{p^s-1} < t_{p^s} = u$.

Then no factor of this product is equal to zero. The assumptions and the definition of A_r imply that a_{t_0,t_1} is not an entry of A_{11}, so $t_1 > r$. Since $r < t_1 < t_2$ and $a_{t_1,t_2} \neq 0$ (that is a_{t_1,t_2} is not an entry of A_{22} and A_{11}), $t_2 > 2r$. Similarly $t_3 > 3r$ and by an easy induction $u = t_{p^s} > p^s r \geqslant kr \geqslant$ $\geqslant (k-1)r + l = n$ which is impossible. Since $c_{tu} = 0$ for $u-t \geqslant p^s$, it follows that $A^{p^s} = E$.

Lemma 5. Let $\exp UT(n,F) = p^m$, $A = (A_{ij}) \in \mathbb{A}_{p^r}$ for $\frac{m}{2} < r \leqslant m$. If $A_{11} = (a_{ij})$ is a matrix such that

$$a_{ij} = \begin{cases} 1 & \text{for } i = j \\ 0 & \text{for } i>j \text{ or } j = i+1 \text{ and } i=p^{r-1}+1,\ldots,p^r-1 \text{ or } j > i+1 \end{cases}$$

then $o(A) \leqslant p^r$.

Proof. From the assumptions $n = (k-1)p^r + l$, $n \leqslant p^m$, $0 < l \leqslant p^r$, and $m < 2r$, i.e. $m \leqslant 2r-1$, it follows that $k = 1 + (n-l)p^{-r} < 1 + np^{-r} \leqslant 1 + p^{r-1}$. Thus $k \leqslant p^r$.

Let $A = (A_{ij})$ be an element of $\mathbb{A}_{p^r}$ satisfying the assumptions of the Lemma. Using (2) we have $A^{p^r} = (c_{tu})$ where $c_{tu} \in F$ and

$$(5)\quad c_{tu} = \begin{cases} \sum\limits_{t<t_1<\ldots<t_{p^r-1}<u} a_{t,t_1}\ldots a_{t_{p^r-1},u} & \text{for } u-t \geqslant p^r, \\ 1 & \text{for } u = t, \\ 0 & \text{otherwise.} \end{cases}$$

We show that all the products $a_{t,t_1}\ldots a_{t_{p^r-1},u}$ occurring in the above sum are equal to zero ($a_{t_i,t_j} \in F$ are entries of A). Suppose by way of contradiction that $a_{t,t_1}\ldots a_{t_{p^r-1},u} \neq 0$ and $t = t_0 < t_1 < \ldots < t_{p^r-1} < t_{p^r} = u$. Then no factor of this product is equal to zero. Let m denote the number of entries $0 \neq a_{t_i,t_{i+1}} \in A_{11}$ in this product. Of course this entries occur at the first m positions of the product. Since there are at most p^{r-1} entries not equal to zero over the main diagonal, we have $m \leqslant p^{r-1}$. Let $a_{t_j,t_{j+1}}$ be the first entry $0 \neq a_{t_j,t_{j+1}} \notin A_{11}$ of this product. Thus $t_{j+1} > p^r$. Since $p^r < t_{j+1} < t_{j+2}$ and $a_{t_{j+1},t_{j+2}} \neq 0$, (that is $a_{t_{j+1},t_{j+2}}$ is not entry of A_{11}, A_{22}) $t_{j+2} > 2p^r$. Similarly $t_{j+3} > 3p^r$ and by easy induction $n > (p^r-m)p^r \geqslant (p^r-p^{r-1})p^r = p^{r-1}(p-1)p^r \geqslant$ $\geqslant p^{r-1}p^r \geqslant kp^r \geqslant (k-1)p^r + 1 = n$ which is impossible. Since $c_{tu} = 0$ for $u-t \geqslant p^r$, it follows that $A^{p^r} = E$.

Lemma 6. If $B = (B_{ij}) \in B_r$, then for $p^s \geqslant r$ $o(B) \leqslant p^s$.

Proof. Let $B = (B_{ij})$ be an element of B_r. From the definition of B_r it follows $B_{ii} \in UT(r,F)$ for $i = 1,\ldots,k-1$, $B_{kk} \in UT(1,F)$ and $B_{ij} = 0$ for $i \neq j$. Let $B^{p^s} = C = (C_{ij})$ where $p^s \leqslant r$. Observe that $C_{ii} = B_{ii}^{p^s} \in UT(r,F)$ for $i = 1,\ldots,k-1$, $B_{kk}^{p^s} \in UT(1,F)$ and $C_{ij} = 0$ for $i \neq j$. Using Lemma 3 we obtain $B_{ii} = E$ for $i = 1,\ldots,k$. Hence $o(B) \leqslant p^s$.

Proof of Theorem 1. Let $X = (x_{ij})$, $x_{ij} \in F$ be an element of the group UT(n,F). For each $\frac{m}{2} < r \leq m$ we find matrices $A = (a_{ij})$ and $B = (b_{ij})$, with $a_{ij}, b_{ij} \in F$ such that $X = AB$ and $A, B \in K_r$.

Let $y_{i,i+1}$, $z_{i,i+1}$ be nonzero elements of F such that $x_{i,i+1} = y_{i,i+1} + z_{i,i+1}$ for $i = 1,\dots,p^{r-1}$ (if $F \neq GF(2)$ such entries $y_{i,i+1}$, $z_{i,i+1}$ always exist).

We define matrices $A = (a_{ij})$ and $B = (b_{ij})$ with a_{ij}, $b_{ij} \in F$ in the following way:

$$A = (A_{ij}) \in A_{p^r} \quad \text{for} \quad n = (k-1)p^r + l, \quad 0 < l \leq p^r,$$

where

$$A_{ij} = \begin{cases} A_{11} & \text{for } i = j = 1 \\ E & \text{for } i = j = 2,\dots,k \\ X_{ij}X_{jj}^{-1} & \text{otherwise,} \end{cases}$$

and

$$A_{11} = (a_{ij}), \quad a_{ij} = \begin{cases} y_{ij} & \text{for } i = 1,\dots,p^{r-1},\ j = i+1 \\ 1 & \text{for } i = j \\ 0 & \text{otherwise,} \end{cases}$$

$$B = (B_{ij}) \in B_{p^r} \quad \text{for} \quad n = (k-1)p^r + 1, \quad 0 < l \leq p^r,$$

where

$$B_{ij} = \begin{cases} A_{11}^{-1}X_{11} & \text{for } i = j = 1 \\ X_{ii} & \text{for } i = j = 2,\dots,k \\ 0 & \text{otherwise.} \end{cases}$$

Notice that for the matrix $A_{11}^{-1}X_{11} = (b_{ij})$ we have $b_{ij} = z_{ij}$ for $i = 1,\dots,p^{r-1}$, $j = i + 1$. Hence we have $AB = X$.

Now we prove that $o(A) = o(B) = p^r$. From Lemma 5 it follows easily that $o(A) \leq p^r$. Using (2) we have $A^{p^{r-1}} = (d_{ij})$ with $d_{1,p^{r-1}+1} = y_{12} \dots y_{p^{r-1},p^{r-1}+1} \neq 0$. Thus $A^{p^{r-1}} \neq E$ and $o(A) = p^r$. Using Lemma 6 we obtain $B^{p^r} = E$. On the other hand $B^{p^{r-1}} = (g_{ij}) \neq E$, because by (2) $g_{1,p^{r-1}+1} = x_{12} \dots x_{p^{r-1},p^{r-1}+1} \neq 0$. Hence $o(B) = p^r$. The theorem is proved.

Now we show that this result cannot be extended to $F = GF(2)$.

Lemma 7. If r is a natural number and K_{r+1} is the set of elements of $UT(2^r+1, GF(2))$ of order 2^{r+1} then

$$K_{r+1}K_{r+1} = \left\{ A \in UT(2^r+1,GF(2)) \; : \; a_{ij} = 0 \text{ for } j=i+1,\ i=1,\dots,2^r \right\}.$$

Proof. By Lemma 3 $\exp UT(2^r+1,GF(2)) = 2^{r+1}$. Let $A = (a_{ij}) \in UT(2^r+1,GF(2))$ and $A^{2^r} = B = (b_{ij})$. The equality (2) gives

$$b_{ij} = \begin{cases} a_{12} \dots a_{2^r,2^r+1} & \text{for } i = 1,\ j = 2^r+1, \\ 1 & \text{for } i = j \\ 0 & \text{otherwise.} \end{cases}$$

Thus $K_{r+1} = \left\{ A = (a_{ij}) \in UT(2^r+1,GF(2)) \; : \; a_{12} \dots a_{2^r,2^r+1} \neq 0 \right\} = \left\{ A = (a_{ij}) \in UT(2^r+1,GF(2)) \; : \; a_{i,i+1} = 1 \text{ for } i = 1,\dots,2^r \right\}$.

Let $H = \left\{ A = (a_{ij}) \in UT(2^r+1,GF(2)) : a_{ij} = 0 \text{ for } j=i+1,\ i=1,\dots,2^r \right\}$.

First we show that $K_{r+1}K_{r+1} \subseteq H$.

If $A = (a_{ij})$, $B = (b_{ij})$ are matrices from K_{r+1} then $AB \in H$ (because $AB = C = (c_{ij})$ where $c_{i,i+1} = a_{i,i+1} + b_{i,i+1} = 0$ for $i = 1,\dots,2^r$).

On the other hand every element $X = (x_{ij})$ from H can be written as the product of the following matrices $A = (a_{ij})$, $B = (b_{ij})$ from K_{r+1}

$$A = (a_{ij}) \text{ where } a_{ij} = \begin{cases} 1 & \text{for } i = j \text{ or } j = i+1 \\ x_{ij} - a_{i,j-1} & \text{for } j > i+1 \\ 0 & \text{for } i > j, \end{cases}$$

$$B = (b_{ij}) \text{ where } b_{ij} = \begin{cases} 1 & \text{for } i = j \text{ or } j = i+1 \\ 0 & \text{otherwise,} \end{cases}$$

$$X = AB \in K_{r+1}K_{r+1}.$$

Hence $K_{r+1}K_{r+1} = H$.

Lemma 8. Let G be p-group, $K_1 = \{x \in G : o(x) = p\}$. Then

a) $K_1 \subseteq K_1K_1$ for $p \neq 2$,

b) $K_1 \subset K_1K_1$ for $p = 2$ and $|K_1| > 1$.

Proof.

a) For any $x \in K_1$ we have $x = x^2x^{-1} \in K_1K_1$.

b) Let g be an element of $Z(G) \cap K_1$. For $x \in K_1 \setminus \{g\}$ we have $x = gxg = (gx)g \in K_1K_1$ and $g = xgx = (xg)x \in K_1K_1$.

Theorem 2. If $n \leqslant p^2$, then UT(n,F) is a W-group.

Proof. If $n \leqslant p$ then the theorem is a consequence of the above quoted results on regular p-group. (It is also a simple consequence of Lemmas 3 and 8).

Suppose now that $p < n \leqslant p^2$. Thus $UT(n,F) = \{E\} \cup K_1 \cup K_2$. Obviously $E \in K_1K_1$ and by Lemma 8 $K_1 \subseteq K_1K_1$. Let $X = (x_{tu}) = (X_{ij})$ be an element of K_2 where $x_{tu} \in F$ and X_{ij} is the same as in (3) with the additional assumption $r = p$.

Now we define elements $A, B \in K_1$ such that $X = AB$. Let $A = (A_{ij}) \in A_p$ where

$$A_{ij} = \begin{cases} X_{ij}X_{jj}^{-1} & \text{for } i \leqslant j \\ 0 & \text{for } i > j, \end{cases}$$

and $B = (B_{ij}) \in B_p$ where

$$B_{ij} = \begin{cases} X_{ij} & \text{for } i = j \\ 0 & \text{otherwise.} \end{cases}$$

From Lemmas 4 and 6 we have $o(A) \leqslant p$, $o(B) \leqslant p$. This and the fact that $o(X) = p^2$ imply that $A \neq E$, $B \neq E$. Hence $X = AB \in K_1K_1$. Using Theorem 1 one obtains $K_2K_2 = UT(n,F)$ for $F \neq GF(2)$. If $F = GF(2)$ we have to consider two groups $UT(3,GF(2))$ and $UT(4,GF(2))$.

The first one is the dihedral group of order 8. Since $3 = 2^1 + 1$, we have clearly by Lemma 7 $K_2K_2 = Z(UT(3,GF(2)))$.

Let us consider now the group $UT(4,GF(2))$. Obviously $UT(4,GF(2)) = \{E\} \cup K_2 \cup K_1$. Using (2) one obtains

$K_1 = \{A \in UT(4,GF(2)) : a_{12}a_{23}=0,\ a_{23}a_{34}=0,\ a_{12}a_{24}+a_{13}a_{34}=0,\ A \neq E\}$.

Thus $K_2 = \{A \in UT(4,GF(2)) : a_{12}a_{23} \neq 0 \text{ or } a_{23}a_{34} \neq 0 \text{ or } a_{12}a_{24}+a_{13}a_{34} \neq 0\}$. We give below some subsets of the set K_2:

$Z_1 = \{A \in UT(4,GF(2)) : a_{12}=1,\ a_{23}=1,\ a_{34}=0\}$,

$Z_2 = \{A \in UT(4,GF(2)) : a_{12}=1,\ a_{23}=1,\ a_{34}=1\}$,

$Z_3 = \{A \in UT(4,GF(2)) : a_{12}=0,\ a_{23}=1,\ a_{34}=1\}$,

$Z_4 = \{A \in UT(4,GF)2)) : a_{12}=0,\ a_{23}=0,\ a_{34}=1,\ a_{13}=1\}$,

$Z_5 = \{A \in UT(4,GF(2)) : a_{12}=1,\ a_{23}=0,\ a_{34}=0,\ a_{24}=1\}$,

$Z_6 = \{A \in UT(4,GF(2)) : a_{12}=1,\ a_{23}=0,\ a_{34}:1,\ a_{13}=1,\ a_{24}=0\}$,

$Z_7 = \{A \in UT(4,GF(2)) : a_{12}=1,\ a_{23}=0,\ a_{34}=1,\ a_{13}=0,\ a_{24}=1\}$.

It is easy to check that $Z_i \cap Z_j = \emptyset$ for $i \neq j$, $i,j=1,\dots,7$ and that $|Z_i| = 8$ for $i = 1,2,3$, $|Z_i| = 4$ for $i = 4,5$, $|Z_i| = 2$ for $i = 6,7$. Thus $|K_2| \geqslant \sum_{i=1}^{7} |Z_i| = 8 + 8 + 8 + 4 + 4 + 2 + 2 = 36$ and $|UT(4,GF(2))| = 2^6 = 64$. Therefore $|K_2| > \frac{|UT(4,GF(2))|}{2}$. This implies that for each $X \in UT(4,GF(2))$ we have

$$|XK_2| = |K_2| > \frac{|UT(4,GF(2))|}{2} .$$

Thus $XK_2 \cap K_2 \neq \emptyset$ and there exist matrices $A, B \in K_2$ such that $XA = B$. Hence $X = BA^{-1} \in K_2K_2$. Therefore $UT(4,GF(2)) \in K_2K_2$ and the Theorem is proved.

REFERENCES

[1] E. Ambrosiewicz : On groups having the property W. Demonstratio Math. 10 (1977) 361-383.

[2] E. Ambrosiewicz : The property W for regular p-groups and for nilpotent groups of degree 2, Demonstratio Math. 13 (1980) 613-617.

[3] G. Birkhoff, S. Mac Lane : A survey of modern algebra. New York (1962) 216-220.

[4] B. Huppert : Endliche Gruppen. Berlin (1967) 384-385.

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TWO REMARKS ABOUT HOMOTHETIC COVERINGS

Introduction

In this paper a body means a convex compact set of points (of E^n) which has non-empty interior.

I.C.Hochberg and A.S.Markus [5] had introduced a notion of a homothetic covering of a body F in E^n, i.e. such a covering of F whose each member is homothetic to F with a homothety-coefficient $k \in (0,1)$. Recall that a body F′ is called homothetic with respect to the body F if there exists such a homothety H_s^k (s is a centre of this homothety and k is its coefficient) that $F' = H_s^k(F)$.

The smallest number of bodies homothetic to F which cover F is denoted by b(F). In [5] the authors had shown that for a plane body F different from a parallelogram we have b(F) = 3 and for a parallelogram Q there is b(Q) = 4. They also conjectured there that for every n-dimensional body $Q \subset E^n$ there is $b(Q) \geqslant n+1$ and for every integer $m \in \{n+1,\dots,2^n\}$ there exists in E^n a convex bounded body F such that b(F) = m. The same is the contents of Hadwiger′s conjecture [4], formulated in other terms.

Let F be a body covered with $F_1,\dots,F_{n+1}$ homothetic to F and let k_i denote a homothety-coefficient of F_i ($i \in \{1,\dots,n+1\}$). It is known that $k_i \in (0,1)$ for every i but a quantity of each k_i depends on the others homothety-coefficients, i.e. k_i depends on $k_1,\dots,k_{i-1},k_{i+1},\dots,k_{n+1}$. If the quantities of k_i are independent one from another, then the family of bodies $F_1,\dots,F_{n+1}$ need not be a homothetic covering of F.

Yu. Belousov [1] has shown that a plane body F different from a parallelogram cannot be covered with three homothetic ones if $k_i < \frac{2}{3}$ for every $i \in \{1,2,3\}$. In the present paper we prove a generalization of this theorem for a body $F \subset E^n$ such that $b(F) = n+1$ (see Theorem 1). In [1] Belousov has also shown that any centrally-symetric plane body cannot be covered with three homothetic ones if $k_i < \frac{3}{4}$ for every i. It implies that for every class of plane bodies F there exists a liminal number $k_2^3(F) \in (0,1)$, i.e. such a number that if every $k_i < k_2^3(F)$, then no body F of this class can be covered with three homothetic ones and if every $k_i \geq k_2^3(F)$, then each F can be; e.g. for the class of triangles T we have $k_2^3(T) = \frac{2}{3}$, for the class of regular hexagons S we have $k_2^3(S) = \frac{3}{4}$ and $k_2^3(C) = \frac{\sqrt{3}}{2}$ for the class of circles C (see [1], [2]).

For an n-dimensional body F we can introduce a notion of m-th liminal number of F ($m \in \{n+1,\ldots,2^n\}$) (see [3], p.135), i.e. such a number $k_n^m(F) \in (0,1)$ that if $b(F) = m$ and $k_i < k_n^m(F)$ for every $i \in \{1,\ldots,m\}$, then F cannot be covered with m bodies homothetic to F and if $k_i \geq k_n^m(F)$, then F can be. As it follows from Belousov s theorem [1] the number $\frac{2}{3}$ is the 3-rd liminal number for the class of plane bodies different from parallelogram. The following question appears: does there exist for every $k \in \langle \frac{2}{3}, 1)$ a class of plane bodies such that k is the 3-rd liminal number for this class? The answer to this question is positive [2, Theorem 7, p.226]. In the present paper it will be shown that the answer to the same question for the n-dimensional bodies is also positive (see Theorem 2).

1. Let F be such a body in E^n that $b(F) = n+1$. Hence there exist n+1 centres $o_1, o_2, \ldots, o_{n+1}$ of homotheties that F is covered with the bodies $F_1, F_2, \ldots, F_{n+1}$ which are homothetic to F, each of which is the image of F under a respective homothety with the centre o_i ($i \in \{1,\ldots,n+1\}$). The points $o_1,\ldots,o_{n+1}$ cannot all belong to one hyperplane in E^n.

If they belong, then F cannot be covered with $F_1,\ldots,F_{n+1}$; a point of F, the distance of which to this hyperplane is the greatest, is covered with no homothetic body. The points $o_1,\ldots,o_{n+1}$ form an n-dimensional simplex, which will be called the simplex of homothety-centres for F.

Lemma. If $b(F) = n+1$ for some body $F \subset E^n$, then F is contained in its simplex of homothety-centres.

Proof. Assume that it is not so. Then in the semispace, determined by the hyperplane H^* through, e.g. $o_1,\ldots,o_n$ and not containing the simplex of homothety-centres for F, there is a part F of F (it may be even $F' = F$). Take the hyperplane H, supporting to F' and parallel to H^*, and let q belong to $H \cap F$. Denote by q_i the image of q under the homothety with the centre o_i ($i \in \{1,\ldots,n+1\}$) and remark that the simplex of homothety-centres is contained in the semispace (determined by H^*) not containing the points of H. Hence none of F_i has a common point with H and for every i the point q_i belongs to none of F_i which contradicts to hypothesis that $b(F) = n+1$.

Theorem 1. Any bounded convex body $F \subset E^n$, for which $b(F) = n+1$, cannot be covered with n+1 bodies homothetic to F if every homothety-coefficient is less than $\frac{n}{n+1}$.

Proof. Assume that it is not so, i.e. F is covered with $F_1,\ldots,F_{n+1}$ and for every homothety-coefficient k_i, where $i \in \{1,\ldots,n+1\}$, we have $k_i < \frac{n}{n+1}$. Then, in view of Lemma, F is contained in its simplex of homothety-centres S_o. Let a simplex S_i be the image of S_o under the homothety with the centre o_i and coefficient equal to $\frac{n}{n+1}$. The point o, which is the barycentre of S_o, belongs to every S_i and, moreover, belongs to this face of S_i which does not contain o_i. Hence, if $o \in F$, then for $k_i < \frac{n}{n+1}$ this point is not covered with S_i for each i, and it cannot be covered with F_i because $F_i \subset S_i$.

If $o \notin F$, then either none of (n-1)-dimensional faces of the simplex $B_i = (o_1 \ldots o_{i-1} o o_{i+1} \ldots o_{n+1})$ (for some i) intersects F or at least one of them does. In the first case at least one of $F_1,\dots,F_{n+1}$, namely F_i, has no common point with F because F is contained in B_i which is disjoint with S_i out of the point o, while $F_i \subset S_i$. In the second case denote by B_{ij} a face of B_i not containing o_j ($j \in \{1,\dots,n+1\}$, $j \neq i$) and let $q \in B_{ij} \cap \mathrm{bd}F$ be such a point that $\mathrm{dist}(o,q) = \min\{\mathrm{dist}(o,x) : x \in B_{ij} \cap \mathrm{bd}F\}$. Then $q \notin S_i$ and $q \notin S_j$ because $B_{ij} \cap S_i = \{o\}$ and $B_{ij} \cap S_j = \{o\}$. It implies that $q \notin F_i$ and $q \notin F_j$. Each of rays $\langle o_m, q)$ ($m \in \{1,\dots,n+1\}$, $i \neq m \neq j$) emanated from o_m and passing q is contained in a hyperplane including B_{ij}. Therefore, by convexity of F, this ray does not meet F behind q. Hence, $F_m \cap \langle o_m, q)$ lies nearer o_m than q which implies $q \notin F_m$ for each m.

2. It is known ([2], Theorem 7, p.226) that for every $k \in \langle \frac{2}{3}, 1)$ there exists a class of plane bodies such that k is the 3-rd liminal number of this class. We also have

Theorem 2. For every $k \in \langle \frac{n}{n+1}, 1)$ there exists in E^n a class of n-dimensional bodies that k is the (n+1)-th liminal number for it.

Proof. Let S^p be a frustum of an n-dimensional simplex S, where the ratio of measures of the upper base and the lower one is equal to p^{n-1} ($p \in \langle 0, 1)$. It is shown in ([3], Theorem 1) that

$$k_n^{n+1}(S^p) = \begin{cases} \dfrac{n}{n+1} & \text{if } p \in \langle 0, \frac{n-1}{n} \rangle \\[2ex] \dfrac{1}{2-p} & \text{if } p \in \left(\frac{n-1}{n}, 1\right) \end{cases}$$

is the (n+1)-th liminal number of the class of such frustums. The function $f(p) = \frac{1}{2-p}$ is continuous on $\langle \frac{n-1}{n}, 1 \rangle$ and it increases from $\frac{n}{n+1}$ (for $p = \frac{n-1}{n}$) to 1 (for p = 1). It implies that for each real $p \in \langle \frac{n-1}{n}, 1)$ there exists S^p whose (n+1)-th liminal number is equal to $\frac{1}{2-p} \in \langle \frac{n}{n+1}, 1)$.

In particular, $k_n^{n+1}(S) = \frac{n}{n+1}$ is the (n+1)-th liminal number of the class of n-dimensional simplexes S.

3. Some open questions appear there:

(a) From Belousov's paper [1] it follows that for each class of plane bodies there exists a respective 3-rd liminal number. He has shown that for the class of plane bodies Q different from a parallelogram $k_2^3(Q) = \frac{2}{3}$ and for the class of plane centrally-symmetric bodies D there is $k_2^3(D) = \frac{3}{4}$. Moreover, it is known [2] that for the class of circles C there is $k_2^3(C) = \frac{\sqrt{3}}{2}$. This same, probably, can be observed for different classes of n-dimensional bodies, e.g. for class of centrally-symmetric bodies, for bodies with a smooth boundary etc. What is (n+1)-th liminal number for each of these classes?

(b) What is the liminal number of a class of all bodies $F \subset E^n$, for which $b(F) = n+j$, where $j \in \{2,3,\dots,2^n-n\}$?

(c) If $k_n^{n+j}(F)$ is the (n+j)-th liminal number of a class of all bodies F, for which $b(F) = n+j$, does there exists a subclass of bodies G such that $k_n^{n+j}(G) = k$ for every $k \in \langle k_n^{n+j}(F), 1)$?

(d) Remark that a quantity of the liminal number of a body F depends not only from a form of F but also from a ratio of lengths of respective segments in F. All n-dimensional simplexes, in particular triangles, have the same (n+1)-th liminal number, namely $\frac{n}{n+1}$, while they have different forms. But the frustums of simplexes, obtained from the same simplexes, have different (n+1)-th liminal numbers, depending from a ratio of measures of the upper base and lower one. It is interesting that this ratio has no influence on the quantity of the (n+1)-th liminal number if it is not greater than $\left(\frac{n-1}{n}\right)^{n-1}$. What is the reason of this?

(e) The set of all bodies F, for which $b(F) = n+1$, can be divided into uncountably many classes, each of them has a different (n+1)-th liminal number (it follows from Theorem 2). The set of all bodies F, for which $b(F) = 2^n$, forms only one class, each member of which has the same 2^n-th liminal number (it is so in the case n = 2 - see [5] - but probably the same holds in the cases $n \geqslant 3$, too). Does there exist $j \in \left\{2,3,\dots,2^n-n\right\}$ that the set of bodies F, for which $b(F) = n+j$, can be divided into countably many or finitely many (at least two) classes, each of which has a different $(n+j)$-th liminal number?

REFERENCES

[1] Ю.Ф. Белоусов: Теоремы о покрытии плоских фигур, Ukrain. Geometr. Sb. 20 (1977), 10-17.

[2] S. Fudali: O pokryciach homotetycznych figur wypukłych, Matematyka 34 (1981), 221-228.

[3] S. Fudali: Homothetic covering of frustums of simplicies, Demonstratio Math. 16 (1983), 133-150.

[4] H. Hadwiger: Ungeloste Probleme, Elem. der Math. 12 (1957), 121.

[5] И.Ц. Гохберг, А.С. Маркус: Одна задача о покрытии выпуклых фигур подобными, Izv. Mołdav. Fil. Akad. Nauk SSSR 10 (1960), 87-90.

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Paweł Felcyn

LIMINAL NUMBER OF A CONVEX BODY WITH A SMOOTH BOUNDARY

1. Introduction

Boltianski [1] has proved that any n-dimensional convex body with a smooth boundary can be covered with n+1 smaller homothetic copies. Fudali [3] has shown that the (n+1)-th liminal number of a convex body in E^n is equal to $\frac{n}{n+1}$ and he asked there about such number for an n-dimensional convex body with a smooth boundary. In the present paper an answer to this question is given.

2. Preliminaries

In the paper, a homothetic copy of a body $Q \subset E^n$ will be denoted by k_jQ, where k_j is the ray-ratio of a respective homothety. Recall the notion of the m-th liminal number of Q [2, p.135]: Let $K^m(Q)$ denote the set of numbers $k \in (0,1)$, for which families $\{k_1Q,\ldots,k_mQ\}$ with $k_j \leq k$ do not cover Q. The least upper bound of $K^m(Q)$ is called the m-th liminal number of Q and denoted by $K_o^m(Q)$.

Note the following: For every $\varepsilon \in (0,1/2)$:

a) there is a smooth convex function $g_\varepsilon : [0,\varepsilon] \to R$ such that

$$g_\varepsilon(0) = g_\varepsilon(\varepsilon) = 1 - \varepsilon,$$

$$g'_{\varepsilon-}(\varepsilon) = -1,$$

$$g^{(n)}_{\varepsilon-}(\varepsilon) = 0 \quad \text{for} \quad n > 1,$$

$$g^{(n)}_{\varepsilon+}(0) = \infty \quad \text{for} \quad n \in N,$$

b) there is a smooth convex function $h_\varepsilon : [1-\varepsilon, 1-\varepsilon/2] \to R$ such that

$$h_\varepsilon(1-\varepsilon) = \varepsilon,$$
$$h_\varepsilon(1-\varepsilon/2) = 0,$$
$$h'_{\varepsilon+}(1-\varepsilon) = -1,$$
$$h^{(n)}_{\varepsilon+}(1-\varepsilon) = 0 \quad \text{for} \quad n > 1,$$
$$h^{(n)}_{\varepsilon-}(1-\varepsilon/2) = -\infty \quad \text{for} \quad n \in N,$$

where $g^{(n)}_{\varepsilon-}$ $(h^{(n)}_{\varepsilon-})$ and $g^{(n)}_{\varepsilon+}$ $(h^{(n)}_{\varepsilon+})$ denote left and right-hand-side n-th derivatives respectively.

Corollary. The function $k_\varepsilon : [0, 1-\varepsilon/2] \to R$ for $\varepsilon \in (0, 1/2)$ given by the formula

$$k_\varepsilon(t) = \begin{cases} g_\varepsilon(t) & \text{for} \quad t \in [0, \varepsilon] \\ 1-t & \text{for} \quad t \in [\varepsilon, 1-\varepsilon] \\ h_\varepsilon(t) & \text{for} \quad t \in [1-\varepsilon, 1-\varepsilon/2] \end{cases}$$

is smooth and convex. Moreover, for any $t \in [0, 1-\varepsilon/2]$ the inequalities

$$1-t-\varepsilon \leqslant k_\varepsilon(t) \leqslant 1-t$$

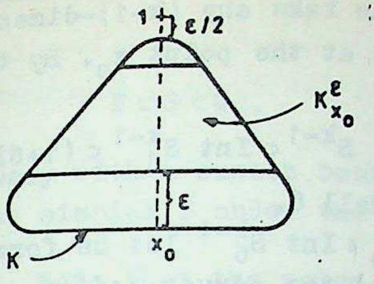
are satisfied.

3. Homothetic covering of a body with a smooth boundary

Lemma 1. Let $K \subset E^n$ be an n-dimensional convex bounded body with a smooth boundary. Let x_0 be an interior point of K. For any $t \in [0, 1-\varepsilon/2]$ let $k_\varepsilon(t)K$ be a body homothetic to K with the ray-ratio $k_\varepsilon(t)$ and with the centre at x_0. Then

$$(1) \quad K^\varepsilon_{x_0} = \{(x_1, \ldots, x_n, t) \in E^{n+1} : t \in [0, 1-\varepsilon/2], (x_1, \ldots, x_n) \in k_\varepsilon(t)K\}$$

is a n+1-dimensional convex body with a smooth boundary.



The proof follows from the construction of the function k_{ε}.

Remark. It is easy to check using the Corollary above that if

$$(2)\qquad K^{\varepsilon}_{x_0,1} = \left\{(x_1,\ldots,x_n,t) \in E^{n+1} : t \in [\varepsilon, 1-\varepsilon], \; (x_1,\ldots,x_n) \in (1-t-\varepsilon)K\right\},$$

$$(3)\qquad K^{\varepsilon}_{x_0,2} = \left\{(x_1,\ldots,x_n,t) \in E^{n+1} : t \in [0,1], (x_1,\ldots,x_n \in (1-t)K\right\},$$

then $K^{\varepsilon}_{x_0,2}$ is homothetic to $K^{\varepsilon}_{x_0,1}$ with the ray-ratio $\frac{1/2}{1/2-\varepsilon} = \frac{1}{1-2\varepsilon}$ and with the centre at $(x_0, 1/2) \in E^{n+1}$. Moreover, $K^{\varepsilon}_{x_0,1} \subset \operatorname{Int} K^{\varepsilon}_{x_0} \subset K^{\varepsilon}_{x_0,2}$.

Lemma 2. For any n-dimensional simplex S^n contained in E^n and for any sufficiently small number $\varepsilon > 0$ there exists a convex body S^n_{ε} with a smooth boundary such that its interior contains S^n and it is contained in the simplex $(1+\varepsilon)S^n$ homothetic to S^n. The respective homothety has the ray-ratio equal to $1+\varepsilon$ and the barycentre of the simplex S^n as the centre.

Proof. For $n = 1$ the Lemma is obvious. Let us assume that the Lemma holds for $n = k-1$. It is sufficient to prove the Lemma for one k-dimensional simplex only, since a homothety and a linear map commute, and an arbitrary k-dimensional simplex can be obtained from the considered one (the barycentre of a simplex S^k linearly depends on the vertices of S^k). Let us take any $(k-1)$-dimensional simplex S^{k-1} with the barycentre at the point x_0. By the inductive assumption we have

$$S^{k-1} \subset \operatorname{Int} S_{\sigma}^{k-1} \subset (1+\sigma) S^{k-1}$$

for sufficiently small σ.

For S_σ^{k-1} and $x_0 \in \operatorname{Int} S_\sigma^{k-1}$ let us form (by Lemma 1 and the Remark following it) the convex bodies

$$S = \left(S_\sigma^{k-1}\right)^{\sigma}_{x_0}, \quad S_1 = \left(S_\sigma^{k-1}\right)^{\sigma}_{x_0,1}, \quad S_2 = \left(S_\sigma^{k-1}\right)^{\sigma}_{x_0,2}$$

(compare (1), (2), (3)).

The body S_1 contains a k-dimensional simplex P such that: Each generatrix of the simplex P is parallel to the respective generatrix of the body S_1.

A base of the simplex P is the $(k-1)$-dimensional simplex $\left\{(S^{k-1},h)\right\}$, where h denotes the root of the equation $\frac{1}{1-2h} = (1+\sigma)\frac{1}{1-2\sigma}$. Note that $h \to 0$ iff $\sigma \to 0$. In this way the image of the base of P under the homothety with the centre at $(x_0, 1/2) \in E^k$ and with the ray-ratio $\frac{1}{1-2h}$ contains the base of S_2.

Let Q be the image of P under this homothety. Then we have

$$Q \supset S_2.$$

For a sufficiently small σ the barycentre p of P lies below the point $(x_0, 1/2)$. Therefore the image Q_1 of the simplex P under the homothetic transformation with the centre at p and with the ray-ratio

$$\frac{\overline{(x_0,0)p}}{\overline{(x_0,0)p} - h}$$

contains the simplex Q. Thus we have inclusions

$$P \subset S_1 \subset S \subset S_2 \subset Q \subset Q_1.$$

Therefore

$$P \subset S \subset Q_1, \tag{4}$$

where S is a convex body with a smooth boundary and Q_1 denotes the image of the simplex P under the homothetic transformation given above. Note that the value of $\frac{\overline{(x_0,0)p}}{\overline{(x_0,0)p} - h}$ tends to 1 if h tends to 0 (or if δ tends to 0).

Now it is sufficient to choose the number δ such that

$$\frac{\overline{(x_0,0)p}}{\overline{(x_0,0)p} - h} = 1 + \varepsilon.$$

By the virtue of (4) we have the inclusions $S^k \subset S \subset (1+\varepsilon)S^k$. The above inclusions hold for any dimension, because k was chosen arbitrarily.

4. Liminal number of a body with a smooth boundary

Theorem. The (n+1)-th liminal number of the family of all n-dimensional bodies in E^n with a smooth boundary is equal to $\frac{n}{n+1}$.

Proof. For any sufficiently small $\varepsilon > 0$ it is sufficient to find a n-dimensional body with a smooth boundary which may be covered with n+1 its homothetic copies, each of them with the ray-ratio $\frac{n}{n+1} + \varepsilon$.

Let S^n be any n-dimensional simplex with the barycentre at x_0. For any $i = 1,2,\dots,n+1$ let P_i' be an image of the simplex S^n under the homothetic transformation with the ray-ratio $\frac{n}{n+1}$ and with the centre at i-th vertex of S^n. Let P_i be an image of P_i under the homothetic transformation with the ray-ratio $1+\varepsilon\cdot\frac{n}{n+1}$ and with the centre at x_0.

By Lemma 2 there exists a convex body S with a smooth boundary for which we have

$$S^n \subset \operatorname{Int} S \subset \left(1+\varepsilon\cdot\frac{n+1}{n}\right) S^n.$$

We are looking for the homothetic covering of the body S. For any $i = 1,\ldots,n+1$ let f_i be a homothetic transformation with the ray-ratio $\left(1+\varepsilon\cdot\frac{n+1}{n}\right)\frac{n}{n+1} = \frac{n}{n+1}+\varepsilon$, which maps the S^n onto the simplex P_i'.

Now it is easy to notice that the family of sets $f_i(S)$ is just the required covering of S. Each $f_i(S)$ contains the simplex P_i' and the simplices P_i' cover the simplex $\left(1+\varepsilon\cdot\frac{n+1}{n}\right)S^n$ which contains the body S.

REFERENCES

[1] V.G. Boltianski: Zadacza ob oswieszczenii granicy wypukłowo tieła, Izv. Mołdav. Fil. Akad. Nauk SSSR 10 (1960), 79-85.

[2] S. Fudali: Homothetic coverings of frustums of simplicies, Demonstratio Math. 16 (1983) 133-150.

[3] S. Fudali: Two remarks about homothetic coverings Demonstratio Math. 21 (1988), 387-392.

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A GENERALIZATION OF SARD'S THEOREM AND A JACOBIAN CONDITION FOR INTERIORITY

Throughout this paper $D, D' \subset R^n$ are open and $f:D \to R^n$ is a map. We denote by $Z_f = \{x \in D \mid f$ is differentiable in x and $J_f(x) = 0\}$. A well-known theorem of Sard says that if $f:D \to R^n$ is a C^1 map, then $m(f(Z_f)) = 0$ (m denotes the Lebesgue measure in R^4). Also, a well-known theorem of Luzin says that if we have $f:(a,b) \to R$ continuous, then $m(f(Z_f)) = 0$, hence without assuming continuous differentiation on (a,b). Since in many problems in higher dimensions the Jacobian plays the same role as the derivative, it is natural to ask if Sard's theorem remains true if we drop the continuous differentiation condition on D. The answer is positive and it is surprising that we do not need even the continuity of f on D. More precisely, we shall prove that if $f:D \to R^n$ is a map, then $m(f(Z_f)) = 0$.

Our second aim is to find conditions for a map $f:D \to R^n$ to have the property N of Luzin. We shall prove that if $A \subset D$, $m(A) = 0$ and f is differentiable on A, then $m(f(A)) = 0$. Also it is known that not every homeomorphism $f:D \to D'$ has the property N of Luzin. We shall prove that f has the property N of Luzin with the possible exception of a "pathological" set of null measure, i.e. we find that there exists $K \subset D$ such that $m(K) = 0$ and for every $A \subset D$ such that $m(A) = 0$ and $A \cap K = \emptyset$, we have $m(f(A)) = 0$.

A known theorem of Titus and Yound [5], which generalizes a theorem of S.Stoilow [4], may be announced in this way:

Theorem A1. Let $f:D \to R^n$ be a C^1 map, light, such that $J_f(x) \geq 0$ for every $x \in D$ or $J_f(x) \leq 0$ for every $x \in D$. Then f is open on D.

We shall apply our results to prove the following generalization of the preceding theorem: "Let $f:D \to R^n$ be differentiable and light such that $J_f(x) \geq 0$ a.e. on D or $J_f(x) \leq 0$ a.e. on D. Then f is open on D".

We shall need the following definitions, notations and theorems:

Definition 1. Let $f:D \to R^n$. We denote $L(x,f) = \limsup_{h \to 0} \frac{\|f(x+h)-f(x)\|}{\|h\|}$.

Definition 2. Let $f:D \to R^n$. We say that

$\lim_{r \to 0} \frac{m(\overline{f(\bar{B}(a,r))})}{m(B(a,r))} = 0$ if, for every $\varepsilon > 0$, there exists $r_\varepsilon > 0$ such that for $r \leq r_\varepsilon$, $\frac{m(\overline{f(\bar{B}(a,r))})}{m(B(a,r))} < \varepsilon$.

Definition 3. Let $f:D \to R^n$. We denote $H_f = \{x \in D \mid f$ has partial derivatives in x and $J_f(x) = 0\}$.

Definition 4. Let $f:D \to R^n$. We say that f is light if $\dim f^{-1}(f(x)) = 0$ for every $x \in D$.

We denote by V(n) the volume of the unit ball in R^n and $N = (4[\sqrt{n}]+5)^n+1$.

Theorem A2. Let $f:D \to D'$ be continuous and light. Then for every $U \subset D$ open there follows $\operatorname{int} f(U) \neq \emptyset$.

Theorem A3. Let $f:D \to D'$ be a homeomorphism. Then there exists $K \subset D$ such that $m(K) = 0$ and, for every $a \in D \setminus K$, $\lim_{r \to 0} \frac{m(f(\bar{B}(a,r)))}{m(B(a,r))} < \infty$.

Theorem (Rademacher-Stepanov). Let $f:D \to R^n$ be continuous such that $L(x,f) < \infty$ a.e. Then f is differentiable a.e.

Theorem A2 may be found in [7], Theorem A3 may be found in [6] page 82 and the theorem of Rademacher and Stepanov may be found in [1] page 38.

I wish to thank Professor Solomon Marcus for his reading to manuscript and for all advices I received during the writing of this paper.

Lemma 1. Let $Q \subset R^n$ be a n-dimensional cube, $A \subset Q$ and $a > 0$ such that Q can be divided in subcubes of side a, the sides of every such subcube being parallel with the sides of Q. Then there exist $x_1, x_2, \dots, x_p \in A$ such that $A \subset \bigcup_{i=1}^{p} B(x_i, a\sqrt{n})$ and every point from Q is contained in at most N such balls.

Proof. Let us choose, in every subcube of the division which intersects A, a point from A. We denote this set by $x_1, x_2, \dots, x_p$ and we show that this set satisfies the conditions from the theorem. Indeed, if $x \in A$, there exists $1 \leqslant i \leqslant p$ such that $d(x, x_i) \leqslant a\sqrt{n}$, hence $x \in B(x_i, a\sqrt{n})$. Let us fix $j \leqslant p$. If $B(x_i, a\sqrt{n}) \cap B(x_j, a\sqrt{n}) \neq \emptyset$, then $d(x_i, x_j) \leqslant 2a\sqrt{n}$, hence x_i is contained in a cube of side $a(4[\sqrt{n}]+5)$ which has the middle point in the middle point of the subcube of side a containing x_j and the sides parallel with the sides of Q. Since this cube contains $N-1$ subcubes of side a from our division, we find that $B(x_j, a\sqrt{n})$ may be intersected by at most $N-1$ of the balls $B(x_i, a\sqrt{n})$, $1 \leqslant i \leqslant p$, and the theorem is proved.

Theorem 1. Let $f : D \to R^n$ and $A = \left\{ a \in D \,\middle|\, \lim_{r \to 0} \frac{\overline{m(f(\bar{B}(a,r))}}{m(B(a,r))} = 0 \right\}$. Then $m(f(A)) = 0$.

Proof. Let $Q, Q' \subset D$ be cubes such that $\bar{Q} \subset \operatorname{int} Q' \subset D$; the middle point of Q is the middle point of Q' and the sides of Q are parallel with the sides of Q'. Let d be the length of the side Q, d' be the length of the side of Q' and $k_0 \in N$ such that for $k \geqslant k_0$, $\frac{\sqrt{n}}{k} \leqslant \frac{d'-d}{2}$. Let $A_{mk} = \left\{ x \in Q \cap A \,\middle|\, \frac{m(\overline{f(\bar{B}(x,r))})}{m(B(x,r))} < \frac{1}{m} \text{ for every } r \leqslant \frac{\sqrt{n}}{k} \right\}$, $k \geqslant k_0$, $m \in N$. Then $A \cap Q = \bigcup_{k=k_0}^{\infty} A_{mk}$ for every $m \in N$ and, since $A_{mk} \subset A_{m(k+1)}$, it follows that $A_{mk} \nearrow A \cap Q$. We apply Lemma 1 for Q and A_{mk} and $a \leqslant \frac{1}{k}$ such that $d/a \in N$ and we find $x_1, x_2, \dots, x_p \in A_{mk} \subset Q \cap A$ such that $\frac{m(\overline{f(\bar{B}(x_i,r))})}{m(B(x_i,r))} < \frac{1}{m}$

for every $r \leq a\sqrt{n}$, $i = 1,2,\dots,p$, $A_{mk} \subset \bigcup_{i=1}^{p} B(x_i, a\sqrt{n})$ and every point from Q is contained in at most N such balls. Since $\overline{f(A_{mk})}$ is closed, it is measurable and we have: $m(\overline{f(A_{mk})}) \leq$ $\leq m(\overline{f(\bigcup_{i=1}^{p} B(x_i, a\sqrt{n}))}) \leq m(\bigcup_{i=1}^{p} \overline{f(B(x_i, a\sqrt{n}))}) \leq m(\bigcup_{i=1}^{p} \overline{f(\bar{B}(x_i, a\sqrt{n}))}) \leq$ $\leq \sum_{i=1}^{p} m(\overline{f(\bar{B}(x_i, a\sqrt{n}))}) \leq \frac{1}{m} \sum_{i=1}^{p} m(B(x_i, a\sqrt{n})) \leq \frac{N}{m} \cdot m(\bigcup_{i=1}^{p} B(x_i, a\sqrt{n})) \leq$ $\leq \frac{N}{m} \cdot m(Q')$ for every $k \geq k_0$ and $m \in N$. Let $H_m = \bigcup_{k=k_0}^{\infty} \overline{f(A_{mk})}$. Then H_m is measurable and $\overline{f(A_{mk})} \nearrow H_m$, hence $m(\overline{f(A_{mk})}) \nearrow m(H_m)$ and $m(H_m) \leq \frac{N \cdot m(Q')}{m}$. Also, $f(A \cap Q) = \bigcup_{k=k_0}^{\infty} f(A_{mk}) \subset$ $\subset \bigcup_{k=k_0}^{\infty} \overline{f(A_{mk})} = H_m$, hence $m^*(f(A \cap Q)) \leq m^*(H_m) = m(H_m) \leq \frac{N}{m} \cdot m(Q')$ for every $m \in N$, therefore $f(A \cap Q)$ is measurable and $m(f(A \cap Q)) =$ $= 0$. Since Q is arbitrary chosen, we obtain that $m(f(A)) = 0$, q.e.d.

Theorem 2. Let $f: D \longrightarrow R^n$. Then $m(f(Z_f)) = 0$.

Proof. Let $a \in Z_f$ and $0 < \varepsilon < 1$. There exists $r_\varepsilon > 0$ such that, for $\|x-a\| \leq r_\varepsilon$, $\|f(x)-f(a)-f'(a)(x-a)\| \leq \varepsilon \cdot \|x-a\|$. Then, for $\|x-a\| < r_\varepsilon$, we obtain that $\|f(x)-f(a)\| \leq (\|f'(a)\|+1) \cdot$ $\cdot \|x-a\|$. Since $J_f(a) = 0$, it results that $f(a) + f'(a)(x-a)$ belongs to a hyperplane H. Then, for $r \leq r_\varepsilon$, $\overline{f(\bar{B}(a,r))}$ is contained in a cylinder of height $2\varepsilon r$ and a basis which has the area equal to the area of the n-1 dimensional ball $H \cap B(f(a),$ $(\|f'(a)\|+1)r)$. Let $M = \frac{2 \cdot V(n-1)(\|f''(a)\|+1)^{n-1}}{V(n)}$. We proved that, for $r \leq r_\varepsilon$, $\frac{m(\overline{f(\bar{B}(a,r))})}{m(B(a,r))} < \varepsilon M$, hence $\lim_{r \to 0} \frac{m(\overline{f(\bar{B}(a,r))})}{m(B(a,r))} = 0$ and we apply Theorem 1.

Theorem 3. Let $f: D \longrightarrow R^n$ and $K \subset D$ be such that, for every $x \in K$, there exist $r_x > 0$ and $M_x > 0$ such that, for $r \leq r_x$, $\frac{m(\overline{f(\bar{B}(x,r))})}{m(B(x,r))} < M_x$. Then, for every $A \subset K$ such that $m(A) = 0$, it results that $m(f(A)) = 0$.

Proof. Let $A \subset K$ be such that $m(A)=0$. Let $H_m = \left\{ x \in A \mid \text{there exists } r_x > 0 \text{ such that, for } r \leq r_x,\ \frac{m(\overline{f(\bar{B}(x,r))})}{m(B(x,r))} < m \right\}$. Then $A = \bigcup_{m=1}^{\infty} H_m$, hence $f(A) = \bigcup_{m=1}^{\infty} f(H_m)$. Since if we prove that $m(f(H_m)) = 0$ for every $m \in N$ we obtain that $m(f(A)) = 0$, we can suppose that there exists $M > 0$ such that, for every $x \in A$, there exists $r_x > 0$ such that, for every $r \leq r_x$, $\frac{m(\overline{f(\bar{B}(x,r))})}{m(B(x,r))} < M$. Let Q, Q' and $k_0 \in N$ be as in Theorem 1. Since $m(Q \cap A) = 0$, we can find, for every $m \in N$, $U_m \subset D$ open sets such that $Q \cap A \subset U_m$ and $m(U_m) < \frac{1}{m}$. Let $A_{mk} = \left\{ x \in Q \cap A \mid B\left(x, \frac{\sqrt{n}}{k}\right) \subset U_m \text{ and } \frac{m(\overline{f(\bar{B}(x,r))})}{m(B(x,r))} < M \text{ for every } r \leq \frac{\sqrt{n}}{k} \right\}$, $k \geq k_0$, $m \in N$. Then $Q \cap A = \bigcup_{k=k_0}^{\infty} A_{mk}$ for every $m \in N$ and $A_{mk} \nearrow Q \cap A$. We apply Lemma 1 for Q and A_{mk} and $a \leq \frac{1}{k}$ such that $d/a \in N$ and we find $x_1, x_2, \ldots, x_p \in A_{mk} \subset Q \cap A$ such that $B(x_i, a\sqrt{n}) \subset U_m$ and, for every $r \leq a\sqrt{n}$, $\frac{m(\overline{f(\bar{B}(x_i,r))})}{m(B(x_i,r))} < M$, $i = 1,2,\ldots,p$, $A_{mk} \subset \bigcup_{i=1}^{p} B(x_i, a\sqrt{n})$ and every point from Q is contained in at most N such balls. Since $\overline{f(A_{mk})}$ is closed, it is measurable and we have:

$$m(\overline{f(A_{mk})}) \leq m(\overline{f(\bigcup_{i=1}^{p} \bar{B}(x_i, a\sqrt{n}))}) \leq m(\overline{\bigcup_{i=1}^{p} f(\bar{B}(x_i, a\sqrt{n}))}) \leq$$

$$\leq m(\bigcup_{i=1}^{p} \overline{f(\bar{B}(x_i, a\sqrt{n}))}) \leq \sum_{i=1}^{p} m(\overline{f(\bar{B}(x_i, a\sqrt{n}))}) \leq M \sum_{i=1}^{p} m(B(x_i, a\sqrt{n})) \leq$$

$$\leq M \cdot N \cdot m(\bigcup_{i=1}^{p} B(x_i, a\sqrt{n})) \leq M \cdot N \cdot m(U_m) \leq \frac{M \cdot N}{m}$$ for every $m \in N$ and $k \geq k_0$. Let $H_m = \bigcup_{k=k_0}^{\infty} \overline{f(A_{mk})}$. Then H_m is measurable and $\overline{f(A_{mk})} \nearrow H_m$, hence $m(\overline{f(A_{mk})}) \nearrow m(H_m)$ and $m(H_m) \leq \frac{M \cdot N}{m}$ for every $m \in N$. Also, $f(A \cap Q) = f(\bigcup_{k=k_0}^{\infty} A_{mk}) = \bigcup_{k=k_0}^{\infty} f(A_{mk}) \subset \bigcup_{k=k_0}^{\infty} \overline{f(A_{mk})} =$

$= H_m$, hence $m^*(f(A \cap Q)) \leqslant m^*(H_m) = m(H_m) \leqslant \frac{M \cdot N}{m}$ for every $m \in N$, therefore $f(A \cap Q)$ is measurable and $m(f(A \cap Q)) = 0$. Since Q is arbitrary chosen, we obtain that $m(f(A)) = 0$, q.e.d.

From Theorem 3 we can easily obtain the following theorems:

Theorem 4. Let $f:D \longrightarrow R^n$ be such that there exists $K \subset D$ such that, for every $x \in K$, $L(x,f) < \infty$. Then, for every $A \subset K$ such that $m(A) = 0$, we have $m(f(A)) = 0$.

Theorem 5. Let $f:D \longrightarrow R^n$ be differentiable. Then f has the property N of Luzin.

Theorem 6. Let $f:D \longrightarrow R^n$ be such that $D(x,f) < \infty$ for every $x \in D$. Then $m(f(H_f)) = 0$.

Proof. From the theorem of Rademacher and Stepanov, f is differentiable a.e. and we apply Theorem 2 and Theorem 4.

Theorem 7. Let $f:D \longrightarrow D'$ be a homeomorphism. Then there exists $K \subset D$ such that $m(K) = 0$ and for every $A \subset D$ such that $A \cap K = \emptyset$ and $m(A) = 0$ we obtain that $m(f(A)) = 0$.

Proof. We apply Theorem A3 and Theorem 3.

Theorem 8. Let $f:D \longrightarrow R^n$ be light such that $L(x,f) < \infty$ for every $x \in D$, f has partial derivatives a.e. in D and $J_f(x) \geqslant 0$ a.e. in D or $J_f(x) \leqslant 0$ a.e. in D. Then f is open on D.

Proof. From the theorem of Rademacher and Stepanov, f is differentiable on $D \setminus K$, where $K \subset D$ is such that $m(K) = 0$. From Theorem 2 and Theorem 4, $m(f(Z_f \cup K) = 0$. We can suppose that $J_f(x) \geqslant 0$ on $D \setminus K$ or $J_f(x) \leqslant 0$ on $D \setminus K$. Let us suppose that $J_f(x) \geqslant 0$ on $D \setminus K$. If $a \in D$ is such that $J_f(a) \neq 0$, then $i(f,a,f(a)) = 1$ if $J_f(a) > 0$ and $i(f,a,f(a)) = -1$ if $J_f(a) < 0$ (see [2] page 81). Let now $x \in D$ be fixed. Since f is light, there exists $U \in \vartheta_x$ bounded such that $U \subset D$ and $f(x) \notin f(\mathrm{Fr}\,U)$. Let $V \in \vartheta_{f(x)}$ be such a ball that $V \cap f(\mathrm{Fr}\,U) = \emptyset$ and $Q = f^{-1}(V) \cap U$. Then $Q \in \vartheta_x$ and, from Theorem A2, $\mathrm{int}\, f(Q) \neq \emptyset$, hence there exists $y \in V \cap f(U) \setminus f(Z_f \cup K)$. Since $f^{-1}(y) \cap U$ is discrete, we have: $d(f,U,f(x)) = d(f,U,y) =$

$= \sum_{c\in f^{-1}(y)\cap U} i(f,c,f(c)) = \text{card}\, f^{-1}(y)\cap U \geq 1$, hence, for $z \in V$ arbitrary, $d(f,U,z) = d(f,U,f(x)) \neq 0$ and, from [3] page 26, we obtain that $z \in f(U)$. Since we can choose a fundamental system of neighbourhoods of x denoted by $\mathcal{U}$ such that, for every $U \in \mathcal{U}$, $f(U) \in \vartheta_{f(x)}$, we obtain that f is open in x and the theorem is proved. If we suppose that $J_f(x) \leq 0$ a.e., the proof is similar.

Theorem 9. Let $f:D \to R^n$ be differentiable and light such that $J_f(x) \geq 0$ a.e. in D or $J_f(x) \leq 0$ a.e. in D. Then f is open on D.

REFERENCES

[1] P. Caraman: Homoeomorfisme quasiconforme n-dimensionale. Edit. Acad. R.S.R. 1968.

[2] K.G. Lloyd: Degree Theory, Cambridge, Univ. Press. Cambridge 1978.

[3] S. Sburlan: Gradul topologic, Edit. Acad. R.S.R. 1983.

[4] S. Stoilow: Sur l'inversion des transformations dont le déterminant fonctionnel s'annulle sans changer le signe. Bull. Math. l'École Polytechnique de Bucareşt, 10 (1939) 19-22.

[5] Titus and Young: A Jacobian condition for interiority, Michigan Math. J. 1 (1952) 89-94.

[6] J. Väisälä: Lectures on n-dimensional quasiconformal mappings, Springer-Verlag 1971.

[7] Wallman, Hurewitz: Dimension Theory 1948.

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THE THREE-PARABOLIC PROBLEM FOR HALF-SPACE WITH BOUNDARY CONDITIONS OF LAURICELLA TYPE

1. Introduction

The subject of the present paper is the construction of a solution u of the equation

$$P^3u(x,t) = f(x,t), \tag{1}$$

where $P = \Delta - D_t$, $\Delta = \sum_1^n D^2_{x_i}$, $P^3 = P(P^2)$, $P^2 = P(P)$, in the domain

$$D = D_1 \times (0,T),\ D_1 = \left\{x=(\bar{x},x_n): \bar{x} = (x_1,\dots,x_{n-1}) \in E_{n-1}, x_n > 0\right\},$$

satisfying the initial conditions

$$D^i_t u(x,0) = f_i(x),\ i = 0,1,2,\ x \in D_1, \tag{2}$$

and the boundary conditions

$$D^i_{x_n} u(\bar{x},0,t) = h_i(\bar{x},t),\ \ i = 0,1,2,\ (\bar{x},t) \in E_{n-1} \times (0,T). \tag{3}$$

In order to solve the problem (1)-(3) with given functions f, f_i, h_i, we shall apply the convenient Green function G.

In the paper [1] the similar problem for bicaloric equation by another method is solved.

2. Green's function

Let us consider the function

$$G(x,t;y,s) = G_1(x,t;y,s) - G_2(x,t;y,s),$$

where

$$G_1(x,t;y,s) = K(\bar{x},t;\bar{y},s)(t-s)^{-1/2}\exp\left[(x_n-y_n)^2B(t,s)\right],$$

$$G_2(x,t;y,s) = K(\bar{x},t;\bar{y},s)(t-s)^{-1/2}\exp\left[(x_n+y_n)^2B(t,s)\right],$$

$$\bar{y} = (y_1,\dots,y_{n-1}),\ K(\bar{x},t;\bar{y},s) = U_1U_2\dots U_{n-1},$$

$$U_i(\bar{x},t;\bar{y},s) = (t-s)^{-1/2}\exp\left[(\bar{x}_i-\bar{y}_i)^2B(t,s)\right],\ i=1,\dots,n-1,$$

$$B(t,s) = \left[-4(t-s)\right]^{-1}.$$

Lemma 1. If $(x_n,t,s) \in (0,\infty)\times(0,T)\times(0,T)$ and the function $Q(x_n,t,s)$ is of class C^2 with respect to x_n and of class C^1 with respect to t,s, $0 < s < t$, then

$$P_{x,t}\left[K(\bar{x},t;\bar{y},s)Q(x_n,t,s)\right] = K(\bar{x},t;\bar{y},s)P_{x_n,t}Q(x_n,t,s),$$

where $P_{x,t} = \Delta - D_t$, $P_{x_n,t} = D^2_{x_n} - D_t$.

Proof.

$$P_{x,t}(KQ) =$$

$$= Q(D^2_{x_1}U_1)U_2\dots U_{n-1} + \dots + Q(D^2_{x_{n-1}}U_{n-1})U_1\dots U_{n-2} +$$

$$+ K(D^2_{x_n}Q) - Q(D_tU_1)U_2\dots U_{n-1} - \dots - Q(D_tU_{n-1})U_1\dots U_{n-2} - K(D_tQ) =$$

$$= Q(D^2_{x_1}U_1 - D_tU_1)U_2\dots U_{n-1} + \dots + Q(D^2_{x_{n-1}}U_{n-1} - D_tU_{n-1})U_1\dots U_{n-2} +$$

$$+ K(D^2_{x_n}Q - D_tQ) = K(P_{x_n,t}Q),$$

because $(D^2_{x_i} - D_t)U_i = 0$, $i = 1,\dots,n-1$.

3. Reduction of the problem (1)-(3) to the problem with homogeneous boundary conditions

Definition 1. Denote by (K) the class of all functions u such that $D^k_{x_i}D^j_t u \in C(D)$ for $i = 1,\dots,n$; $k = 0,1,\dots,6$; $j = 0,1,2,3$; $0 \leq i+j \leq 6$ and $D^i_{x_n} u \in C(\bar{D})$, $i = 0,1,2$, all these derivatives being bounded in D.

Definition 2. Denote by (F) the class of all functions f such that: 1° $D^i_{x_j} f$, $i = 0,1,\dots,6$; $j = 1,\dots,n$, are continuous and bounded in D_1, 2° $D^i_{x_n} f(\bar{x},0) = 0$, $i = 0,1,2$, for $\bar{x} \in E_{n-1}$.

Let us introduce the unknown function $w(x,t)$ by formula

(4) $$w(x,t) = u(x,t) - r(x,t),$$

where $u \in (K)$ is the solution of the problem (1)-(3) and

(5) $$r(x,t) = h_o(x,t) + x_n h_1(x,t) + \frac{1}{2} x_n^2 h_2(x,t)$$

and let

$$S = \left\{ (\bar{x},t) : \bar{x} \in E_{n-1},\ t \in (0,T) \right\}.$$

Lemma 2. If the function f is continuous and bounded for $(x,t) \in D$ and $f_i \in (F)$, $f_i(\bar{x},0) = 0$, $h_i \in C^{6-2i}(S)$, $i = 0,1,2$, and the conditions $h_i(\bar{x},0) = 0$, $i = 0,1,2$, hold for $\bar{x} \in E_{n-1}$, then

(1a) $$P^3 w(x,t) = F(x,t), \quad (x,t) \in D,$$

where $F(x,t) = f(x,t) - P^3 r(x,t)$,

(2a) $$D_t^i w(x,0) = D_t^i u(x,0) - D_t^i r(x,0) = F_i(x), \quad i=0,1,2,\ x \in D_1,$$

(3a) $$D^i_{x_n} w(\bar{x},0,t) = 0, \quad i=0,1,2,\ \bar{x} \in E_{n-1}.$$

Proof. Using (4), (5), we have:

ad (1a)

$$P^3 w(x,t) = P^3 u(x,t) - P^3 r(x,t) = F(x,t) \quad \text{for} \quad (x,t) \in D;$$

ad (2a)

$$w(x,0) = u(x,0) - r(x,0) - f_o(x) - h_o(x,0) + x_n h_1(x,0) + \frac{1}{2} x_n^2 h_2(x,0) = F_o(x) \quad \text{for} \quad x \in D_1,$$

$$D_t w(x,0) = D_t u(x,0) - D_t r(x,0) = f_1(x) - D_t h_0(x,0) - x_n D_t h_1(x,0) -$$

$$- \frac{1}{2} x_n^2 D_t h_2(x,0) = F_1(x) \quad \text{for} \quad x \in D_1,$$

$$D_t^2 w(x,0) = D_t^2 u(x,0) - D_t^2 r(x,0) = f_2(x) - D_t^2 h_0(x,0) - x_n D_t^2 h_1(x,0) -$$

$$- \frac{1}{2} x_n^2 D_t^2 h_2(x,0) = F_2(x) \quad \text{for} \quad x \in D_1;$$

ad (3a)

$$D_{x_n}^i w(\bar{x},0,t) = D_{x_n}^i u(\bar{x},0,t) - D_{x_n}^i r(\bar{x},0,t) = 0,$$

$$i=0,1,2, \text{ for } (\bar{x},t) \in S.$$

4. The properties of the integral w_3

Let

$$w_3(x,t) = A_n \int_0^t \int_{D_1} F(y,s)(t-s)^2 G(x,t;y,s)\,dy\,ds = w_{3,1} - w_{3,2},$$

where

$$w_{3,j} = A_n \int_0^t \int_{D_1} F(y,s)(t-s)^2 G_j(x,t;y,s)\,dy\,ds, \quad j=1,2,$$

and $A_n = (2\sqrt{\pi})^{-n}$, and consider the integrals

$$w_{3,j}^i = A_n \int_0^t \int_{D_1} F(y,s)(t-s)^2 D_{x_n}^i \left[G_j(x,t;y,s)\right] dy\,ds,$$

$$i=0,1,2; \; j=1,2,$$

$$D_t w_{3,j} = A_n D_t \left[\int_0^t \int_{D_1} F(y,s)(t-s)^2 G_j(x,t;y,s)\,dy\,ds\right], \quad j=1,2.$$

Definition 3. Denote by (F_1) the class of functions F such that $D_{y_n}^i F$, $i = 0,1$, $D_s F$ are continuous and bounded in the set D and the conditions $D_{y_n}^i F(\bar{y},0,s) = 0$,

$i = 0,1,2,3$, for $\bar{y} \in E_{n-1}$, $s \in (0,T)$ and $D_s F(y,0) = 0$ for $y \in D_1$ are fulfilled.

Lemma 3. If $F \in (F_1)$, then:

1^0 the integrals $w^i_{3,j}$, $i = 0,1,2$; $j = 1,2$, $D_t w_{3,j}$, $j = 1,2$, are locally uniformly convergent in the set D,

2^0 $$Pw_{3,j} = -2A_n \int_0^t \int_{D_1} F(y,s)(t-s)G_j(x,t;y,s)\,dyds, \quad j=1,2,$$

for $(x,t) \in D$,

3^0 $w_3(\bar{x},0,t) = 0$ for $\bar{x} \in E_{n-1}$, $t \in (0,T)$,

4^0 there exist the continuous derivatives $D_{x_n} w_3(\bar{x},0,t) = S_1(\bar{x},t)$, $D^2_{x_n}\left[w_3(\bar{x},0,t)\right]$ and besides $D^2_{x_n}\left[w_3(\bar{x},0,t)\right] = 0$ for $(\bar{x},t) \in S$.

Proof. Ad 1^0. For the integral $w^0_{3,1}$ we have the estimate

$$\left|w^0_{3,1}\right| \leqslant A_n C^1 \int_0^t \int_{-\infty}^{\infty} (t-s)^{3/2} \exp\left[(x_n-y_n)^2 B(t,s)\right] dy_n ds,$$

where $C^1 = \sup_D |F(y,s)|$. Putting

$$(x_n-y_n)\,(2\sqrt{t-s})^{-1} = z \tag{6}$$

we obtain

$$\left|w^0_{3,1}\right| \leqslant A_n C^1 C \int_0^t \int_{-\infty}^{\infty} (t-s)^2 \exp(-z^2)\,dzds = 3^{-1}\sqrt{\pi}\, A_n C^1 C t^3,$$

where C is a suitable constant.

For the integral $w^1_{3,1}$ we have

$$w^1_{3,1} = -A_n \int_0^t \int_{D_1} F(y,s)(t-s)^2 D_{y_n}\left[G_1(x,t;y,s)\right] dyds =$$

$$= -A_n \int_0^t \int_{E_{n-1}} F(y,s)(t-s)^2 G_1(x,t;y,s)\Big|_{y_n=0}^{y_n=\infty} d\bar{y}ds +$$

$$+ A_n \int_0^t \int_{D_1} D_{y_n}\left[F(y,s)\right](t-s)^2 G_1(x,t;y,s)\,dyds =$$

$$= A_n \int_0^t \int_{D_1} D_{y_n}\left[F(y,s)\right](t-s)^2 G_1(x,t;y,s)\,dyds.$$

The last integral can be estimated similarly as $w^0_{3,1}$.

For the integral $w^2_{3,1}$ we have

$$w^2_{3,1} = A_n \int_0^t \int_{D_1} F(y,s)(t-s)^2 D^2_{y_n}\left[G_1(x,t;y,s)\right]dyds =$$

$$= A_n \int_0^t \int_{E_{n-1}} F(y,s)(t-s)^2 D_{y_n}\left[G_1(x,t;y,s)\right]\Big|_{y_n=0}^{y_n=\infty} d\bar{y}ds -$$

$$- A_n \int_0^t \int_{D_1} D_{y_n}\left[F(y,s)\right](t-s)^2 D_{y_n}\left[G_1(x,t;y,s)\right]dyds =$$

$$= -A_n \int_0^t \int_{E_{n-1}} D_{y_n}\left[F(y,s)\right](t-s)^2 G_1(x,t;y,s)\Big|_{y_n=0}^{y_n=\infty} d\bar{y}ds +$$

$$+ A_n \int_0^t \int_{D_1} D^2_{y_n}\left[F(y,s)\right](t-s)^2 G_1(x,t;y,s)\,dyds =$$

$$= A_n \int_0^t \int_{D_1} D^2_{y_n}\left[F(y,s)\right](t-s)^2 G_1(x,t;y,s)\,dyds,$$

the last integral having the similar majorant as $w^0_{3,1}$.

Let $D_t w_{3,1} = L_1 + L_2$, where

$$L_1 = \lim_{s \to t} A_n \int_{D_1} F(y,s)(t-s)^2 G_1(x,t;y,s)\,dy,$$

$$L_2 = A_n \int_0^t \int_{D_1} F(y,s) D_t \left[(t-s)^2 G_1(x,t;y,s) \right] dyds.$$

Thus

$$|L_1| \leq A_n C^1 \lim_{s \to t} \int_{D_1} (t-s)^2 K(t-s)^{-\frac{1}{2}} \exp\left[(x_n-y_n)^2 B(t,s) \right] dy \leq$$

$$\leq A_n C^1 \lim_{s \to t} \left\{ (t-s)^2 \int_{E_{n-1}} K d\bar{y} \int_0^\infty (t-s)^{-\frac{1}{2}} \exp\left[(x_n-y_n)^2 B(t,s) \right] dy_n \right\}.$$

Applying the transformation (6) we obtain

$$|L_1| \leq A_n C^1 \lim_{s \to t} (t-s)^2 C_3 = 0,$$

where C_3 is a suitable positive constant.

Integrating L_2 by parts we obtain

$$L_2 = A_n \int_{D_1} F(y,s)(t-s)^2 G_1(x,t;y,s) \Big|_{s=0}^{s=t} dy -$$

$$- A_n \int_0^t \int_{D_1} D_s \left[F(y,s) \right] (t-s)^2 G_1(x,t;y,s) dyds =$$

$$= -A_n \int_0^t \int_{D_1} D_s \left[F(y,s) \right] (t-s)^2 G_1(x,t;y,s) dyds.$$

The last integral can be estimated as $w^0_{3,1}$.

For the integrals $w^i_{3,2}$, $i = 0,1,2$, $D_t w_{3,2}$ we obtain the similar estimates.

Ad 2°. By assertion 1°, it follows that

$$P w_{3,j} = A_n \int_0^t \int_{D_1} F(y,s) P_{x,t} \left[(t-s)^2 G_j(x,t;y,s) \right] dyds =$$

$$= -2A_n \int_0^t \int_{D_1} F(y,s)(t-s) G_j(x,t;y,s) dyds, \quad j=1,2.$$

Ad 3^0. We have

$$w_3(\bar{x},0,t) = A_n \int_0^t \int_{D_1} F(y,s)(t-s)^2 G(\bar{x},0,t;y,s)\,dyds = 0,$$

since $G(\bar{x},0,t;y,s) = 0$.

Ad 4^0. We have

$$D_{x_n}\left[w_3(\bar{x},0,t)\right] = -A_n \int_0^t \int_{D_1} F(y,s) D_{y_n}\left[G_1(\bar{x},0,t;y,s)\right] dyds -$$

$$- A_n \int_0^t \int_{D_1} F(y,s) D_{y_n}\left[G_2(\bar{x},0,t;y,s)\right] dyds.$$

Integrating by parts we get

$$D_{x_n}\left[w_3(\bar{x},0,t)\right] = A_n \int_0^t \int_{D_1} D_{y_n}\left[F(y,s)\right]\left[G_1(\bar{x},0,t;y,s) + \right.$$

$$\left. + G_2(\bar{x},0,t;y,s)\right] dyds = S^1(\bar{x},t)$$

and further

$$D^2_{x_n}\left[w_3(\bar{x},0,t)\right] = A_n \int_0^t \int_{D_1} D^2_{y_n}\left[F(y,s)\right] G(\bar{x},0,t;y,s)\,dyds = 0.$$

Lemma 4. If the assumptions of Lemma 3 are satisfied, then:

1^0 $P^2 w_3(x,t) = A_n \int_0^t \int_{D_1} F(y,s)\left[G_1(x,t;y,s) - G_2(x,t;y,s)\right] dyds$ for $(x,t) \in D$,

2^0 $P^3 w_3(x,t) = F(x,t)$ for $(x,t) \in D$.

Proof. Ad 1^0. By Lemma 3 (the assertion 2^0), we obtain

$$P^2w_3(x,t) = \lim_{s\to t} \int_{D_1} F(y,s)(t-s)G(x,t;y,s)dy +$$
$$+ A_n\int_0^t\int_{D_1} F(y,s)D_t\left[(t-s)G(x,t;y,s)\right]dyds$$

and, by Lemma 1, we get

$$P^2w_3(x,t) = A_n\int_0^t\int_{D_1} F(y,s)G(x,t;y,s)dyds = Q_1 - Q_2,$$

where

$$Q_j = \int_0^t\int_{D_1} F(y,s)G_j(x,t;y,s)dyds, \quad j=1,2.$$

Ad 2^0. By Poisson's theorem, we obtain

$$PQ_1(x,t) = F(x,t) \quad \text{for} \quad (x,t)\in D,$$

$$PQ_2(x,t) = A_n\int_0^t\int_{D_1} F(y,s)P_{x,t}\left[G_2(x,t;y,s)\right]dyds = 0 \text{ for } (x,t)\in D.$$

5. Construction of the potential compatibile to the equation (1a) and initial conditions (2a)

Let us consider the function defined as follows

(7) $$v(x,t) = v_0(x,t) + tv_1(x,t) + \frac{1}{2}t^2v_2(x,t),$$

where

$$v_0(x,t) = A_n\int_{D_1} F_0(y)G(x,t;y,s)dy,$$

$$v_1(x,t) = A_n\int_{D_1}\left[F_1(y) - \Delta F_0(y)\right]G(x,t;y,0)dy,$$

$$v_2(x,t) = A_n \int_{D_1} \left[F_2(y) - 2\Delta F_1(y) + \Delta^2 F_0(y)\right] G(x,t;y,0)\,dy.$$

Lemma 5. If the functions $F_i \in (F)$, $i = 0,1,2$, $F_0(\bar{y},0) = 0$, $F_1(\bar{y},0) - \Delta F_0(\bar{y},0) = 0$, $F_2(\bar{y},0) - 2\Delta F_1(\bar{y},0) + \Delta^2 F_0(\bar{y},0) = 0$, the functions $\Delta^i F_0$, $i = 0,1,2$, $\Delta^i F_1$, $i=0,1$, F_2 are absolutely integrable in D_1, then the function v satisfies:

1° $P^3 v(x,t) = 0$ in D,

2° the initial conditions (2a) in D_1,

3° the boundary conditions

(a) $v(\bar{x},0,t) = 0$ in S, (b) $D^2_{x_n}\left[v(\bar{x},0,t)\right] = 0$ in S, and besides

(c) $\lim\limits_{x_n \to 0} D_{x_n}\left[v(x,t)\right] = D_{x_n}\left[v(\bar{x},0,t)\right]$ for $x \in E_{n-1}$, $t > 0$.

Proof. Ad 1°. By Lemma 1, we have

$$P_{x,t}\left[v_0(x,t)\right] = \int_{D_1} F_0(y) P_{x,t}\left[G(x,t;y,0)\right] dy =$$

$$= \int_{D_1} F_0(y) K(\bar{x},t;\bar{y},s) P_{x,t}\left[B_1(x_n,t;y_n,0)\right] dy = 0,$$

since the function

$$B_1 = t^{-\frac{1}{2}}\left[\exp(-(4t)^{-1}(x_n-y_n)^2) - \exp(-(4t)^{-1}(x_n+y_n)^2)\right],$$

satisfies the equation $Pu = 0$.

Next

$$P\left[tv_1(x,t)\right] = -v_1(x,t),$$

$$P^2\left[tv_1(x,t)\right] = -P\left[v_1(x,t)\right] = 0.$$

Thus

$$P^3\left[tv_1(x,t)\right] = 0,\ P\left[t^2 v_2(x,t)\right] = P\left[t(tv_2(x,t))\right] = tv_2(x,t),$$

$$P^2\left[t^2 v_2(x,t)\right] = v_2(x,t).$$

Hence

$$P^3\left[t^2 v_2(x,t)\right] = P\left[v_2(x,t)\right] = 0 \quad \text{for} \quad (x,t)\in D$$

and consequently (1a) holds.

Ad 2°. Let $v_0 = v_0^1 + v_0^2$, where

$$v_0^1(x,t) = A_n \int_{D_1} F_0(y)G_1(x,t;y,0)dy,$$

$$v_0^2(x,t) = A_n \int_{D_1} F_0(y)G_2(x,t;y,0)dy.$$

The integral v_0^1 being Fourier-Poisson integral, by [2], p.454, we obtain $v_0^1(x,0) = F_0(x)$ for $x \in D_1$.

For the integral v_0^2 we obtain the estimate

$$|v_0^2(x,t)| \leqslant \text{const.} \int_{-\infty}^{\infty} t^{\frac{1}{2}}\left[(x_n+y_n)^2 t\right]^{-1}(x_n+y_n)^2 \exp\left[B(t,s)(x_n+y_n)^2\right]dy_n \leqslant$$

$$\leqslant \text{Const.}\, x_n^{-2}\, t^{\frac{1}{2}} \int_{-\infty}^{\infty} t^{-1}(x_n+y_n)^2 \exp\left[B(t,s)(x_n+y_n)^2\right]dy_n.$$

Putting $(4t)^{-1}(x_n+y_n)^2 = z$, we obtain

$$|v_0^2(x,t)| \leqslant \text{const.}\, x_n^{-2} t^{\frac{1}{2}} \int_{-\infty}^{\infty} \exp(-z^2)dz \to 0 \quad \text{as} \quad t \to 0.$$

Hence $v_0^2(x,0) = 0$ for $x \in D_1$ and $v_0(x,0) = F_0(x)$ for $x \in D_1$.

By (7), we find

$$(8) \quad D_t v(x,t) = D_t\left[v_0(x,t)\right] + v_1(x,t) + tD_t\left[v_1(x,t)\right] + 2tv_2(x,t) + t^2 D_t\left[v_2(x,t)\right].$$

Since $D_tG = \Delta_xG = \Delta_yG$, we have

$$D_t\left[v_0(x,t)\right] = A_n \int\limits_{D_1} F_0(y)D_t\left[G(x,t;y,0)\right]dy =$$

$$= A_n \int\limits_{D_1} \Delta F_0(y)G(x,t;y,0)dy.$$

Hence, by (7), (8), we obtain

$$D_t\left[v(x,0)\right] = \Delta F_0(x) + F_1(x) - \Delta F_0(x) = F_1(x).$$

By (7), (8), we have

$$(9)\quad D_t^2\left[v(x,t)\right] = D_t^2\left[v_0(x,t)\right] + 2D_t\left[v_1(x,t)\right] + tD_t^2\left[v_1(x,t)\right] +$$

$$+ 2v_2(x,t) + 4tD_t\left[v_2(x,t)\right] + t^2D_t^2\left[v_2(x,t)\right],$$

and

$$D_t^2\left[v_0(x,t)\right] = A_n \int\limits_{D_1} \Delta^2F_0(y)G(x,t;y,0)dy,$$

$$D_t\left[v_1(x,t)\right] = A_n \int\limits_{D_1} \left[F_1(y)-\Delta F_0(y)\right]D_t\left[G(x,t;y,0)\right]dy =$$

$$= 2A_n \int\limits_{D_1} \left[F_1(y) - \Delta F_0(y)\right]\Delta_y\left[G(x,t;y,0)\right]dy =$$

$$= 2A_n \int\limits_{D_1} \left[\Delta F_1(y) - \Delta^2F_0(y)\right]G(x,t;y,0)dy,$$

$$v_2(x,t) = A_n \int\limits_{D_1} \left[F_2(y)-2\Delta F_1(y)+\Delta^2F_0(y)\right]G(x,t;y,0)dy.$$

Hence, by (9), we obtain $D_t^2v(x,0) = F_2(x)$ for $x \in D_1$.

Ad 3^o(a), (b). We shall prove that $v_o(\bar{x},0,t) = 0$ (the proof of the conditions $v_i(\bar{x},0,t) = 0$, $i=2,3$, is similar). We have

$$|v_o(x,t)| \leq A_n \text{ const. } \int_{D_1} |F_o(y)|\,dy.$$

Consequently, the integral v_o is uniformly convergent with respect to x_n and continuous for $x_n \geq 0$. Hence

$$v_o(x,t) \to v_o(\bar{x},0,t) \quad \text{as} \quad x_n \to 0.$$

On the other hand $G(\bar{x},0,t;y,0) = 0$ and $v_o(\bar{x},0,t) = 0$ for $t > 0$.

Let us consider the integral

$$v_o^2(x,t) = A_n \int_{D_1} F_o(y) D^2_{x_n}\left[G(x,t;y,0)\right] dy =$$

$$= A_n \int_{D_1} F_o(y) D^2_{y_n}\left[G(x,t;y,0)\right] dy.$$

Integrating twice by parts, we obtain

$$v_o^2(x,t) = A_n \int_{E_{n-1}} F_o(y) D_{y_n}\left[G(x,t;y,0)\right]\Big|_{y_n=0}^{y_n=\infty} d\bar{y} -$$

$$- A_n \int_{D_1} D_{y_n}\left[F_o(y)\right] D_{y_n}\left[G(x,t;y,0)\right] dy =$$

$$= -A_n \int_{E_{n-1}} D_{y_n}\left[F_o(y)\right] G(x,t;y,0)\Big|_{y_n=0}^{y_n=\infty} d\bar{y} +$$

$$+ A_n \int_{D_1} D^2_{y_n}\left[F_o(y)\right] G(x,t;y,0)\,dy =$$

$$= A_n \int_{D_1} D^2_{y_n}\left[F_o(y)\right] G(x,t;y,0)\,dy.$$

The last integral tends to the integral

$$A_n \int\limits_{D_1} D^2_{y_n}\left[F_o(y)\right] G(\bar{x},0,t;y,0)\,dy = 0 \quad \text{as} \quad x_n \to 0.$$

Ad $3^o(c)$. Let

$$v_{o,1}(x,t) = A_n \int\limits_{D_1} F_o(y) D_{x_n}\left[G(x,t;y,0)\right] dy = v^1_{o,1}(x,t) + v^2_{o,1}(x,t),$$

where

$$v^1_{o,1}(x,t) = A_n \int\limits_{D_1} F_o(y) D_{x_n}\left[G_1(x,t;y,0)\right] dy =$$

$$= -A_n \int\limits_{D_1} F_o(y) D_{y_n}\left[G_1(x,t;y,0)\right] dy =$$

$$= -A_n \int\limits_{E_{n-1}} F_o(y) G_1(x,t;y,0)\Big|_{y_n=0}^{y_n=\infty} d\bar{y} +$$

$$+ A_n \int\limits_{D_1} D_{y_n}\left[F_o(y)\right] G_1(x,t;y,0)\,dy =$$

$$= A_n \int\limits_{D_1} D_{y_n}\left[F_o(y)\right] G_1(x,t;y,0)\,dy,$$

$$w^2_{o,1}(x,t) = A_n \int\limits_{D_1} F_o(y) D_{x_n}\left[G_2(x,t;y,0)\right] dy =$$

$$= -A_n \int\limits_{D_1} F_o(y) D_{y_n}\left[G_2(x,t;y,0)\right] dy =$$

$$= -A_n \int\limits_{D_1} D_{y_n}\left[F_o(y)\right] G_2(x,t;y,0)\,dy.$$

Similarly as in 3^o(a), (b), we can verify that the integrals $v^i_{0,1}$, $i = 1,2$, are locally uniformly convergent at every point $(\bar{x},0,t)$ and

$$D_{x_n}\left[v_0(\bar{x},0,t)\right] = v^1_{0,1}(\bar{x},0,t) + v^2_{0,1}(\bar{x},0,t) = S_1(\bar{x},t).$$

Similarly, we obtain

$$D_{x_n}\left[v_j(\bar{x},0,t)\right] = S_j(\bar{x},t), \quad j=2,3.$$

Consequently, by (7), we obtain

$$D_{x_n}\left[v(\bar{x},0,t)\right] = S_1(\bar{x},t) + tS_2(\bar{x},t) + \frac{1}{2}\,t^2S_3(\bar{x},t).$$

6. The fundamental theorem

By Lemmas 2-5 we obtain the following result.

Theorem 1. If the assumptions of Lemmas 2-5 are satisfied and $S^1(\bar{x},t) + S_1(\bar{x},t) + tS_2(\bar{x},t) + \frac{1}{2}\,t^2S_3(\bar{x},t)=0$, then the function

$$w(x,t) = w_3(x,t) + v_0(x,t) + tv_1(x,t) + \frac{1}{2}\,t^2v_2(x,t)$$

is the solution of the problem (1a)-(3a) and the function $u(x,t) = w(x,t) + r(x,t)$ is the solution of the problem (1)-(3).

REFERENCES

[1] A. Borzymowski : The first bicaloric problem in a half-space, Revue Roumaine Math. Pures App. 18 (1973), 1521-1534.

[2] M. Krzyżański : Partial differential equations of second order, Vol. I, II. Warszawa 1971.

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ON THE EXISTENCE OF SOLUTIONS OF A SYSTEM OF DIFFERENTIAL INEQUALITIES

Introduction

Let C and D be matrices of dimensions kxm. Let v(t) be continuous function for $t \in [t_0, t_1] =: T$, with values in R^k.

We will be dealing with the problem of existence of a solution of the following system of differential inequalities:

$$(1) \qquad C\dot{x}(t) + Dx(t) \leqslant v(t),$$

$$(2) \qquad x(0) = \bar{x}, \quad \text{for} \quad t \in T.$$

In this paper we show that under some natural conditions, system (1)-(2) admits a solution.

The proof will be based on some recently obtained results for differential inclusions.

The purpose of this paper is to obtain the following theorem.

Theorem. If the rows of matrix C are non-negative and nonzero (see Application), then there exists a function x(t) absolutely continuous on the time interval T such, that (1) is satisfied almost everywhere on this interval and (2) holds.

The proof of this theorem consists of several lemmas, which we shall be dealing with in the sequel.

Main Result

Let E^m be the m-dimensional vector space composed of vectors $x = (x_1,\dots,x_m)$ with the norm $\|x\| = \sqrt{\sum_{i=1}^{m} x_i^2}$ and let $P(E^m)$ be the metric space of all nonempty closed subsets of E^m with the Hausdorff metric:

$$h(W,Z) = \min\left\{d : W \subset S_d(Z),\ \ Z \subset S_d(W)\right\},$$

where $W,Z \in P(E^m)$ and $S_d(M)$ denotes a d-neighbourhood of the set M in the space E^m.

For a given multimapping $H : E^1 \times E^m \longrightarrow P(E^m)$ we consider the relation:

$$(3) \qquad \dot{x} \in H(t,x), \quad x(0) = \overline{x}.$$

An absolutely continuous function $x(t)$ is said to be a solution of (3) on the time interval T if the condition $\dot{x}(t) \in H(t,x(t))$ holds almost everywhere on this interval.

We consider a function H given in the form:

$$(4) \qquad H(t,x) = \left\{z \in E^m : Cz \leqslant -Dx + v(t)\right\},$$

where the function $v(t)$, $t \in T$ is given.

Let us remark that a function $x(t)$ which is defined on the time interval T is a solution of the differential inequalities (1) - (2) if and only if it satisfies (3) with the function $H(t,x)$ defined by (4).

Now we reformulate the inequalities (1) in terms of so called differential inclusions.

In connection with the above, the problem of existence of any solution of differential inequalities (1) - (2) can be reduced to the problem of existence of a solution of the differential inclusion (3), where $H(t,x)$ is a function defined by (4).

Lemma 1. If the rows $r_i(C)$ of the matrix C are non-negative and nonzero, then the set $H(t,x)$ is nonempty and closed for all $x \in E^m$ and $t \in T$.

Proof. From the assumption of the lemma, the set $H(t,x)$ is a common part of k closed half-spaces defined by the inequalities:

(i) $(r_i(C), z) \leqslant (r_i(-D), x) + v_i(t) =: \gamma_i(t,x)$,

where $i = 1,2,\ldots,k$ and (a,b) denotes the scalar product of vectors a and b.

Because each vector $r_i(C)$ is non-negative and nonzero, the intersection of these half-spaces is non-empty for each γ_i, $i = 1,\ldots,k$.

Let λ_i be the smallest positive number such that the vector $-\lambda_i\,(1,1,\ldots,1)$ belongs to the half-space defined by inequality (i). Then $-\mu(1,1\ldots,1) \in H(t,x)$, where

$$\mu = \max\left\{\lambda_1,\ldots,\lambda_k\right\}.$$

It follows that for each $t \in T$ and for each $x \in E^m$ $H(t,x)$ is a nonempty set, independent of the value of the function $v(t)$. This completes the proof of Lemma 1.

Let us observe that if one of the rows of the matrix C, for example $r_i(C)$, is zero for some i, then the condition $(r_i(C), z) \leqslant \gamma_i$ determines a nonempty set (the whole space E^m) only if $\gamma_i \geqslant 0$. So, in the case where some rows of matrix C are zero, $H(t,x)$ is nonempty with additional limitations of x and $v(t)$. We observe that $H(t,x)$ is an unbounded set. Thus to show the existence of a solution of the problem (3) we may apply the theorem on the existence of solutions for differential inclusions, when the right side is unbounded.

The following theorem is included in the work of J. Himmelberg and F.S. Van Vleck [1].

Theorem (Himmelberg, Van Vleck). Let a multimapping $H : T \times E^m \to P(E^m)$ be such that:

a) $H(t,x)$ is a closed set for each $(t,x) \in T \times E^m$;

b) $H(\cdot,x)$ is measurable for each $x \in E^m$;

c) there exists a function φ integrable on the time interval T, such that $h\,(H(t,x), H(t,y)) \leqslant \varphi(t)\,\|x-y\|$ for each x, y, t;

d) there exists an absolutely continuous function $w:T \times E^m$ such that: $\sup\{d(\dot{w}(t), H(t,w(t))) \mid t \in T\} < \infty$.

Then the problem (3) has a solution on the time interval T (in the sense of absolutely continuous functions).

We shall prove the assumptions of this theorem are satisfied by the function defined by (4).

We know that $H(t,x)$ is a closed and nonempty subset E^m. So the condition a) of the Himmelberg and Van Vleck's theorem is satisfied.

Now we shall prove that $H(\cdot,x)$ is a measurable function for each $x \in E^m$ in the sense of the following definition:

Definition. A multivalued function $F:E^m \to \Omega(E^m)$ is called measurable if for any closed set $P \subset E^m$ the set $\{x : F(x) \cap P \neq \emptyset\}$ is Lebesque measurable.

Continuity and lipschitzianity of a multivalued function $F(x)$ are defined in the usual way. For example a function $F(x)$ satisfies a Lipschitz condition with constant I if for any points $x, x' \in E^m$ the inequality

$$h(F(x), F(x')) \leqslant l \cdot \|x-x'\|$$

holds. The number $|F| = h(\{0\}, F)$ is called the modulus of the set F.

Lemma 2. If Z is a closed set of E^m, $H(t) = \{z \in E^m : (c,z) \leqslant \gamma(t)\}$, $\gamma(t)$ is continuous on the time interval T, c is a nonzero vector with non-negative coordinates, then $t(Z) = \{t \in T : H(t) \cap Z \neq \emptyset\}$ is a measurable set.

Proof. Let $\underline{\gamma} = \inf\{\gamma(t) : t \in T\}$, $\bar{\gamma} = \sup\{\gamma(t) : t \in T\}$. There exists $\underline{t} \in T$ and $\bar{t} \in T$ such that $\gamma(\underline{t}) = \underline{\gamma}$ and $\gamma(\bar{t}) = \bar{\gamma}$.
If $H(\bar{t}) \cap Z = \emptyset$, then $t(Z) = \emptyset$.
If $H(\underline{t}) \cap Z \neq \emptyset$, then $t(Z) = T$.
If $H(\underline{t}) \cap Z = \emptyset$ and $H(\bar{t}) \cap Z \neq \emptyset$, then there exists a number γ_Z such that $t(Z) = \{t \in T : \gamma(t) \geqslant \gamma_Z\}$.

From Lemma 2 it follows that assumption b) is satisfied because the intersection of measurable sets is also a measurable set.

Lemma 3. There exists a constant δ such that $h(H(t,x), H(t,y)) \leqslant \delta \|x-y\|$ for all x and y.

The distance ε_i between the following parallel hyperplanes $(r_i(C), z) = \gamma_i(t,x)$ and $r_i(C),\ z = \gamma_i(t,y)$, bounding respectively the sets $H(t,x)$ and $H(t,y)$ is equal to:

$$\varepsilon_i = \frac{|\gamma_i(t,x) - \gamma_i(t,y)|}{\|r_i(C)\|} = \frac{|(r_i(-D),\ x-y)|}{\|r_i(C)\|} \leqslant$$

$$\leqslant \frac{\|r_i(-D)\|}{\|r_i(C)\|}\ \|x-y\| = \mu_i \|x-y\|, \quad i=1,\dots,k.$$

$H(t,y)$ is a polyhedron which is obtained from the polyhedron $H(t,x)$ by a parallel displacement of its faces.

From geometrical considerations it follows that $h(H(t,x), H(t,y)) \leqslant \sqrt{m} \max\{\varepsilon_1,\dots,\varepsilon_k\}$. Thus we have

$$h(H(t,x), H(t,y)) \leqslant \sqrt{m} \max\{\mu_1,\dots,\mu_m\} \|x-y\|$$

which shows that the multifunction $H(t,x)$ satisfies the assumption c) of Himmelberg and Van Vleck's theorem.

Now we shall prove that there exists an absolutely continuous function $w : T \to E^m$ such that:

$$\sup\{d(\dot{w}(t), H(t,w(t))) \mid t \in T\} < \infty .$$

Let $v(t)$ be continuous function on the time interval T. Then the functions $\gamma_i(t,x)$ reach a minimum on the interval T for each fixed $x \in E^m$. Let

$$\tilde{\gamma}_i(x) = \min_T \gamma_i(t,x).$$

Then the inequalities (i) for $\gamma_i = \tilde{\gamma}_i(x)$ define a nonempty set $G(x)$ such that $G(x) \subset H(t,x)$ for each $t \in T$ and $x \in E^m$. Thus $d(0, H(t,x)) \leqslant d(0, G(x))$ for $t \in T$.

Let $w(t) = \text{const} = w_0 \in E^m$, then we have:

$d(\dot{w}(t), H(t,w(t))) = d(0, H(t,w_0)) \leqslant d(0, G(w_0)) < \infty$, because $G(w_0)$ is a nonempty set.

Thus the assumptions of Himmelberg and Van Vleck's theorem are satisfied and there exists a solution of the problem $\dot{x} \in H(t,x)$, $x(0) = x_0$ for $t \in T$, where $H(t,x)$ is defined by (4). In this way we have completed the proof of our theorem.

Application

A dynamical open economic model of Leontief's type (see [2]) is given by a system of linear differential equations of the first order:

$$\begin{cases} x(t) = Ax(t) + B\dot{x}(t) + u(t) \\ y(t) = Fx(t) + G\dot{x}(t). \end{cases}$$

We want to find such vector-functions $u(t)$ and $y(t)$ of final output and of consumption of primary factors that satisfy inequalities

$$u(t) \geqslant \bar{u}(t) \quad \text{and} \quad y(t) \leqslant \bar{y}(t) \quad \text{for} \quad t \in T$$

when $\bar{u}(t)$ and $\bar{y}(t)$ are given continuous vector-functions, and $x(0) = x_0$.

The problem of existence of a plan of total output $x(t)$ given constraints $\bar{u}(t)$ and $\bar{y}(t)$ reduces to a problem of existence of a solution of the system of differential inequalities

$$(5) \quad \left\{ \begin{array}{l} \begin{bmatrix} B \\ G \end{bmatrix} \dot{x}(t) + \begin{bmatrix} A - I \\ F \end{bmatrix} x(t) \leqslant \begin{bmatrix} -\bar{u}(t) \\ \bar{y}(t) \end{bmatrix}, \\ \qquad x(0) = x_0, \quad t \in T. \end{array} \right.$$

We have proved that if the rows of matrices B and G (which are non-negative by definition) are non-zero then (5) has a solution $x(t)$. But there is not to ensured that $x(t)$ be non-negative for all $t \in T$.

Remark. If $\dot{x}(0) = 0$, $x(0) = \bar{x} > 0$ and $v(t) = v(0) = \bar{v} > 0$, where $\bar{v} = D\bar{x}$, then $x(t) = \bar{x}$, $t \in T$, is a positive solution of the problem $\dot{x} \in H(t,x)$, $x(0) = x_0$ for $t \in T$, where $H(t,x)$ is defined by (4).

The question whether there exists a non-negative solution of this problem when v(t) is not constant has not been solved so far.

BIBLIOGRAPHY

[1] C.J. Himmelberg, F.S. Van Vleck: Lipschitzian generalized differential equations, Rend. Sem. Mat. Univ. Ped., 48 (1973) 159-169.

[2] K. Lancaster: Mathematical Economics. New York 1969. Macmillan Series in Economics.

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INEQUALITIES FOR CONDITIONED NORMAL APPROXIMATIONS

1. Introduction

Let $\{X_n, n \geqslant 1\}$ be a sequence of independent random variables defined on a probability space $(\Omega, \mathcal{A}, P)$. Assume $EX_n = 0$, $EX_n^2 = \sigma_n^2 < \infty$, $n \geqslant 1$. Let $S_n = \sum_{k=1}^{n} X_k$, $s_n^2 = \sum_{k=1}^{n} \sigma_k^2$, $n \geqslant 1$.

It is well known (cf. [6], p.467) that if $s_n^2 \to \infty$, as $n \to \infty$, and for every $t \in R$

$$\left|P(S_n < t\, s_n) - \Phi(t)\right| \to 0 \quad \text{as} \quad n \to \infty,$$

then for all $B \in \mathcal{A}$ with $P(B) > 0$

(1) $$\alpha_n(B) = \sup_t \left|P(S_n < t\, s_n \,|\, B) = \Phi(t)\right| \to 0 \quad \text{as} \quad n \to \infty,$$

where $\Phi(t)$ denotes the standard normal distribution function, i.e.,

$$\Phi(t) = (2\pi)^{-\frac{1}{2}} \int_{-\infty}^{t} \exp(-u^2/2)\,du.$$

On the other hand, if $E|X_n|^{2+s} = \beta_n^{2+s} < \infty$, $n \geqslant 1$, for some $0 < s \leqslant 1$, then by Theorems 3 [7] there exists a constant b (depending only on s) such that for all $n \geqslant 1$

(2) $$\alpha_n(\Omega) \leqslant bL_n^s,$$

where $L_n^s = \sum_{k=1}^{n} \beta_n^{2+s} / s_n^{2+s}$. Thus, by (2), the sequence $\{\alpha_n(\Omega)(L_n^s)^{-1},\ n \geq 1\}$ is bounded. It would be worthwile to determine a sequence $\delta_n \to \infty$ - and if possible the "best" - such that the sequence $\{\delta_n \alpha_n(B),\ n \geq 1\}$ is bounded for each $B \in \mathcal{A}$ with $P(B) > 0$. But taking into account the results presented in [2] one can note that no sequence of independent, even identically distributed, random variables admits such a sequence $\delta_n \to \infty$. This means that each rate of convergence for $\alpha_n(B)$ can be destroyed by a suitable $B \in \mathcal{A}$, so that only convergence rates depending on the set B are available.

In this paper we investigate the convergence order in conditioned central limit theorems, that is, the convergence order of $\sup_t |P(S_n < t\, s_n\, B) - \Phi(t)|$. The results presented extend, to nonidentically distributed random variables $\{X_n,\ n \geq 1\}$ some theorems given by Landers and Rogge [2] and [3].

It is well known that central limit theorems play the most important role in probability theory and statistics. The results which present the rate of convergence in the central limit theorems are also very important in practical and theoretical applications. Especially, Berry-Esseen type results are very well known in probability theory and statistics.

The conditional central limit theorem play, for example, a prominent role in probability theory [5]. On the other hand some applications of conditional limit theorems can be found in the classical occupancy problem, sampling without replacement, the Wilcoxon statistic, the Poisson index of dispersion, testing geometric versus Poisson distribution. Thus the results presented in this paper can also be useful, similarly as Berry-Esseen type results, in probability theory and statistics.

2. Inequalities for conditioned normal approximation

Let $\{X_n,\ n \geq 1\}$ be a sequence of independent random variables such that $EX_n = 0$ and $EX_n^2 = \sigma_n^2 < \infty$, $n \geq 1$. Denote in the following $\mathcal{F}_n = \sigma\{X_1,\dots,X_n\}$, $S_{kn} = S_n - S_k$, $\mathfrak{s}^2_{kn} = \mathfrak{s}^2_n - \mathfrak{s}^2_k = \sigma^2(S_{kn})$, $L^s_{kn} = \Big(\sum\limits_{i=k+1}^{n} \beta_i^{2+s}\Big)/\mathfrak{s}^{2+s}_{kn}$, $1 \leqslant k < n$, $n \geqslant 1$.

Theorem 1. Let $\{X_n, n \geqslant 1\}$ be a sequence of independent random variables such that $EX_n = 0$ and $EX_n^2 = \sigma_n^2 < \infty$, $n \geqslant 1$. Then for $k < n$ we have P – a.e.:

$$(1)\quad \sup_t \left|P(S_n < t\,\mathfrak{s}_n \mid \mathcal{F}_k) - \Phi(t)\right| \leqslant \sup_t \left|P(S_{kn} < t\,\mathfrak{s}_{kn}) - \Phi(t)\right| + (2\pi)^{-\frac{1}{2}} |S_k|/\mathfrak{s}_{kn} + (8\pi e)^{-\frac{1}{2}} \mathfrak{s}^2_k/\mathfrak{s}^2_{kn}.$$

From Theorem 1 we get the following

Corollary. Let $\{X_n, n \geqslant 1\}$ be a sequence of independent random variables such that $EX_n = 0$, $EX_n^2 = \sigma_n^2$, $E|X_n|^{2+s} = \beta_n^{2+s} < \infty$, $n \geqslant 1$, for some fixed $s > 0$. Then there exists a constant b (only depending on s) such that for $k < n$ we have P – a.e.:

$$(2)\quad \sup_t \left|P(S_n < t\,\mathfrak{s}_n \mid \mathcal{F}_k) - \Phi(t)\right| \leqslant b(L^s_{kn} + L^{s^*}_{kn}) + (2\pi)^{-\frac{1}{2}} |S_k|/\mathfrak{s}_{kn} + (8\pi e)^{-\frac{1}{2}} \mathfrak{s}^2_k/\mathfrak{s}^2_{kn},$$

where $s^* = \min(1,s)$.

Theorem 2. Let $\{X_n, n \geqslant 1\}$ be a sequence of independent random variables such that $EX_n = 0$, $EX_n^2 = \sigma_n^2$, $E|X_n|^{2+s} = \beta_n^{2+s} < \infty$, $n \geqslant 1$, for some fixed $s \geqslant 1$. If $L^s_n \to 0$ as $n \to \infty$, then for each r with $2 \leqslant r \leqslant 2+s$ there exists a constant C_r (only depending on r) such that for all $B \in \mathcal{F}_k$ with $P(B) > 0$

$$(3)\quad \sup_t \left|P(S_n < t\,\mathfrak{s}_n \mid B) - \Phi(t)\right| \leqslant (P(B))^{-\frac{1}{r}} C_r \left\{\mathfrak{s}_k/\mathfrak{s}_{kn} + L^s_{kn} + L^1_{kn}\right\}.$$

Let us observe that in Theorem 2 it is not possible to obtain an inequality of the form

$$(4)\quad \sup_t \left|P(S_n < t\, s_n \mid B) - \Phi(t)\right| \leq b\left\{ s_k/s_{kn} + L^s_{kn} + L^3_{kn}\right\},$$

where b is a constant not depending on $B \in \mathcal{F}_k$. Since if $P(X_1 < t) < 1$ for every t, then

$$\lim_{t\to\infty} P(S_n/s_n > 0 \mid X_1 > t) = 1$$

which contradicts (4).

From Theorem 1 and Theorem 1.1 [8] we get the following

Theorem 3. Let $\{X_n, n \geq 1\}$ be a sequence of independent random variables such that $EX_n = 0$ and $EX_n^2 = \sigma_n^2 < \infty$ for all $n \geq 1$. Suppose $s_n \to \infty$, $(\log\log s_n^2)^{1/2}|X_n|/s_n \to 0$ a.s. and, for every t, $|P(S_n < t\, s_n) - \Phi(t)| \to 0$ as $n \to \infty$. If $\{k(n), n \geq 1\}$ is a sequence of integers with

$$(5)\quad s^2_{k(n)} \log\log s^2_{k(n)}/s^2_n \to 0 \quad \text{as} \quad n \to \infty,$$

then

$$(6)\quad \lim_{n\to\infty} \sup_t \left|P(S_n < t\, s_n \mid \mathcal{F}_{k(n)}) - \Phi(t)\right| = 0 \quad \text{a.s.}$$

If, in addition, $E|X_n|^{2+s} < \infty$, $n \geq 1$, for some fixed $0 < s \leq 1$ and the sequence $\{L^s_n s_n/(s^2_{k(n)} \log\log s^2_{k(n)})^{1/2}, n \geq 1\}$ is bounded, then the sequence

$$(7)\quad \left\{(s_n^2/s^2_{k(n)} \log\log s^2_{k(n)})^{\frac{1}{2}} \sup_t \left|P(S_n < t\, s_n \mid \mathcal{F}_{k(n)}) - \Phi(t)\right|, n \geq 1\right\}$$

is a.s. bounded.

Let us observe that if $\{X_n, n \geq 1\}$ is a sequence of independent random variables such that $EX_n^2 = 1$, $n \geq 1$, then the condition (5) is as follows

$$k(n) \log\log k(n)/n \to 0 \quad \text{as} \quad n \to \infty.$$

If, in addition, $EX_n^{2+s} = b_0 < \infty$, $n \geqslant 1$, $0 < s \leqslant 1$, then $L_n^s = b_0 n^{-s/2}$ and

$$L_n^s \mathfrak{s}_n/(\mathfrak{s}^2_{k(n)} \log\log \mathfrak{s}^2_{k(n)})^{\frac{1}{2}} =$$

$$= b_0 n^{(1-s)/2}/(k(n) \log\log k(n))^{\frac{1}{2}}.$$

From Theorem 1 we also obtain the conditioned central limit theorem of Renyi [5] or ([6], p. 467).

Theorem 4. Let $\{X_n,\ n \geqslant 1\}$ be a sequence of independent random variables such that $EX_n = 0$ and $EX_n^2 = \sigma_n^2 < \infty$ for all $n \geqslant 1$. If $\mathfrak{s}_n \to \infty$ and, for every t, $P(S_n < t\,\mathfrak{s}_n) \to \Phi(t)$ as $n \to \infty$, then for every $B \in \mathcal{A}$ with $P(B) > 0$

$$\lim_{n\to\infty} P(S_n < t\,\mathfrak{s}_n | B) = \Phi(t).$$

Let us observe that if $\{X_n,\ n \geqslant 1\}$ is a sequence of independent and identically distributed random variables, then in the case $s = 1$ the theorems presented above reduce to those obtained by Langers and Rogge [2].

3. Proofs

In this section we prove the results presented in Section 2.

Proof of Theorem 1. First we remark that the function

$$\omega \to F_{k+1}^{(n)} * F_{k+2}^{(n)} * \dots * F_n^{(n)}(t\,\mathfrak{s}_n/\mathfrak{s}_{kn} - S_k(\omega)/\mathfrak{s}_{kn}) =$$

$$= F_{k+1,n}^{(n)}(t\,\mathfrak{s}_n/\ kn - S_k(\omega)/\mathfrak{s}_{kn})$$

is a version of the conditional expectation

$$E(I_{[S_n < t\mathfrak{s}_n]} | \mathcal{F}_k) = P(S_n < t\,\mathfrak{s}_n | \mathcal{F}_k),$$

where $F_i^{(n)}(t) = P(X_i < t\,\mathfrak{s}_{kn})$.

Hence

$$\sup_t \left|P(S_n < t\, \mathfrak{s}_n | \mathcal{F}_k) - \Phi(t)\right| \leqslant \sup_t \left|F^{(n)}_{k+1,n}(t\, \mathfrak{s}_n/\mathfrak{s}_{kn} -$$

$$- S_k/\mathfrak{s}_{kn}) - \Phi(t\, \mathfrak{s}_n/\mathfrak{s}_{kn} - S_k/\mathfrak{s}_{kn})\right| +$$

$$+ \sup_t \left|\Phi(t\, \mathfrak{s}_n/\mathfrak{s}_{kn} - S_k/\mathfrak{s}_{kn}) - \Phi(t)\right| \leqslant \sup_t \left|F^{(n)}_{k+1,n}(t) - \Phi(t)\right| +$$

$$+ \sup_t \left|\Phi(t\, \mathfrak{s}_n/\mathfrak{s}_{kn} - S_k/\mathfrak{s}_{kn}) - \Phi(t\, \mathfrak{s}_n/\mathfrak{s}_{kn})\right| +$$

$$+ \sup_t \left|\Phi(t\, \mathfrak{s}_n/\mathfrak{s}_{kn}) - \Phi(t)\right| \leqslant \sup_t \left|F^{(n)}_{k+1,n}(t) - \Phi(t)\right| +$$

$$+ (2\pi)^{-\frac{1}{2}} |S_k|/\mathfrak{s}_{kn} + (8\pi e)^{-\frac{1}{2}} \mathfrak{s}_k^2/\mathfrak{s}_{kn}^2 .$$

The last inequalities are consequences of the inequalities given in ([4], p.143). Thus (1) is proved.

Now taking into account Theorem 1 and Theorem 3 [7] one can prove Corollary, so that we omitted the details.

Proof of Theorem 2. According to the Hölder inequality

$$\sup_t \left|P(S_n < t\, \mathfrak{s}_n, B) - \Phi(t)P(B)\right| = \sup_t \left|E\left\{I_{[S_n < t\, _n, B]} - I_B \Phi(t)\right\}\right| =$$

$$= \sup_t \left|E(I_B E(\left\{I_{[S_n < t\, \mathfrak{s}_n]} - \Phi(t)\right\} | \mathcal{F}_k))\right| =$$

$$= \sup_t \left|E(P(S_n < t\, \mathfrak{s}_n | \mathcal{F}_k) - \Phi(t)) I_B\right| \leqslant$$

$$\leqslant \sup_t (E(I_B)^{r/(r-1)})^{(r-1)/r} (E\left|P(S_n < t\, \mathfrak{s}_n | \mathcal{F}_k) - \Phi(t)\right|^r)^{1/r} \leqslant$$

$$\leqslant (P(B))^{1-1/r} \left\| \sup_t \left|P(S_n < t\, \mathfrak{s}_n | \mathcal{F}_k) - \Phi(t)\right\|_r \right.,$$

where $\|Z\|_r = (E|Z|^r)^{1/r}$.

Now, by (2) and the triangle inequality, we get

$$\|\sup_t|P(S_n < t\, s_n \mid \mathcal{F}_k) - \Phi(t)\|_r \leqslant b\{L^s_{kn} + L^1_{kn} + s_k/s_{kn} +$$

$$+ \|S_k/s_k\|_r(s_k/s_{kn}).$$

On the other hand, if $L^s_n \to 0$ as $n \to \infty$, then for every $2 \leqslant r \leqslant 2+s$ and all $\varepsilon > 0$

$$s_n^{-r}\sum_{k=1}^{n} E|X_k|^r I(|X_k| \geqslant \varepsilon\, s_n) \longrightarrow 0 \quad \text{as} \quad n \to \infty.$$

Thus, taking into account the results given in [1], or Theorem 4 [7] in the case $r > 2$, we note that there exists a constant K_r, depending only on r, such that $\sup_k\|S_k/s_k\|_r \leqslant K_r$, so that (3) follows from the inequalities given above.

Proof of Theorem 3. At first we note that, by (5), $s_{k(n)}/s_s \to 0$ as $n \to \infty$, so that $S_{k(n)}/s_n \xrightarrow{P} 0$ as $n \to \infty$. Thus the assumption $\sup_t|P(S_n < t\, s_n) - \Phi(t)| \to 0$, as $n \to \infty$, implies

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$$\sup_t|P(S_{k(n)n} < t\, s_{k(n)n}) - \Phi(t)| \to 0 \tag{8}$$

as $n \to \infty$.

On the other hand, the law of the iterated logarithm, Theorem 1.1 [8], implies

$$H_- \leqslant \limsup_{n\to\infty} |S_{k(n)}|/(2s^2_{k(n)} \log\log s^2_{k(n)})^{\frac{1}{2}} \leqslant H_+ \quad \text{a.s.}, \tag{9}$$

where H_- and H_+ are numbers satisfying $0 \leqslant H_- \leqslant H_+ \leqslant 1$, .

$$H_-^2 = \liminf_{n\to\infty} H_n(x), \qquad H_+^2 = \limsup_{n\to\infty} H_n(x)$$

and

$$H_n(x) = s_n^{-2} \sum_{j=1}^{n} EX_j^2 I(|X_j| \leqslant x\, s_j(2 \log \log s_j^2)^{-\frac{1}{2}}).$$

The values H_- and H_+ are independent of x. Hence (9) and (5) imply

$$(10) \qquad \limsup_{n \to \infty} |S_{k(n)}|/s_{k(n)n} \leqslant$$

$$\leqslant H_+ \limsup_{n \to \infty} (2s^2_{k(n)} \log \log s^2_{k(n)})/s_{k(n)n} = 0.$$

Now Theorem 1, (8) and (10) imply (6). Furthermore, by Theorem 3 [7], there exists a positive constant b (only depending on s) such that

$$\sup_t |P(S_{k(n)n} < t\, s_{k(n)n}) - \Phi(t)| \leqslant bL^s_{k(n)n} \leqslant bL_n^s\, s_n^{2+s}/s^{2+s}_{k(n)n}.$$

Hence, taking into account the assumptions of Theorem 3, one can note that the sequence

$$(11) \quad \{(s_n^2/s^2_{k(n)} \log \log s^2_{k(n)})^{\frac{1}{2}} \sup_t |P(S_{k(n)n} < t\, s_{k(n)n}) - \Phi(t)|, \quad n \geqslant 1\}$$

is bounded. On the other hand, by (9), for all sufficiently large n

$$(12) \quad |S_{k(n)}|(s_n^2/s^2_{k(n)} \log \log s^2_{k(n)})^{\frac{1}{2}}/s_{k(n)n} \leqslant b_1 H_+ \quad \text{a.s.,}$$

where b_1 is an absolute constant, so that (7) follows from Theorem 1, (11) and (12). Thus the proof of Theorem 3 is completed.

Proof of Theorem 4. Let $\mathcal{F}_\infty = \sigma\{X_n,\ n \geq 1\}$. Let $B \in \mathcal{A}$, with $P(B) > 0$, be given. Then there exist $\mathcal{F}_n$-measurable functions φ_n such that $0 \leq \varphi_n \leq 1$ and

$$\lim_{n \to \infty} E\left|E(I_B \mid \mathcal{F}_\infty) - \varphi_n\right| = 0.$$

Let $\varepsilon > 0$ be given. Then there exists k_0 such that for every $k \geq k_0$

$$E\left|E(I_B \mid \mathcal{F}_\infty) - \varphi_k\right| < \varepsilon,$$

so that for every $k \geq k_0$

(13) $$\left|P(S_n < t\,\mathfrak{s}_n, B) - P(B)\Phi(t)\right| = \left|EI(S_n < t\,\mathfrak{s}_n)I(B) - P(B)\Phi(t)\right| =$$

$$= \left|E(E(I(S_n < t\,\mathfrak{s}_n)I(B) \mid \mathcal{F}_\infty)) - P(B)\Phi(t)\right| \leq$$

$$\leq \left|E(I(S_n < t\,\mathfrak{s}_n)E(I(B) \mid \mathcal{F}_\infty)) - EI(S_n < t\,\mathfrak{s}_n)\,\varphi_k\right| +$$

$$+ \left|EI(S_n < t\,\mathfrak{s}_n)\varphi_k - \Phi(t)E\varphi_k\right| + \left|\Phi(t)(E\varphi_k - P(B))\right| =$$

$$= \left|EI(S_n < t\,\mathfrak{s}_n)(E(I(B) \mid \mathcal{F}_\infty) - \varphi_k)\right| +$$

$$+ \left|E(\varphi_k E(I(S_n < t\,\mathfrak{s}_n)\ \mathcal{F}_k)) - \Phi(t)E\varphi_k\right| + \left|\Phi(t)\right|\left|E\varphi_k - P(B)\right| \leq$$

$$\leq E\left|E(I(B) \mid \mathcal{F}_\infty) - \varphi_k\right| + E(\varphi_k\left|E(I(S_n < t\,\mathfrak{s}_s) \mid \mathcal{F}_k) - \Phi(t)\right|) +$$

$$+ E\left|\varphi_k - E(I(B) \mid \mathcal{F}_\infty)\right| \leq 2\varepsilon + E(\varphi_k\left|P(S_n < t\,\mathfrak{s}_n \mid \mathcal{F}_k) - \Phi(t)\right| \leq$$

$$\leq 2\varepsilon + \|\varphi_k\|_2 \|P(S_n < t\,\mathfrak{s}_n \mid \mathcal{F}_k) - \Phi(t)\|_2.$$

On the other hand, by Theorem 1, we get

(14) $$\|P(S_n < t\,\mathfrak{s}_n \mid \mathcal{F}_k) - \Phi(t)\|_2 \leq \sup_t \left|P(S_{kn} < t\,\mathfrak{s}_{kn}) - \Phi(t)\right| +$$

$$+ (2\pi)^{-\frac{1}{2}} (ES_k^2)^{\frac{1}{2}} / \mathfrak{s}_{kn} + (8\pi e)^{-\frac{1}{2}}\, \mathfrak{s}_k^2 / \mathfrak{s}_{kn}^2.$$

But $(ES_k^2)^{1/2} = \delta_k$. Furthermore, $\delta_n^2 \to \infty$ as $n \to \infty$, so that for every fixed k

$$\sup_t \left| P(S_{kn} < t\,\delta_{kn}) - \Phi(t) \right| \to 0 \quad \text{as} \quad n \to \infty .$$

Thus Theorem 4 follows from (13) and (14).

REFERENCES

[1] В.М. Круглов : Центральная предельная теорема и чакон больших чисел в среднем, Teor. Verojatnost. i Primenen. 26 (1981) 824-827.

[2] D. Landers, L. Rogge : Inequalities for conditional normal approximations, Ann. Probability, 5 (1977), 595-600.

[3] D. Landers, L. Rogge : Exact approximation orders in the conditional central limit theorem, Z. Wahrsch. Verw. Gebiete, 66 (1984), 227-244.

[4] В.В. Петров : Суммы независимых случайных величин. Moskwa 1972.

[5] A. Renyi : On mixing sequences of sets, Acta Math. Acad. Sci. Hungar. 9 (1958), 215-228.

[6] A. Renyi : Probability theory, Budapest 1970.

[7] Z. Rychlik : Nonuniform central limit bounds with applications to probabilities of deviations. Teor. Verojatnost. i Primenen. 28 (1983), 646-652.

[8] R.J. Tomkins : Lindeberg functions and the law of the iterated logarithm, Z. Wahrsch. Verw. Gebiete, 65 (1983), 135-143.

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ASYMPTOTISCHE ENTWICKLUNGEN DER HYPERGEOMETRISCHEN FUNKTION F(a, b, c, z) FÜR $|c| \to \infty$ UND KONSTANTE WERTE a, b UND z

1. Einleitung

Asymptotische Entwicklungen der hypergeometrischen Funktion $F(a,b;c;z)$ für betragsmäßig große reelle oder komplexe Werte eines oder mehrerer Parameter wurden erstmalig von O. Perron [7] und G.N. Watson [12] untersucht. Für $|c| \to \infty$, bei konstanten Werten von a, b, z werden in [7] unter der einschränkenden Bedingung Im c = const asymptotische Entwicklungen angegeben, während in [12] zwar eine Konzept zur Herleitung derartiger Entwicklungen für beliebig gegen Unendlich strebendes c vorgestellt wird, darüber hinaus aber nur die sehr enfach zu beweisende Entwicklung

$$(1) \quad F(a,b;c+l;z) \approx \frac{\Gamma(c+l)}{\Gamma(c+l-b)} \sum_{\nu=0}^{\infty} \frac{k_\nu \Gamma(b+\nu)}{\Gamma(b)} l^{-b-\nu} \qquad (|l| \to \infty)$$

(mit einer erzeugenden Funktion für die Koeffizienten k_ν) angegeben ist. Die den Geltungsbereich von (1) definierenden Bedingungen sind in [12] nur zum Teil konkret formuliert und zudem mit einem wesentlichen Vorzeichenfehler behaftet. Das Konzept von Watson wird in [4] etwas weiter ausgeführt und dabei (mit einem nicht ganz lückenlosen Beweis) gezeigt, daß in einem explizit angegebenen, von z abhängigen Winkelraum der komplexen c-Ebene aus (1) die für $|z| < 1$ sogar mit dem Gleichheitszeichen geltende Entwicklung

$$(2)\quad F(a,b;c;z) \approx \sum_{\nu=0}^{\infty} \frac{(a)_\nu (b)_\nu}{\nu!\,(c)_\nu} z^\nu \quad (|c| \to \infty,\ a,b,z \text{ fest},\ |\arg(1-z)| \leqslant \pi)$$

folgt. (Hier und im folgenden verwenden wir die üblichen Bezeichnungen $(a)_\nu = \Gamma(a+\nu)/\Gamma(a)$, also $(a)_0 = 1$, $(a)_\nu = a(a+1)(a+2)\ldots(a+\nu-1)$ für $\nu \geqq 1$). Nicht richtig ist die in [4] wie in [1] aufgestellte, unbewiesene Behauptung, daß (2) bei beliebig gegen Unendlich strebendem c gilt, falls nur $|z| < 1$ ist.

In der vorliegenden Arbeit werden asymptotische Entwicklungen für $F(a,b;c;z)$ bei beliebig gegen Unendlich strebendem c hergeleitet, deren Gültigkeitsbereiche bei jeweils beliebigen Werten a, b, z ($|\arg(1-z)| \leqslant \pi$) die komplexe c-Ebene voll ausschöpfen, selbstverständlich bis auf die nicht zum Definitionsbereich von F gehörenden Punkte $c = 0,-1,-2,\ldots$.

2. Bezeichnungen

Ist $\operatorname{Re} z > 1/2$, $z \neq 1$ mit $|\arg(1-z)| \leqslant \pi$, so sei

$$(3)\qquad \alpha = \alpha(z) = \arctan \frac{\arg(1-z) - \operatorname{Arg} z + \pi}{|\ln 1-1/z|},$$

$$(4)\qquad \beta = \beta(z) = \arctan \frac{\arg(1-z) - \operatorname{Arg} z - \pi}{\ln|1-1/z|},$$

wobei Arg z den Hauptwert des Arguments bezeichnet, d.h. $\operatorname{Arg} z \in (-\pi,\pi]$. Wegen $\ln|1-1/z| < 0$ und $\arg(1-z)-\operatorname{Arg} z \in [-\pi,\pi]$ für $\operatorname{Re} z > 1/2$ gelten die Ungleichungen

$$(5)\qquad -\pi/2 < \alpha \leqslant 0 \leqslant \beta < \pi/2,$$

wobei offenbar α und β nicht gleichzeitig verschwinden können. Mit ε und δ werden im folgenden stets positive Konstanten bezeichnet, die beliebig klein gewählt werden können, während k, m und n stets beliebige, voneinander unabhängige, nichtnegative ganze Zahlen bedeuten.

3. Resultate

Die folgenden Sätze gelten unter den generellen Voraussetzungen $c \neq -n$, $|\arg(1-z)| \leqslant \pi$, $z \neq 0$ und $z \neq 1$. Die Parameter a und b sowie z seien beliebige feste komplexe Zahlen.

Satz 1: Es gilt

$$F(a,b;c;z) \approx \sum_{\nu=0}^{\infty} \frac{(a)_\nu (b)_\nu}{\nu!\,(c)_\nu} z^\nu \qquad (|c| \to \infty) \tag{6}$$

in jedem der folgenden Fälle:

(i) $a = -m$ oder $b = -n$;

(ii) $\operatorname{Re} z < 1/2$ und $|c+n| \geqslant \delta > 0$;

(iii) $\operatorname{Re} z = 1/2$ und $|\arg c| \leqslant \pi - \varepsilon$;

(iv) $\operatorname{Re} z > 1/2$ und

$$-\beta - \pi/2 + \varepsilon \leqslant \arg c \leqslant -\alpha + \pi/2 - \varepsilon\,. \tag{7}$$

Satz 2: Sind $a \neq -m$ und $b \neq -n$ sowie $\operatorname{Re} z > 1/2$, so gilt im Winkelraum

$$-\alpha - \pi/2 + \varepsilon \leqslant \arg(-c) \leqslant -\beta + \pi/2 - \varepsilon \tag{8}$$

für $|c| \to \infty$ die asymptotische Entwicklung

$$F(a,b;c;z) \approx \tag{9}$$

$$\approx \frac{\pi}{\sin(\pi c)} \frac{z^{1-c}(1-z)^{c-b-a}}{\Gamma(a)\,\Gamma(b)} \frac{\Gamma(a+b-c)}{\Gamma(1-c)} \sum_{\nu=0}^{\infty} \frac{(1-a)_\nu (1-b)_\nu}{\nu!\,(c-b-a+1)_\nu} (1-z)^\nu .$$

Satz 3: Sind $a \neq -m$ und $b \neq -n$, so gilt für $|c| \to \infty$

$$F(a,b;c;z) = \sum_{\nu=0}^{k} \frac{(a)_\nu (b)_\nu}{\nu!\,(c)_\nu} z^\nu + O(c^{-k-1}) + \tag{10}$$

$$+ \frac{\pi}{\sin(\pi c)} \frac{z^{1-c}(1-z)^{c-b-a}}{\Gamma(a)\,\Gamma(b)} \frac{\Gamma(a+b-c)}{\Gamma(1-c)} \left[\sum_{\nu=0}^{k} \frac{(1-a)_\nu (1-b)_\nu}{\nu!\,(c-b-a+1)_\nu} (1-z)^\nu + O(c^{-k-1}) \right]$$

in jedem der folgenden Fälle:

(i) Re $z < 1/2$ und $c = -n + o(1)$;

(ii) Re $z = 1/2$ und $\arg(-c) = o(1)$;

(iii) Re $z > 1/2$ und $\arg c = -\alpha + \pi/2 + o(1)$ oder $\arg c = -\beta - \pi/2 + o(1)$.

Zu Satz 3 bleibt zu bemerken, daß in Abhängigkeit von den Werten der Konstanten a, b, z eine der beiden Summen in (10) gegenüber der anderen vernachlässigbar sein kann, aber darüber keine einfache allgemeine Aussage möglich ist. Das hängt offenbar davon ab, ob der Vorfaktor der zweiten Summe mit $|c| \to \infty$ exponentiell wächst oder fällt.

4. Beweise

4.1. Der Fall (i) von Satz 1 ist trivial, da (6) nach Definition der hypergeometrischen Funktion sogar mit dem Gleichheitszeichen gilt. Wir können deshalb im folgenden annehmen, daß weder a noch be gleich Null oder einer negativen ganzen Zahl ist.

4.2. Für $c > 0$ und $\nu > n$ gilt

$$(c)_\nu = c(c+1)\dots(c+n-1)(c+n)\dots(c+\nu-1) \geqslant c^n n(n+1)\dots(\nu-1) = c^n \frac{\nu!}{\nu(n-1)!}.$$

Daraus folgt für $|z| < 1$ $(z \neq 0)$, beliebige komplexe a, b und $c > 0$ die Abschätzung

$$\left| F(a,b;c;z) - \sum_{\nu=0}^{n} \frac{(a)_\nu (b)_\nu}{\nu!(c)_\nu} z^\nu \right| \leqslant \frac{(n-1)!}{c^n} \sum_{\nu=n+1}^{\infty} \frac{\nu(|a|)_\nu (|b|)_\nu}{(\nu!)^2} |z|^\nu \leqslant$$

$$\leqslant \frac{(n-1)!\,|z|}{c^n} \frac{d}{dz} F(|a|,|b|;1;|z|) = O(c^{-n}).$$

Damit ist (6) für $|z| < 1$ und c (reell) $\to +\infty$ bewiesen.

4.3. Zum Beweis der nichttrivialen Fälle gehen wir von der bekannten Integraldarstellung

$$F(a,b;c;z) =$$

(11)
$$= \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a}dt \quad (\mathrm{Re}\ c > \mathrm{Re}\ b > 0)$$

aus, wobei für $\arg(1-z) = -\pi$ oder $\arg(1-z) = \pi$ (oberes bzw. unteres Ufer des Schnittes $z > 1$) der Punkt $1/z$ des Integrationsintervalls auf einem kleinen oberen bzw. unteren Halbkreis umgangen wird. Mit der Substitution $1-t = e^{-\tau}$ erhält man aus (11)

(12)
$$F(a,b;c;z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^\infty e^{-c\tau}e^{b\tau}(1-e^{-\tau})^{b-1}(1-z+ze^{-\tau})^{-a}d\tau.$$

Das bedeutet: F ist für $\mathrm{Re}\ c > \mathrm{Re}\ b\ (>0)$ bis auf den Faktor vor dem Integral die Laplace-Transformierte der Funktion

(13)
$$f(\tau) = e^{b\tau}(1-e^{-\tau})^{b-1}(1-z+ze^{-\tau})^{-a}.$$

Die Funktion f ist holomorph bezüglich τ in jedem Winkelraum mit dem Scheitel 0, der keine Nullstellen $+2k\pi i$ und $-\mathrm{Log}(1-1/z) + 2k\pi i$ von $1-e^{-\tau}$ bzw. $1-z+ze^{-\tau}$ enthält, insbesondere also für

(14)
$$-\pi/2 < \mathrm{Arg}\ \tau < \pi/2, \quad \text{falls } \mathrm{Re}\ z \leqslant 1/2 \text{ ist,}$$

und

(15)
$$\alpha < \mathrm{Arg}\ \tau < \beta, \quad \text{falls } \mathrm{Re}\ z > 1/2 \text{ ist.}$$

In jedem abgeschlossenen Teilwinkelraum von (14) bzw. (15) ist offenbar $e^{-b\tau}f(\tau) = O(1)$ für $\tau \to \infty$, so daß der Integrationsweg in (12) innerhalb der Winkelräume (14) bzw. (15) beliebig um den Nullpunkt gedreht werden kann. Bezeichnet φ den Drehwinkel, so erhält man die analytische Fortsetzung der Bildfunktion in die Halbebene $\mathrm{Re}\left[e^{i\varphi}(c-b)\right] > 0$, [2]. Da

$F(a,b;c;z)/\Gamma(c)$ eine ganze holomorphe Funktion von c ist, gilt demnach

$$(16)\qquad F(a,b;c;z) =$$

$$= \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^{\infty e^{i\varphi}} e^{-c\tau} f(\tau)\,d\tau \quad (\operatorname{Re} b > 0,\ \operatorname{Re}\left[(c-b)e^{i\varphi}\right] > 0)$$

mit

$$(17)\qquad \varphi \in \begin{cases} (-\pi/2, \pi/2) & \text{für} \quad \operatorname{Re} z \leqslant 1/2, \\ (\alpha, \beta) & \text{für} \quad \operatorname{Re} z > 1/2. \end{cases}$$

Die Funktion $f(\tau)\tau^{-b+1}$ ist holomorph in einer Umgebung von $\tau = 0$, so daß f eine asymptotische Entwicklung

$$(18)\qquad f(\tau) \approx \tau^{b-1} \sum_{\nu=0}^{\infty} P_\nu(a,b,z)\tau^\nu \quad (\tau \to 0)$$

besitzt. Die Koeffizienten $P_\nu(a,b,z)$ sind, wie man aus (13) leicht erkennt, Polynome in a, b und z. Aus (18) folgt nach einer bekannten Verallgemeinerung des Lemmas von Watson [3] in jedem Winkelraum $\left|\arg\left[e^{i\varphi}(c-b)\right]\right| \leqslant \pi/2 - \varepsilon$ die bereits von Watson [12] (vgl. (1) in der vorliegenden Arbeti) angegebene asymptotische Entwicklung

$$(19)\qquad F(a,b;c;z) \approx$$

$$\approx \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \sum_{\nu=0}^{\infty} P_\nu(a,b,z)\Gamma(\nu+b)c^{-\nu-b} \quad (|c| \to \infty).$$

4.4. Nach [8] gilt für beliebige komplexe Zahlen s und t

$$(20)\ \frac{\Gamma(c+s)}{\Gamma(c+t)} \approx \sum_{k=0}^{\infty} (-1)^k \frac{B_k^{(s-t+1)}(s)(t-s)_k}{k!} c^{s-t-k}$$

$$(c \to \infty,\ |\arg(c+s)| \leqslant \pi-\varepsilon),$$

wobei die $B_k^{(s)}(t)$ die durch die erzeugende Funktion

$$z^s e^{tz}(e^z-1)^{-s} = \sum_{k=0}^{\infty} \frac{B_k^{(s)}(t)}{k!} z^k \qquad (|z| < 2\pi) \tag{21}$$

definierten verallgemeinerten Bernoullischen Polynome (in s und t) sind.

4.5. Setzt man für den Faktor $\Gamma(c)/\Gamma(c-b)$ in (19) die nach 4.4. für $|\arg c| \leqslant \pi-\varepsilon$ geltende asymptotische Entwicklung (20) ein und multipliziert anschließend beide Entwicklungen miteinander, so erhält man für $\left|\arg\left[(c-b)e^{i\varphi}\right]\right| \leqslant \pi/2-\varepsilon$

$$F(a,b;c;z) \approx \sum_{\nu=0}^{\infty} Q_\nu(a,b,z)c^{-\nu} \qquad (|c| \to \infty) \tag{22}$$

mit Koeffizienten Q_ν, die Polynome in a, b und z sind. Werden die Potenzen $c^{-\nu}$ nach der äquivalenten asymptotischen Skala $\left\{[(c)_\mu]^{-1}\right\}_{\mu\geqslant 0}$ entwickelt, so folgt aus (22)

$$F(a,b;c;z) \approx \tag{23}$$

$$\approx \sum_{\nu=0}^{\infty} R_\nu(a,b,z)\left[(c)_\nu\right]^{-1} \qquad \left(c \to \infty, \left|\arg\left[(c-b)e^{i\varphi}\right]\right| \leqslant \pi/2-\varepsilon\right)$$

mit in a,b,z ganzrationalen Koeffizienten R_ν.

Für $|z| < 1$, $\varphi = 0$ und c (reell) $\to +\infty$ müssen nach 4.2 die Entwicklungen (6) und (23) identisch sein, also $R_\nu = (a)_\nu(b)_\nu z^\nu/\nu!$. Berücksichtigt man noch (17), so ist gezeigt, daß (6) im Fall $\operatorname{Re} z \leqslant 1/2$ im Winkelraum $|\arg c| \leqslant \pi-\varepsilon$ und im Fall $\operatorname{Re} z > 1/2$ im Winkelraum $-\beta-\pi/2+\varepsilon \leqslant \arg c \leqslant -\alpha+\pi/2-\varepsilon$ gilt, aber jeweils noch unter der einschränkenden Bedingung $\operatorname{Re} b > 0$.

4.6. Setzt man in der Gaußschen Beziehung [1]

$$F(a,b;c;z) = (1-z)F(a,b+1;c;z) + (1-a/c)zF(a,b+1;c+1;z)$$

für die hypergeometrischen Funktionen auf der rechten Seite die nach 4.5 für $\operatorname{Re} b > -1$ geltenden Entwicklungen (6) ein, so erhält man für F(a,b;c;z) wieder (6). Durch wiederholte Anwendung dieser Schlußweise kann man sich von der Einschränkung Re b > 0 befreien. Damit ist Satz 1 bis auf den Fall (ii) bewiesen.

4.7. Zum Beweis des für arg(−c) = o(1) noch offenen Falls (ii) von Satz 1 sowie der Sätze 2 und 3 gehen wir aus von der bekannten Beziehung [1]

$$(24)\quad F(a,b;a+b-c+1;1-z) = \frac{\Gamma(a+b-c+1)\Gamma(1-c)}{\Gamma(a-c+1)\Gamma(b-c+1)} F(a,b;c;z) + $$

$$+ \frac{\Gamma(a+b-c+1)\Gamma(c-1)}{\Gamma(a)\Gamma(b)} z^{1-c}(1-z)^{c-b-a} F(1-a,1-b;2-c;z).$$

Wegen $\Gamma(c-1) = -\pi/[\Gamma(2-c)\sin(\pi c)]$ erhält man aus (24)

$$(25)\quad F(a,b;c;z) = k_1(a,b,c)F(a,b;a+b-c+1;1-z) +$$

$$+ k_2(a,b,c,z)F(1-a,1-b;2-c;z)$$

mit

$$(26)\quad k_1 = \Gamma(a-c+1)\Gamma(b-c+1)/[\Gamma(a+b-c+1)\Gamma(1-c)],$$

$$(27)\quad k_2 = \frac{\pi}{\sin(\pi c)} \frac{\Gamma(a-c+1)\Gamma(b-c+1)}{\Gamma(a)\Gamma(b)\Gamma(1-c)\Gamma(2-c)} z^{1-c}(1-z)^{c-b-a}.$$

Nach dem bis einschließlich 4.6 geführten Beweis gilt

$$(28)\quad F(a,b;a+b-c+1;1-z) \approx \sum_{\nu=0}^{\infty} \frac{(a)_\nu (b)_\nu}{\nu!(a+b-c+1)_\nu} (1-z)^\nu \quad (|c| \to \infty)$$

in einem durch die Ungleichungen $|\arg(-c)| \leq \pi - \varepsilon$ für $\mathrm{Re}(1-z) \leq 1/2$ oder $-\beta - \pi/2 + \varepsilon \leq \arg(-c) \leq -\alpha + \pi/2 - \varepsilon$ ($\alpha = \alpha(1-z)$, $\beta = \beta(1-z)$) für $\mathrm{Re}(1-z) > 1/2$ definierten Winkelraum, der mit W_{1-z} bezeichnet werde. Die Reihe in (28) kann in eine asymptotische Reihe nach der äquivalenten Skala $\{c^{-\mu}\}_{\mu\geq 0}$ übergeführt werden, deren Koeffizienten Polynome in a, b und z sind:

$$F(a,b;a+b-c+1;1-z) \approx \sum_{\nu=0}^{\infty} \tilde{P}_\nu(a,b,z)c^{-\nu} \quad (|c| \to \infty,\ c \in W_{1-z}). \tag{29}$$

Die in (26) auftretenden Quotienten $\Gamma(a-c+1)/\Gamma(a+b-c+1)$ und $\Gamma(b-c+1)/\Gamma(1-c)$ besitzen nach 4.4 für $|c| \to \infty$, $|\arg(-c)| \leqslant \pi-\varepsilon$ bis auf sich gegenseitig wegkürzende Faktoren $(-c)^{-b}$ und $(-c)^{b}$ asymptotische Entwicklungen nach Potenzen von 1/c mit Koeffizienten, die Polynome in a und b bzw. in b sind. Multipliziert man beide Entwicklungen miteinander und die Produktreihe anschließend mit (29), so erhält man die Entwicklung

$$k_1(a,b,c)F(a,b;a+b-c+1;1-z) \approx \tag{30}$$

$$\approx \sum_{\nu=0}^{\infty} \tilde{Q}_\nu(a,b,z)c^{-\nu} \quad (|c| \to \infty,\ c \in W_{1-z}),$$

die schließlich noch in die äquivalente asymptotische Reihe

$$k_1(a,b,c)F(a,b:a+b-c+1;1-z) \approx \tag{31}$$

$$\approx \sum_{\nu=0}^{\infty} \tilde{R}_\nu(a,b,z)\{(c)_\nu\}^{-1} \quad (|c| \to \infty,\ c \in W_{1-z})$$

transformiert werden kann. Dabei sind $\tilde{Q}_\nu$ und $\tilde{R}_\nu$ Polynome in a, b und z. Zur Berechnung der $\tilde{R}_\nu$ beschränken wir uns zunächst auf den Fall $\mathrm{Re}\ z < 1/2$, $0 < \gamma_0/2 \leqslant \arg(-c) \leqslant \gamma_0$ mit einer kleinen festen Zahl γ_0. Dann ist offenbar $1/\sin(\pi c) = o(1)$, wegen (20) $\Gamma(a-c+1)/\Gamma(1-c) \sim (-c)^{a}$ und $\Gamma(b-c+1)/\Gamma(2-c) \sim (-c)^{b-1}$ sowie mit $\gamma = \arg(-c)$

$$|z^{-c}(1-z)^{c}| = \tag{32}$$

$$= \exp\{-|c|\cos\gamma[\ln|1-1/z|+\tan\gamma(\mathrm{Arg}\ z - \arg(1-z))]\} \leqslant \tag{33}$$

$$\leqslant \exp\{-|c|\cos\gamma_0[\ln|1-1/z|-\tan\gamma_0|\mathrm{Arg}\ z - \arg(1-z)|]\},$$

so daß k_2 bei hinreichend klein gewähltem γ_0 mit $|c| \to \infty$ exponentiell gegen Null strebt. Da wegen (6) $F(1-a,1-b;2-c;z)$ gleichzeitig beschränkt bleibt, folgt aus (25) und (31)

$$(34) \qquad F(a,b;c;z) \approx \sum_{\nu=0}^{\infty} \tilde{R}_\nu(a,b,z)\left\{(c)_\nu\right\}^{-1}$$

$$(|c| \to \infty,\ 0 < \gamma_0/2 \leq \arg(-c) \leq \gamma_0,\ \mathrm{Re}\ z < 1/2).$$

Andererseits gilt unter den angegebenen Bedingungen nach 4.6 die Entwicklung (6), woraus wegen der Eindeutigkeit der Koeffizienten einer asymptotischen Entwicklung $\tilde{R}_\nu(a,b,z) = (a)_\nu (b)_\nu z^\nu/\nu!$ folgt. Statt (31) kann man also schreiben

$$(35) \qquad k_1(a,b,c)F(a,b;a+b-c+1;1-z) \approx$$

$$\approx \sum_{\nu=0}^{\infty} \frac{(a)_\nu (b)_\nu}{\nu!(c)_\nu}\, z^\nu \quad (|c| \to \infty\ ,\ c \in W_{1-z}).$$

Ersetzt man in (35) a durch 1-a, b durch 1-b, c durch c-b-a+1 und z durch 1-z, so erhält man

$$(36) \qquad \frac{\Gamma(b-c+1)\Gamma(a-c+1)}{\Gamma(2-c)\Gamma(a+b-c)}\, F(1-a,1-b;2-c;z) \approx \sum_{\nu=0}^{\infty} \frac{(1-a)_\nu (1-b)_\nu}{\nu!(c-b-a+1)_\nu}\,(1-z)^\nu$$

$$(|c| \to \infty\ ,\ c \in W_z).$$

Aus (25), (35) und (36) folgt schließlich

$$(37) \qquad F(a,b;c;z) = \sum_{\nu=0}^{n} \frac{(a)_\nu (b)_\nu}{\nu!(c)_\nu}\, z^\nu + O(c^{-n-1}) +$$

$$+ k_3(a,b,c,z)\left[\sum_{\nu=0}^{n} \frac{(1-a)_\nu (1-b)_\nu}{\nu!(c-b-a+1)_\nu}\,(1-z)^\nu + O(c^{-n-1})\right]$$

$$(|c| \to \infty,\ c \in W_z \cap W_{1-z})$$

mit

$$k_3(a,b,c,z) = \frac{\pi}{\sin(\pi c)} \frac{\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)\Gamma(1-c)} z^{1-c}(1-z)^{c-b-a}, \tag{38}$$

Offenbar hängt es von k_3 ab, ob und gegebenenfalls welche Teile in (37) vernachlässigbar sind.

4.8. Ist Re $z \leq 1/2$, so gilt (37) speziell für $\gamma = \arg(-c) = o(1)$. In diesem Fall folgt aus $\Gamma(a+b-c)/\Gamma(1-c) \sim (-c)^{a+b-1}$, (32) und (38)

$$|k_3(a,b,c,z)| \approx \tag{40}$$

$$\approx \frac{\pi}{\sin(\pi c)} \left|(-c)^{a+b-1}\right| \exp\left\{-|c|(1+o(1))\left[\ln|1-1/z|+o(1)\right]\right\}.$$

Daraus wird ersichtlich, daß für Re $z < 1/2$ und $|c+n| \geq \delta > 0$ der Koeffizient k_3 mit $|c| \to \infty$ exponentiell fällt, und sich deshalb (37) auf (6) reduzieren läßt. Damit ist auch der Fall (ii) von Satz 1 vollständig bewiesen.

Ist Re $z < 1/2$ und $c = -n+o(1)$ $(n \to \infty)$ oder Re $z = 1/2$ und $\arg(-c) = o(1)$, so kann über die Größenordnung von k_3 keine allgemeingültige Aussage gemacht und folglich keiner der Anteile in (37) von vornherein vernachlässigt werden. Das stimmt überein mit den Aussagen in den Fällen (i) und (ii) von Satz 3.

4.9. Es sei jetzt Re $z > 1/2$ und c strebe in dem zum Winkelraum (7) komplementären Winkelraum

$$W_z^* : -\alpha-\pi/2-\varepsilon < \arg(-c) = \gamma < -\beta+\pi/2+\varepsilon \tag{41}$$

gegen Unendlich. Da α und β nach (3) und (4) nicht gleichzeitig Null sein können, folgt aus (5) $\beta-\alpha>0$. Also ist W_z^* vollständig in W_z $(-\beta-\pi/2+\varepsilon \leq \gamma \leq -\alpha+\pi/2-\varepsilon)$ enthalten und wegen $W_z \cap W_{1-z} = W_z$ gilt (37) in W_z^*. Aus (3) und (4) erhält man

$$(42)\quad |z^{-c}(1-z)^{c}| = \exp\left\{-|c|\cos\gamma\,\ln|1-1/z|\,[1-\tan\alpha\,\tan\gamma] - -\pi\,|c|\,\sin\gamma\right\}$$

oder

$$(43)\quad |z^{-c}(1-z)^{c}| = \exp\left\{-|c|\cos\gamma\,\ln|1-1/z|\,[1-\tan\beta\,\tan\gamma] + +\pi\,|c|\,\sin\gamma\right\}.$$

Im Teilwinkelraum

$$W_z^{**}\ :\ -\pi/2 < -\alpha - \pi/2 + \varepsilon \leqslant \arg(-c) = \gamma \leqslant -\beta + \pi/2 - \varepsilon < \pi/2$$

von W_z^{*} folgt wegen $1/\tan(\alpha-\varepsilon) \leqslant \tan\gamma \leqslant 1/\tan(\beta+\varepsilon)$ und $\cos\gamma \geqslant \sin\varepsilon$ aus (42) bzw. (43)

$$(44)\quad \left|z^{-c}(1-z)^{c}\right| \geqslant \exp\left\{-\lambda_\varepsilon\,|c|\,\sin\varepsilon\ \ln|1-1/z| + \pi\,|c|\,\sin|\gamma|\right\}$$

mit einer positiven Konstanten $\lambda_\varepsilon = 1 - \tan\alpha/\tan(\alpha-\varepsilon)$ für $\gamma < 0$ und $\lambda_\varepsilon = 1 - \tan\beta/\tan(\beta+\varepsilon)$ für $\gamma \geqslant 0$. Berücksichtigt man noch $|1/\sin(\pi c)| \geqslant \exp\{-\pi|c|\,\sin|\gamma|\}$, so erhält man schließlich für betragsmäßig hinreichend große $c \in W_z^{**}$ die Abschätzung

$$(45)\quad |k_3(a,b,c,z)| \geqslant$$

$$\geqslant \left|\pi\,\frac{(-c)^{a+b-1}}{\Gamma(a)\Gamma(b)}\,z(1-z)^{-a-b}\right|\,\exp\left\{-\lambda_\varepsilon|c|\,\sin\varepsilon\ \ln|1-1/z|\right\}.$$

Aus ihr ist ersichtlich, daß $|k_3|$ mit $|c| \to \infty$ exponentiell gegen Unendlich strebt und mithin die erste Summe in (37) vernachlässigbar ist. Damit ist Satz 2 bewiesen.

4.10. Für $\operatorname{Re} z > 1/2$ und $\gamma = \arg(-c) = -\alpha - \pi/2 + o(1)$ oder $\gamma = \arg(-c) = -\beta + \pi/2 + o(1)$ erhält man aus (42) bzw. (43) wegen $\tan\gamma \sim 1/\tan\alpha$ bzw. $\tan\gamma \sim 1/\tan\beta$

$$(46)\quad |z^{-c}(1-z)^{c}/\sin(\pi c)| = \exp\{o(|c|)\},$$

also in Übereinstimmung mit Fall (iii) von Satz 3 keine ausreichend genaue Aussage über die Größenordnung von k_3.

5. Bemerkungen und Ergänzungen

5.1. Ist Re $z < 1/2$ und $c = -n+\mu(n)$ mit $\mu(n) = o(1)$ für $n \to \infty$ (Fall (i) von Satz 3), so kann man sich, anknüpfend an (40), leicht überlegen, daß sich (10) auf (6) reduzieren läßt, wenn $\ln \mu(n) = o(n)$ ist.

5.2. Es ist bekannt, daß mit den Beziehungen

$$F(a,b;c;z)=(1-z)^{c-b-a}F(c-a,c-b;c;z)=(1-z)^{-a}F(a,c-b;c;z/(z-1))$$

und $F(a,b;c;z) = F(b,a;c;z)$ aus den Entwicklungen (6), (9) und (10) leicht entsprechende asymptotische Entwicklungen in den Fällen

- $|c| \to \infty$, $|b| \to \infty$,
- $|c| \to \infty$, $|a| \to \infty$,
- $|c| \to \infty$, $|a| \to \infty$, $|b| \to \infty$

hergeleitet werden können, jeweils aber unter den stark einschränkenden Bedingungen, daß die Differenzen der gegen Unendlich strebenden Parameter konstant sein müssen.

5.3. Folgt man dem Konzept von Watson [12], so können mittels der Beziehung [1]

$$(47)\qquad e^{\pm\pi i(c-b)} \frac{\Gamma(c-b)\Gamma(1-a)}{\Gamma(c-b-a+1)} z^{1-c}(1-z)^{c-b-a} \times$$

$$\times F(1-a,1-b;c-b-a+1;1-z) = \frac{\Gamma(b)\Gamma(c-b)}{\Gamma(c)} F(a,b;c;z) +$$

$$+ e^{\pm\pi i(1-b)} \frac{\Gamma(1-a)\Gamma(b)}{\Gamma(b-a+1)} (-z)^{a-c}(1-z)^{c-b-a} \times$$

$$\times F(1-a,c-a;b-a+1:1/z)$$

(die oberen Vorzeichen in den Exponenten stehen für Im $z > 0$, die unteren für Im $z < 0$) aus asymptotischen Entwicklungen für $|c| \to \infty$ und feste a, b, z solche für $|b| \to \infty$ mit festen a, c, z oder für $|a| \to \infty$ mit festen b, c, z berechnet werden.

Zu diesem Zweck ist (47) nach $F(1-a,c-a;b-a+1;1/z)$ aufzulösen und a, b, c und z durch 1-a bzw. c-a bzw. b-a+1 bzw. 1/z zu ersetzen. Wenn man anschließend noch a und b vertauscht, ergibt sich nach kurzer Rechnung

$$\frac{F(a,b;c;z)}{\Gamma(c)} = q_1\, F(b,b-c+1;a+b-c+1;1-1/z) + \tag{48}$$

$$+\; q_2\, F(1-b,c-b;a-b+1;1/z)$$

mit

$$q_1 = (-z)^{-b}\Gamma(a+1-c)/[\Gamma(c-b)\Gamma(a+b-c+1)] \tag{49}$$

und

$$q_2 = (1-z)^{c-b-a}\, z^{b-c}\, \frac{\Gamma(a-c+1)}{\Gamma(b)\Gamma(a-b+1)}\;. \tag{50}$$

Aus den Sätzen 1 bis 3 erhält man für $|a| \to \infty$

$$F(b,b-c+1;a+b-c+1;1-1/z) = \tag{51}$$

$$= \lambda_1 \left[\sum_{\nu=0}^{n} \frac{(b)_\nu (b+1-c)_\nu}{\nu!\,(a+b-c+1)_\nu} (1-1/z)^\nu + O(a^{-n-1}) \right] +$$

$$+\; \lambda_2\, k \left[\sum_{\nu=0}^{n} \frac{(1-b)_\nu (c-b)_\nu}{\nu!\,(a-b+1)_\nu} z^{-\nu} + O(a^{-n-1}) \right]$$

mit

$$k = k(a,b,c,z) = \tag{52}$$

$$= \frac{\pi}{\sin(a+b-c+1)\pi}\, \frac{(1-1/z)^{c-b-a}(1/z)^{a-b}}{\Gamma(b)\Gamma(b-c+1)}\, \frac{\Gamma(b-a)}{\Gamma(c-b-a)}$$

und Konstanten λ_1, λ_2, die in Abhängigkeit von z und arg a die Werte 0 oder 1 annehmen. Ersetzt man in (51) a, b, c und z in dieser Reihenfolge durch a-c+1, 1-b, 2-c, $(1-1/z)^{-1}$, so ergibt das

$$(53)\qquad F(1-b,c-b;a-b+1;1/z) =$$

$$= \lambda_3\left[\sum_{\nu=0}^{n}\frac{(1-b)_\nu(c-b)_\nu}{\nu!(a-b+1)_\nu}\,z^{-\nu} + O(a^{-n-1})\right] +$$

$$+ \lambda_4 k^*\left[\sum_{\nu=0}^{k}\frac{(b)_\nu(b-c+1)_\nu}{\nu!(a+b-c+1)_\nu}\,(1-1/z)^{\nu} + O(a^{-n-1})\right]$$

mit $k^* = k(a-c+1,1-b,2-c,(1-1/z)^{-1})$ und Koeffizienten λ_3, λ_4, die in Abhängigkeit von a und z die Werte 0 oder 1 annehmen. Aus (48), (51) und (53) folgt

$$(54)\qquad \frac{F(a,b;c;z)}{\Gamma(c)} =$$

$$= \left[\lambda_1 q_1 + \lambda_4 k^* q_2\right]\left[\sum_{\nu=0}^{n}\frac{(b)_\nu(b-c+1)_\nu}{\nu!(a+b-c+1)_\nu}\,(1-1/z)^{\nu} + O(a^{-n-1})\right] +$$

$$+ \left[\lambda_2 k q_1 + \lambda_3 q_2\right]\left[\sum_{\nu=0}^{n}\frac{(1-b)_\nu(c-b)_\nu}{\nu!(a-b+1)_\nu}\,z^{-\nu} + O(a^{-n-1})\right].$$

Die in q_1, q_2, k und k^* auftretenden Quotienten der Form $\Gamma(a+s)/\Gamma(a+t)$ und $\Gamma(s-a)/\Gamma(t-a)$ (s, t konstant) lassen sich nach (20) bis auf Faktoren a^{s-t} bzw. $(-a)^{s-t}$ in asymptotische Reihen nach der Skala $\left\{a^{-\nu}\right\}_{\nu\geq 0}$ entwickeln. Nach dieser Skala können auch die in (54) auftretenden Summen entwickelt werden. Berücksichtigt man noch, daß sich aus $\lambda_1 q_1 + \lambda_4 k^* q_2$ der Faktor $(-az)^{-b}/\Gamma(c-b)$ und aus $\lambda_2 k q_1 + \lambda_3 q_2$ der Faktor $(-az)^{b-c} \times (1-z)^{c-b-a}/\Gamma(b)$ herausheben läßt, so ergeben sich Entwicklungen der Gestalt

$$(55)\quad \frac{F(a,b;c;z)}{\Gamma(c)} = \frac{(-az)^{-b}}{\Gamma(c-b)}\left[\sum_{\nu=0}^{n} \frac{p_\nu(b,c,1/z)}{a^\nu} + O(a^{-n-1})\right] +$$

$$+\frac{(1-z)^{c-b-a}(-az)^{b-c}}{\Gamma(b)}\left[\sum_{\nu=0}^{n} \frac{\tilde{p}_\nu(b,c,1/z)}{a^\nu} + O(a^{-n-1})\right]$$

mit geeignet zu wählenden Werten der Potenzen von $-az$. Die Koeffizienten p_ν und $\tilde{p}_\nu$ sind auf diesem Weg zwar prinzipiell berechenbar, aber schon die Berechnung von p_2 und $\tilde{p}_2$ würde erheblichen rechnerischen Aufwand erfordern, zumal außerdem in Betracht zu ziehen ist, daß dabei wegen der Abhängigkeit von λ_1 bis λ_4 die den Fällen der Sätze 1 bis 3 entsprechenden Fallunterscheidungen hinsichtlich $\operatorname{Re}(1/z)$ und $\arg a$ zu machen sind. In [9] wird eine Entwicklung der Form (55) auf anderem Wege hergeleitet, der die Berechnung der Koeffizienten mittels einer erzeugenden Funktion oder rekursiv [11] gestattet. Es sei noch bemerkt, daß der Versuch, den umgekehrten Weg zu gehen, also aus (55) mittels geeigneter Funktionalgleichungen die Sätze 1 bis 3 zu beweisen, fehlschlägt, da sich dann in gewissen Winkelräumen der komplexen c-Ebene nur o-Beziehungen ergeben.

5.4. **Läßt man im Fall $|c| \to \infty$ zu, daß auch $|a|$ und $|b|$ unbeschränkt sein können, allerdings** unter den Bedingungen $a^2 = o(c)$ und $b^2 = o(c)$, so bilden die Reihenglieder in (6) ebenfalls eine asymptotische Skala. Es erhebt sich die Frage, ob auch dann (6) und damit (9) und (10) gelten, was in einer nachfolgenden Arbeit untersucht werden wird. Die ensprechende Fragestellung für $|a| \to \infty$ mit $b^2 = o(a)$ und $c^2 = o(a)$ wird in [10] behandelt.

LITERATUR

[1] H. Bateman, A. Erdélyi: Higher transcendental functions, Vol.1. New York-Toronto-London: Mc Graw-Hill Book Company, Inc, 1953.

[2] G. Doetsch: Handbuch der Laplace-Transformation I (Nachdruck der 1. Auflage). Basel-Stuttgart: Birkhäuser Verlag 1971.

[3] G. Doetsch: Handbuch der Laplace-Transformation II. Basel-Stuttgart: Birkhäuser Verlag 1955.

[4] A. Kratzer, W. Franz: Transzendente Funktionen, Leipzig: Akademische Verlagsgesellschaft Geest & Portig K.-G. 1960.

[5] T.M. MacRobert: The asymptotic expansions of the spherical harmonics, Proc. Edinburgh Math. Soc. 41 (1923), 82-93.

[6] T.M. MacRobert: On an asymptotic expansion of the hypergeometric function. Proc. Edinburgh Math. Soc. 42 (1923), 84-87.

[7] O. Perron: Über das Verhalten der hypergeometrischen Reihe bei unbegrenztem Wachstum eines oder mehrerer Parameter I, II. Sitzungsber. Heidelberger Akad. Wiss. 7 no. 9 (1916) und 8 no. 1 (1917).

[8] F.G. Tricomi, A, Erdélyi: The asymptotic expansion of a ratio of gamma functions,,Pacific J. Math. 1 (1951), 133-142.

[9] E. Wagner: Asymptotische Entwicklungen der hypergeometrischen Funktionen $F(a,b;c;z)$ für $|a| \to \infty$ und konstante b,c,z. Z. Anal. Anw. 3 (1984), 213-226.

[10] E. Wagner: Asymptotische Darstellungen der hypergeometrischen Funktion für große Parameter unterschiedlicher Größenordnung, Z. Anal. Anw. 5 (3) (1986), 265-276.

[11] E. Wagner: Zur Asymptotik der hypergeometrischen Funktionen für große Parameterwerte, Wiss. Z. Univ. Halle XXXIV'85 M, H.4 (1985), 38-42.

[12] G.N. Watson : Asymptotic expansions of hypergeometric functions, Trans. Cambridge Philosophical Soc. 22 (1918), 277-308.

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JOINT BROWDER SPECTRA OF D-COMMUTING SYSTEMS

1. Introduction

Let X and Y be two complex Hilbert spaces. Let L(X,Y) denote the set of all closed linear operators, defined on linear subspaces of X, with values in Y. Let C(X,Y) denote the subset of those operators from L(X,Y) which are defined everywhere, and hence continuous. For L(X,X) and C(X,X), we shall simply write L(X) and C(X), respectively. For any operator $A \in L(X,Y)$, D(A), N(A) and R(A) denote, respectively, the domain, null space, and range of A.

Let $\{X^p\}_{p\in Z}$ (where Z is the ring of integers) be a family of Hilbert spaces, and let $a^{(p)} \in L(X^p, Y^{p+1})$ be a family of operators such that $R(a^{(p)}) \subset N(a^{(p+1)})$, for all $p \in Z$. Let the following sequence represent the complex of Hilbert spaces:

$$\cdots \xrightarrow{a^{(p-1)}} X^p \xrightarrow{a^{(p)}} X^{p+1} \xrightarrow{a^{(p+1)}} \cdots .$$

Let $\{H^p(X,a)\}_{p\in Z}$ denote the cohomology of the complex $(X,a) = (X^p, a^{(p)})_{p\in Z}$, that is, $H^p(X,a) = N(a^{(p)})/R(a^{(p-1)})$.

A complex $(X,a) = (X^p, a^{(p)})_{p\in Z}$ is said to be Fredholm if $\inf\{r(a^{(p)})\} > 0$ (where $r(a^{(p)})$ is the reduced minimum mo-

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dulus of $a^{(p)}$), $\dim(H^p(X,a)) < \infty$, for each $p \in \mathbf{Z}$, and $H^p(X,a) \neq 0$ only for a finite number of indices. For detail, see [9].

Note that for a Fredholm complex (X,a), the quotient space $H^p(X,a)$ is isomorphic to the subspace $N(a^{(p)}) \ominus R(a^{(p-1)})$, for all $p \in \mathbf{Z}$, and hence $H^p(X,a)$ will carry this meaning in the sequel. We also assume without loss of generality that $D(a^{(p)})$ is dense in X^p for each $p \in \mathbf{Z}$ so that this emphasis the attached to the complex $(X,a) = (X^p, a^{(p)})_{p\in \mathbf{Z}}$.

We recall some definitions [8]. Let $s = (s_1,\dots,s_n)$ be a system of n indeterminates, and let $E(s)$ be the exterior algebra over the complex field $\mathbf{C}$, generated by the indeterminates $s_1,\dots,s_n$. Then, for any integer p, $0 \leqslant p \leqslant n$, $E^p(s)$ is the vector subspace of $E(s)$ containing all homogeneous exterior forms of degree p $(p = 0,1,\dots,n)$ in $s_1,\dots,s_n$. For any Hilbert space X, $E(X,s)$ (resp. $E^p(X,s)$) shall denote the tensor product $X \otimes E(X,s)$ (resp. $X \otimes E^p(x)$). An element $\xi \in E^p(X,s)$ will be written as

$$\xi = \sum_{1 \leqslant j_1 \leqslant \dots \leqslant j_p \leqslant n} x_{j_1 \dots j_p} ;\, s_{j_1} \wedge \dots \wedge s_{j_p},$$

where $x_{j_1\dots j_p} \in X$, and $x_{j_1\dots j_p};\, s_{j_1} \wedge \dots \wedge s_{j_p}$ stands for $x_{j_1\dots j_p} \otimes s_{j_1} \wedge \dots \wedge s_{j_p}$. The space $E^p(X,s)$ is identified with a direct sum of $\binom{n}{p}$ copies of X, therefore, $E^p(X,s)$ can be endowed with a natural structure of Hilbert space. For two Hilbert spaces X and Y, there is a natural identification of the space $E(X,s) \otimes E(Y,t)$ with space $E(X \bar{\otimes} Y, (s,t))$, where $t = (t_1,\dots,t_n)$ is another system of indeterminates.

Let $H_j : E(s) \to E(s)$ be defined by $H_j \eta = s_j \wedge \eta$, $\eta \in E(S)$. It is obvious that

$$H_j H_k + H_k H_j = 0, \quad \text{for} \quad j,k=1,\dots,n.$$

Definition 1.1. A n-tuple $a=(a_1,\dots,a_n)\in L^n(X)$ is said to be a D-commuting system if there exists a dense subspace D of X in $\bigcap_1^n D(a_j)$ with the following properties [4]:

(i) The restriction $\hat{\delta}_a = (a_1 \otimes H_1 + \dots + a_n \otimes H_n)|E(D,s)$ is closable.

(ii) If δ_a is the canonical closure of $\hat{\delta}_a$, then $R(\delta_a) \subset N(\delta_a)$.

Definition 1.2. A D-commuting system $a=(a_1,\dots,a_n)\in L^n(X)$ is said to be singular (resp. non-singular) if $R(\delta_a) \neq N(\delta_a)$ (resp. $R(\delta_a) = N(\delta_a)$).

Remark 1.3. Note that to each D-commuting system we can associate a complex of Hilbert spaces $\left(E^p(X,s),\ \delta_a^{(p)}\right)_{p=0}^n$, where $\delta_a^{(p)} = \delta_a|E^p(X,s) \cap D(\delta_a)$.

Definition 1.4. Let $a=(a_1,\dots,a_n)\in L^n(X)$ be a D-commuting system associated with a complex of Hilbert spaces $\left(E^p(X,s),\ \delta_a^{(p)}\right)_{p=0}^n$. Then $a=(a_1,\dots,a_n)$ is said to be Fredholm if the corresponding complex is Fredholm.

Definition 1.5. The Taylor spectrum of a D-commuting system $a=(a_1,\dots,a_n)\in L^n(X)$ is the set of those points $\lambda \in \mathbf{C}^n$ such that $a-\lambda I$ is singular, denoted $\sigma^T(a,X)$.

Definition 1.6. The Fredholm spectrum of a D-commuting system $a=(a_1,\dots,a_n)\in L^n(X)$ is the set of those points $\lambda \in \mathbf{C}^n$ such that $a-\lambda I$ is not Fredholm, denoted $\sigma_e^T(a,X)$.

Definition 1.7. The Browder spectrum of a D-commuting system $a=(a_1,\dots,a_n)\in L^n(X)$ is defined by the relation

$$\sigma_B^i(a) = \sigma_e^i(a) \cup \left\{\text{accumulation points of } \sigma^i(a)\right\},$$

for i = 1,D,P,T, where 1,D,P, and T, respectively, denote the commutant spectrum, Dash spectrum, polynomial spectrum, and Taylor spectrum. For detail, see [1] and [2].

Note that it is well-known that

$$\sigma^T(a) \subseteq \sigma^1(a) \subseteq \sigma^D(a) \subseteq \sigma^P(a), \tag{1.1}$$

$$\sigma_e^T(a) \subseteq \sigma_e^1(a) \subseteq \sigma_e^D(a) \subseteq \sigma_e^P(a), \text{ and} \tag{1.2}$$

$$\sigma_B^T(a) \subseteq \sigma_B^1(a) \subseteq \sigma_B^D(a) \subseteq \sigma_B^P(a). \tag{1.3}$$

The aim of this paper is to give the Taylor joint Browder spectrum of a tensor product of D-commuting systems. The generalizations to the commutant and Dash spectra follow as the consequences of the main results. Furthermore, we generalize the spectral mapping theorem to the case of the Taylor joint Browder spectrum on tensor products. Unlike the present case of the closed linear operator systems, the situation is quite different in the case of the bounded linear operator systems. For detail, see [7], [8] and [1].

2. Some auxiliary results

We now state and prove (in certain cases) some results.

Lemma 2.1 (Grosu and Vasilescu [4]). Let $a = (a_1,\dots,a_n) \in L^n(X)$ be a D_a-commuting system, let $b = (b_1,\dots,b_m) \in L^m(Y)$ be a D_b-commuting system, and let $a\otimes b = (a_1\otimes I,\dots,a_n\otimes I, I\otimes b_1,\dots,I\otimes b_m) \in L^{n+m}(X\bar{\otimes}Y)$ be a $D_a\otimes D_b$-commuting system. Then

$$\sigma^T(a\otimes b, X\bar{\otimes}Y) = \sigma^T(a,X) \times \sigma^T(b,Y). \tag{2.1}$$

Lemma 2.2 (Verma [12]). Let (X,a) and (Y,b) be two complexes of Hilbert spaces. The complexes (X,a) and (Y,b) are not Fredholm and exact iff $(X\bar{\otimes}Y, a\otimes b)$ is not Fredholm and exact.

Lemma 2.3 (Verma [12]). Let $a = (a_1,\dots,a_n) \in L^n(X)$ be a D_a-commuting system, let $b = (b_1,\dots,b_m) \in L^m(Y)$ be a D_b-commuting system, and let $a\otimes b = (a_1\otimes I,\dots,a_n\otimes I, I\otimes b_1,\dots,I\otimes b_m) \in L^{n+m}(X\bar{\otimes}Y)$ be $D_a\otimes D_b$-commuting system. Then

$$\sigma_e^T(a\otimes b,\ X\bar{\otimes}Y) = \sigma_e^T(a,X)\times\sigma_e^T(b,Y). \tag{2.2}$$

Proof. We give a sketch of the proof for the sake of the completeness. In order to prove the inclusion

$$\sigma_e^T(a\otimes b,\ X\bar{\otimes}Y) \subseteq \sigma_e^T(a,X)\times\sigma_e^T(b,Y),$$

let us consider $z = (\lambda,\mu)\in \mathbb{C}^{n+m}$, and suppose that $(\lambda,\mu)\in$ $\in\sigma_e^T(a\otimes b,\ X\bar{\otimes}Y)$. We may assume without loss of generality that $\lambda = 0$, $\mu = 0$. Then $(0,0)\in\sigma_e^T(a\otimes b,\ X\bar{\otimes}Y)$ implies that $(0,0)\in\sigma^T(a\otimes b,\ X\bar{\otimes}Y)$. Therefore, a and b are not Fredholm and non-singular (by Lemma 2.2).

Conversely, let us assume that $(0,0)\in\sigma_e^T(a,X)\times\sigma_e^T(b,Y)$. $0\in\sigma_e^T(a,X)$ implies that $0\in\sigma^T(a,X)$. Therefore, a is not Fredholm and non-singular. And, $0\in\sigma_e^T(b,Y)$ implies that $0\in\sigma^T(b,Y)$. Therefore, b is not Fredholm and non-singular. Consequently, $a\otimes b$ is not Fredholm and non-singular (by Lemma 2.2), and $(0,0)\in\sigma_e^T(a\otimes b,\ X\bar{\otimes}Y)$.

Lemma 2.4 (Snow [8]). Let f be a function which is analytic in an open neighborhood of $\sigma(a_1)\times\sigma(a_2)$. Then

$$\sigma_B^T\left[f(a_1,a_2)\right] = f\left[\sigma_B^T(a_1,a_2)\right].$$

3. Main results

We are about to prove the main results.

Theorem 3.1. Let $a = (a_1,\dots,a_n)\in L^n(X)$ be a D_a-commuting system, let $b = (b_1,\dots,b_m)\in L^m(Y)$ be a D_b-commuting system, and let $a\otimes b = (a_1\otimes I,\dots,a_n\otimes I, I\otimes b_1,\dots,I\otimes b_m)\in L^{n+m}(X\bar{\otimes}Y)$ be a $D_a\otimes D_b$-commuting system. Then the Taylor joint Browder spectrum

$$\sigma_B^T(a\otimes b,\ X\bar{\otimes}Y) = \sigma_B^T(a,X)\times\sigma_B^T(b,Y). \tag{3.1}$$

Corollary 3.2. For $a\otimes b = (a_1\otimes I,\ I\otimes b_1) \in L^2(X\bar{\otimes}Y)$, we find the result

$$(3.2)\qquad \sigma_B^T(a\otimes b,\ X\bar{\otimes}Y) = \sigma_B(a_1,X)\times\sigma_B(b_1,Y).$$

Proof of Theorem 3.1. To prove the inclusion

$$(3.4)\qquad \left[\sigma_B^T(a,X)\times\sigma_B^T(b,Y)\right]\subseteq\sigma_B^T(a\otimes b,\ X\bar{\otimes}Y),$$

we assume $z=(\lambda,\mu)\in\left[\sigma_B^T(a,X)\times\sigma_B^T(b,Y)\right]$. If $\lambda\in\sigma_e^T(a,X)$ and $\mu\in\sigma_e^T(b,Y)$, then, by Lemma 2.3, it follows that $(\lambda,\mu)\in\sigma_e^T(a\otimes b,\ X\bar{\otimes}Y)\subseteq\sigma_B^T(a\otimes b,\ X\bar{\otimes}Y)$. Furthermore, if $\lambda\in\sigma^T(a,X)\setminus\sigma_e^T(a,X)$, then λ is not an isolated point and hence (λ,μ) is not an isolated point of $\sigma^T(a\otimes b,\ X\bar{\otimes}Y)$ by Lemma 2.1. This implies that $(\lambda,\mu)\in\sigma_B^T(a\otimes b,\ X\bar{\otimes}Y)$. The something happens when $\mu\in\sigma^T(b,Y)\setminus\sigma_e^T(b,Y)$. Thus, we have $\left[\sigma_B^T(a,X)\times\sigma_B^T(b,Y)\right]\subseteq\sigma_B^T(a\otimes b,\ X\bar{\otimes}Y)$. Conversely, suppose $(\lambda,\mu)\in\sigma_B^T(a\otimes b,\ X\bar{\otimes}Y)\supseteq\sigma_e^T(a\otimes b,\ X\bar{\otimes}Y)$. If $(\lambda,\mu)\in\sigma_e^T(a\otimes b, X\bar{\otimes}Y)$, then $(\lambda,\mu)\in\sigma_e^T(a,X)\times\sigma_e^T(b,Y)\subseteq\sigma_B^T(a,X)\times\sigma_B^T(b,Y)$, by Lemma 2.3. And, if $(\lambda,\mu)\in\sigma^T(a\otimes b,\ X\bar{\otimes}Y)\setminus\sigma_e^T(a\otimes b,\ X\bar{\otimes}Y)$, then (λ,μ) is not an isolated point of $\sigma^T(a\otimes b,\ X\bar{\otimes}Y)$. This, in turn, implies, by Lemma 2.1, that either λ or μ is not an isolated point. This further implies that $(\lambda,\mu)\in\left[\sigma_B^T(a,X)\times\sigma_B^T(b,Y)\right]$, and hence

$$(3.5)\qquad \sigma_B^T(a\otimes b,\ X\bar{\otimes}Y)\subseteq\left[\sigma_B^T(a,X)\times\sigma_B^T(b,Y)\right].$$

This proves the theorem.

Note that the analogs of (3.1) to the cases of the commutant and Dash spectra of n-tuples of operator systems are still open. However, at this stage, we prove the analogs of (3.1) when

$$a\otimes b = (a_1\otimes I,\ I\otimes b_1)\in L^2(X\bar{\otimes}Y).$$

Theorem 3.4. Let $a\otimes b = (a_1\otimes I,\ I\otimes b_1)\in L^2(X\bar{\otimes} Y)$. Then

$$\sigma_B^{j}(a\otimes b,\ X\bar{\otimes}Y) = \sigma_B(a_1,X)\times\sigma_B(b_1,Y), \tag{3.6}$$

for j = 1,D,T, where 1,D and T denote, respectively, the commutant Dash and Taylor spectra.

Proof. The proof follows from a combination of the inclusion relations (3.2) and (1.3):

$$\left[\sigma_B(a_1,X)\times\sigma_B(b_1,Y)\right]\subseteq\sigma_B^{T}(a\otimes b,\ X\bar{\otimes}Y)\subseteq\sigma_B^{1}(a\otimes b,\ X\bar{\otimes}Y)\subseteq$$

$$\subseteq\sigma_B^{D}(a\otimes b,\ X\bar{\otimes}Y) = \left[\sigma_B(a_1,X)\times\sigma_B(b_1,Y)\right].$$

Remark 3.5. The question whether result (3.2) holds for the case of the polynomial spectrum is still open.

Theorem 3.6. Let $a\otimes b = (a_1\otimes I,\ I\otimes a_2)\in L^2(X\bar{\otimes}Y)$ be a $D_a\otimes D_b$-commuting system. Let f be a function which is analytic in an open neighborhood of $\sigma(a_1)\times\sigma(a_2)$. Then

$$\sigma_B^{T}\left[f(a_1\otimes I,\ I\otimes a_2)\right] = f\left[\sigma_B(a_1),\ \sigma_B(a_2)\right]. \tag{3.7}$$

Proof. If we apply Lemma 2.4 to the left hand side of (3.7), we get

$$\sigma_B^{T}\left[f(a_1\otimes I,\ I\otimes a_2)\right] = f\left[\sigma_B^{T}(a_1\otimes I,\ I\otimes a_2)\right],$$

which, in turn, by the application of Theorem 3.1, proves the theorem.

Remark 3.7. The extension of the result (3.7) to n-variables is obvious.

Remark 3.8. The analogs of the spectral mapping Theorem 3.6 hold for the cases of the commutant and dash spectra.

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REFERENCES

[1] A.T. Dash : Joint Browder spectra and tensor products, preprint.

[2] A.T. Dash : Joint spectra, Studia Math. 45 (1973), 225-237.

[3] N. Dunford, J. Schwartz : Linear operators II, Interscience Publ. 1963; III, Wiley-Interscience, 1971.

[4] C. Grosu, F.-H. Vasilescu : The Kunneth formula for Hilbert complexes, Integral Equations and Operator Theory 5 (1982), 1-17.

[5] T. Kato : Perturbation theory for linear operators, 2nd ed., Springer-Verlag, New York, 1976.

[6] S. MacLane : Homology, Springer-Verlag, New-York, 1970.

[7] M. Schechter, M. Snow : The Fredholm spectrum on tensor products, Proc. Roy. Irish Acad. Sect. A 75(1975), 121-127.

[8] M. Snow : A joint Browder essential spectrum, Proc. Roy. Irish Acad. Sect. A 75 (1975), 129-131.

[9] J.L. Taylor : A joint spectrum for several commuting operators, J. Funct. Anal. 6 (1970), 172-191.

[10] F.-H. Vasilescu : The stability of Euler characteristic for Hilbert complexes, Math. Ann. 248 (1980), 109-116.

[11] R.U. Verma : The essential spectra od D-commuting systems, Demonstratio Math. 18 (1985), 26-29.

[12] R.U. Verma : A remark on the Fredholm spectrum of D-commuting systems, Demonstratio Math. 18 (1985), 1163-1173.

[13] R.U. Verma : Identity of Harte, Taylor and Commutant spectra, Anal. St. Univ. Iasi, to appear.

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THE SEMIGROUP OF ENDOMORPHISMS OF THE FREE GROUP OF INFINITE RANK

1. Introduction

It is known [9] that an endomorphism of a free group F_n of a finite rank which is invertible from one side must be an automorphism. We consider the situation in the semigroup of endomorphisms of the free group F of countably infinite rank. Two endomorphisms $\underline{u}$ and $\underline{v}$ are called equivalent if there exists an automorphism $\underline{a}$ such that $\underline{u} = \underline{a} \circ \underline{v}$. The duality of automorphisms and the infinite Nielsen transformations allows us to use Theorem 6 from [7] to state that every endomorphism is equivalent to the so-called free endomorphism which maps a part of a base into 1 and another part into a Nielsen-reduced set. Applying this fact we generalize the proofs of several known theorems from F_n to F. We show also that every endomorphism of F is a product of a monomorphism and an epimorphism. The same is proved for the free abelian group F/F′ which implies that every infinite matrix over Z with a finite number of non-zero components in every row is a product of a matrix with lineary independent rows and a left invertible matrix. We will be dealing with subsemigroups of endomorphisms of F which are listed below:

End - semigroup of all endomorphisms,

End_∞ - subset of endomorphisms with the infinitely generated image,

Epi - semigroup of epimorphisms,
Mono - semigroup of monomorphisms,
Sur - semigroup of surjective endomorphisms,
Inj - semigroup of injective endomorphisms,
M - semigroup of endomorphisms which map a base into a set independent mod F′,
R - semigroup of endomorphisms invertible from the right side,
L - semigroup of endomorphisms invertible from the left side,
Aut - group of invertible endomorphisms that is automorphisms.

2. Definitions and preliminaries

Following [9] we consider a fixed relatively free group G = F/V of countably infinite rank. As the basis of G a set of free generators $x_1, x_2, \dots$ is chosen and will remain fixed. Its characteristic property is that every mapping of these generators into the group can be extended to an endomorphism of the group. We shall use a presentation of an endomorphism $\underline{u}$ by means of its components, namely, the images of the free generators x_i. If $\underline{u}$ maps x_i into u_i, $i=1,2,\dots$, we interpret the ordered set of elements u_i as an infinite vector $\underline{u} = (u_1, u_2, \dots)$. The word infinite will be omitted. If now $\underline{u}$ is arbitrarily chosen, the corresponding endomorphism $\underline{u}$ is defined to map the element $g(x_1,\dots,x_n)$ into $g(x_1,\dots,x_n)\underline{u} = g(x_1\underline{u},\dots,x_n\underline{u}) = g(u_1,\dots,u_n)$ or briefly, $g(\underline{x})\underline{u} = g(\underline{u})$, where $\underline{x} = (x_1, x_2, \dots)$. The set of all endomorphisms $\underline{u}$ of G is represented by the set of all vectors with components $u_i = x_i\underline{u}$. Two vectors represent the same endomorphism if and only if the corresponding components interpreted as elements of the absolutely free group F on the same generators x_i differ by elements of V.

The product $\underline{u} \circ \underline{v}$ of endomorphisms is defined by $x_i(\underline{u} \circ \underline{v}) = (x_i\underline{u})\underline{v} = u_i(\underline{v})$ and is represented by the vector $\underline{u} \circ \underline{v} = (u_1(\underline{v}), u_2(\underline{v}), \dots)$. The vector $\underline{x}$ represents the identity endomorphism. We shall use the same small bold-faced letter to signify either an endomorphism or a corresponding vector with components from G.

2.1. Definition. ([11], 2.2.1). An endomorphism $\underline{u}$ is called a monomorphism (epimorphism) if $\underline{v} \circ \underline{u} = \underline{w} \circ \underline{u}$ implies $v = w$ ($\underline{u} \circ \underline{v} = \underline{u} \circ \underline{w}$ implies $\underline{v} = \underline{w}$).

2.2. Definition. An endomorphism $\underline{u}$ is called invertible from the right side if there exists $\underline{v}$ such that $\underline{u} \circ \underline{v} = \underline{x}$. Respectively, $\underline{v}$ is called invertible from the left side. An endomorphism invertible from both sides is called invertible or an automorphism.

In the category of groups the notions of monomorphism and injective morphism (epimorphism and surjective morphism) coincide ([11], 2.2.4, 2.2.12) but, since we restricted ourselves to the subcategory with one object G and the semigroup of morphisms End G, the situation is not obvious.

2.3. The notions of monomorphism and injective endomorphism coincide. Indeed, $\underline{v} \circ \underline{u} = \underline{w} \circ \underline{u}$ implies $\underline{v} = \underline{w}$ if and only if $v_i\underline{u} = w_i\underline{u}$ implies $v_i = w_i$ for all i which gives the required result.

2.4. There exists an endomorphism $\underline{u}$ of the free group F which is a monomorphism and an epimorphism, but is not surjective.

Proof. We take $\underline{u} = (x_1^2, x_2^2, \ldots)$. If now $\underline{u} \circ \underline{v} = \underline{u} \circ \underline{w}$, then $(v_1^2, v_2^2, \ldots) = (w_1^2, w_2^2, \ldots)$ and, by [4], $v_i^2 = w_i^2$ implies $v_i = w_i$ and hence $\underline{v} = \underline{w}$ which shows that $\underline{u}$ is an epimorphism. Since $\underline{u}$ is injective, by 2.3, $\underline{u}$ is a monomorphism. The endomorphism $\underline{u}$ is not surjective, since $gp(\underline{u}) \neq F$.

2.5. $R \subseteq$ Mono, $L =$ Sur $\subseteq$ Epi for a relatively free group G.

Proof. Let $\underline{u} \circ \underline{v} = \underline{x}$, that is $\underline{u} \in R$, $\underline{v} \in L$. If $\underline{w}_1 \circ \underline{u} = \underline{w}_2 \circ \underline{u}$, then multiplied by $\underline{v}$ we have $\underline{w}_1 = \underline{w}_2$ and hence $R \subseteq$ Mono. Similarily $L \subseteq$ Epi. Now $G = gp(\underline{x}) = gp(\underline{u} \circ \underline{v}) \subseteq gp(\underline{v})$ implies $L \subseteq$ Sur. If $\underline{v} \in$ Sur, then we denote by u_i the contra-image of x_i under $\underline{v}$, that is $u_i\underline{v} = x_i$. For $\underline{u} = (u_1, u_2, \ldots)$ we get $\underline{u} \circ \underline{v} = \underline{x}$ and hence $\underline{v} \in L$ which completes the proof.

In the case of F both inclusions in 2.5 are proper, because of 2.4.

2.6. $\mathrm{Aut} = R \cap L = \mathrm{Mono} \cap L = R \cap \mathrm{Epi} \subseteq \mathrm{Mono} \cap \mathrm{Epi}$.

Proof. All that is required is to ensure that $\underline{u} \in \mathrm{Mono} \cap L$ implies $\underline{u} \in R$ and $\underline{u} \in R \cap \mathrm{Epi}$ implies $\underline{u} \in L$. In both cases there exists $\underline{v}$ such that $\underline{u} \circ \underline{v} \circ \underline{u} = \underline{u}$. The alternative cancellings give the results.

2.7. Definition ([5], [8]). An endomorphism $\underline{r}$ of G onto a subgroup $H = gp(\underline{r})$ is called a retraction and H is called a retract if and only if $\underline{r}^2 = \underline{r}$. The notion of the retract in [11] has a different meaning.

2.8. A monomorphism $\underline{u}$ is invertible from the right side if and only if $gp(u)$ is a retract.

Proof. If $\underline{u} \in R$, then there exists $\underline{v}$ such that $\underline{u} \circ \underline{v} = \underline{x}$. We write $\underline{r} = \underline{v} \circ \underline{u}$, then $gp(\underline{u}) = gp(\underline{u} \circ \underline{v} \circ \underline{u}) = gp(\underline{u} \circ \underline{r}) \subseteq gp(\underline{r}) = gp(\underline{v} \circ \underline{u}) \subseteq gp(\underline{u})$ which gives $gp(\underline{r}) = gp(\underline{u})$. Moreover, $\underline{r}^2 = \underline{v} \circ \underline{u} \circ \underline{v} \circ \underline{u} = \underline{r}$, and hence $gp(\underline{u})$ is a retract. Conversely, if $\underline{u} \in \mathrm{Mono}$ and $gp(\underline{u})$ is a retract, then there exists a retraction $\underline{r}$ (obviously $\underline{r} \neq \underline{u}$ if $\underline{u} \neq \underline{x}$), such that $gp(\underline{u}) = gp(\underline{r})$. Since every u_i can be written as a word in generators from $\underline{r}$ and every r_i can be written in generators from $\underline{u}$, we have $\underline{r} = \underline{v} \circ \underline{u}$, $\underline{u} = \underline{w} \circ \underline{r} = \underline{w} \circ (\underline{v} \circ \underline{u}) = (\underline{w} \circ \underline{v}) \circ \underline{u}$. Because $\underline{u}$ is a monomorphism we conclude that $\underline{w} \circ \underline{v} = \underline{x}$. The property $\underline{r}^2 = \underline{r}$ gives $\underline{v} \circ \underline{u} \circ \underline{v} \circ \underline{u} = \underline{v} \circ \underline{u}$. Multiplying it by $\underline{w}$ from the left side and cancelling $\underline{u}$ from the right side we get $\underline{u} \circ \underline{v} = x$, which means that $\underline{u}$ is invertible from the right side, which completes the proof.

A group G is not a Hopf group if it is isomorphic to a proper quotient group. The natural homomorphism onto this quotient group followed by the isomorphism of the quotient group onto G constitutes a surjective endomorphism $\underline{u}$ which is not an automorphism. Conversely, if $\underline{u} \in \mathrm{Sur} \setminus \mathrm{Aut}$, then $G \cong G/\mathrm{Ker}\, \underline{u}$ and G is not a Hopf group. Baer´s theorem [1] says that, if a relatively free group G is isomorphic to a proper quotient group, then it is also isomorphic to a proper subgroup. We can extend this theorem a little further.

2.9. **Theorem.** If a relatively free group G is isomorphic to a proper quotient group, then it is also isomorphic to a proper retract.

Proof. Since G is not a Hopf group, there exists a surjective endomorphism $\underline{v}$ which is not an automorphism. By 2.5, $\underline{v} \in L$ that is $\underline{u} \circ \underline{v} = \underline{x}$ for an endomorphism $\underline{u}$ such that $\underline{u} \in R \subseteq \text{Mono}$. The $\underline{u}$ is not a surjection, because otherwise, by 2.6, $\underline{u}$ and hence $\underline{v}$ would be automorphisms which is not true. Now, by 2.8, gp($\underline{u}$) is a proper retract of G freely generated by the set $\underline{u}$. Since the endomorphism $\underline{v}$ maps the base of gp($\underline{u}$) onto the base $\underline{x}$ of G, so that $u_i\underline{v} = x_i$, we conclude that gp($\underline{u}$) $\cong$ G, which completes the proof.

2.10. **Definition.** A vector $\underline{u}$ is called free if the set of its non-unit components generates gp($\underline{u}$) freely. By a unit in a relatively free group we mean a word which, interpreted as an element of F, is congruent to 1 modulo V.

The free vector may contain finite or infinite sets of units and non-unit components. Multiplying the free vector by a proper permutation from the left side we can get a so-called standard vector of one of the types given below:

1. $\underline{u} = (1,\dots,1,u_{k+1},u_{k+2},\dots)$, $k \geq 0$,
2. $\underline{u} = (u_1,1,u_3,1,\dots)$,
3. $\underline{u} = (u_1,\dots,u_k,1,1,1,\dots)$, $k \geq 0$,

where $u_i \neq 1$. We say that $\underline{u}$ and $\underline{v}$ are of the same type if the corresponding standard vectors are of the same form for the same k.

2.11. The set of free vectors without units represents the semigroup Mono.

Proof. Let $\underline{u}$ be a free vector without units and $\underline{v} \circ \underline{u} = \underline{w} \circ \underline{u}$, then $v_i\underline{u} = w_i\underline{u}$ and hence $v_iw_i^{-1}\underline{u} = 1$ for all i. This means that $v_iw_i^{-1}$ interpreted as element of F is congruent to 1 mod V. Thus, $\underline{v}$ and $\underline{w}$ represent the same endomorphism of G, and $\underline{u} \in$ Mono. Conversely, if $\underline{u} \in$ Mono, then, by 2.3, $\underline{u}$ is an injection and hence $\underline{u}$ is free without units. The proof is complete.

2.12. The subsets R, L, $R \setminus Aut$, $L \setminus Aut$, $End \setminus (R \cup L)$, $End \setminus R$, $End \setminus L$ form the subsemigroups in End. ([6], VI, 1.8).

2.13. For our future needs we introduce also a subsemigroup M of the vectors, where every finite subset of components is independent modulo F'. Obviously $M \in$ Mono. The natural homomorphism $\alpha: F \to F/F'$ induces an isomorphism of M onto the semigroup of monomorphisms of F/F'. We note also that $Aut \subsetneq M$, because every automorphism of F induces an automorphism of F/F'.

3. Equivalence in the semigroup of endomorphisms of F

In this section we consider endomorphisms of the free group F and use the word vector instead of the word endomorphism.

3.1. Definition. Vectors $\underline{u}$ and $\underline{v}$ are called equivalent if there exists an invertible vector $\underline{a}$ such that $\underline{u} = \underline{a} \circ \underline{v}$.

By 2.10, every free vector is equivalent to a standard vector.

3.2. Equivalent vectors generate the same subgroup.

Proof. If $\underline{u} = \underline{a} \circ \underline{v}$, then $gp(\underline{u}) = gp(\underline{a} \circ \underline{v}) \subseteq gp(\underline{v}) = gp(\underline{a}^{-1} \circ \underline{u}) \subseteq gp(\underline{u})$, which gives $gp(\underline{u}) = gp(\underline{v})$. The converse is not true, since the vectors $\underline{x}$ and $\underline{u} = (1, x_1, x_2, \dots)$ generate F, but are not equivalent.

3.3. The subsemigroups from 2.12, 2.13 are closed under equivalence.

3.4. Theorem. Every vector is equivalent to a standard vector with the Nielsen-reduced set of non-unit components.

Proof. By [7], for every vector $\underline{u}$ with components in F there exists an invertible vector $\underline{a}$ such that the set of non-unit components in $\underline{a} \circ \underline{u}$ is Nielsen-reduced and hence $\underline{a} \circ \underline{u}$ is a free vector equivalent (by 2.10) to a standard vector, thus completing the proof.

Note that the Nielsen-Schreier subgroup theorem for F follows from 3.4 and 3.2.

3.5. Theorem. Every subgroup of F is a free group.

Proof. Let $H \subseteq F$ and $\underline{u}$ be any set of generators in H. Then, by 3.4, $\underline{u}$ is equivalent to a free vector $\underline{v}$, which generates the free subgroup $gp(\underline{v})$. By 3.2, $H = gp(\underline{u}) = gp(\underline{v})$ is free, which completes the proof.

Using Theorem 3.4 we can generalise Proposition 2.12 [5] and Theorem 3.3 [8] from F_n to F. A complicated proof of the following theorem is given in [3], Theorem 6.4.

3.6. Theorem. Let $\underline{u}$ be a homomorphism from F onto a free group H. Then F has a basis $Z = Z_1 \cup Z_2$ such that $\underline{u}$ maps $gp(Z_1)$ isomorphically onto H and maps $gp(Z_2)$ into 1.

Proof. Since the extention F of a group by the free group H splits, we can treat H as a subgroup of F and $\underline{u}$ as the endomorphism of F onto $H = gp(\underline{u})$. By 3.4, there exists an invertible vector $\underline{a}$ such that $\underline{v} = \underline{a} \circ \underline{u}$ is a standard vector. We consider the case of $\underline{v} = (v_1, 1, v_3, 1, v_5, 1 \ldots)$. For other forms of $\underline{v}$ the proof is similar. Now, by 3.2, $H = gp(\underline{u}) = gp(\underline{v})$ is freely generated by the set $\langle v_1, v_3, v_5, \ldots \rangle$. Since $\underline{a}$ corresponds to an automorphism, the set of its components $\langle a_1, a_2, a_3, \ldots \rangle =: Z$ is a free base in F such that under $\underline{u}$ its elements with the odd indices are mapped onto the free generators $v_1, v_3, v_5, \ldots$ of H and the elements with the even indices into 1. The above suggests taking $Z_1 = \langle a_1, a_3, a_5, \ldots \rangle$, $Z_2 = \langle a_2, a_4, a_6, \ldots \rangle$. Just as in the case of F_n [8] it follows that Ker $\underline{u}$ coincides with the normal closure $\overline{gp(Z_2)}$ of Z_2 in F. The proof is complete.

3.7. Two free vectors are equivalent if and only if they are of the same type and generate the same subgroup.

Proof. It follows from the previous proof that the orders of the sets Z_1 and Z_2 are in the one-to-one correspondence with the type of the standard vector $\underline{v}$ to which $\underline{u}$ is equivalent. This shows that if $\underline{u}$ and $\underline{w}$ are equivalent free vectors, then they are equivalent to the same standard vector and hence have the same type. By 3.2, they generate the same subgroup. If now $gp(\underline{u}) = gp(\underline{w})$, where $\underline{u}$ and $\underline{w}$ are standard vectors of the same type, then we shall show their equivalence. Every word u_i can be expressed through the non-unit elements of $\underline{w}$ and every w_i can be expressed through the non-unit elements of $\underline{u}$, giving $\underline{u} = \underline{a} \circ \underline{w}$ and $\underline{w} = \underline{b} \circ \underline{u}$, where we take $a_i = b_i = x_i$ if $u_i = w_i = 1$. We have now $\underline{u} = (\underline{a} \circ \underline{b}) \circ \underline{u}$, $\underline{w} = (\underline{b} \circ \underline{a}) \circ \underline{w}$. Since the non-unit elements of $\underline{u}$ and $\underline{w}$ build the sets of free generators in $gp(\underline{u})$, $gp(\underline{w})$, we conclude that $(\underline{a} \circ \underline{b})_i = x_i = (\underline{b} \circ \underline{a})_i$ if $u_i, w_i \neq 1$ and, if $u_i = w_i = 1$, then we also have $(\underline{a} \circ \underline{b})_i = a_i(\underline{b}) = x_i(\underline{b}) = b_i = x_i$. Similarly, $(\underline{b} \circ \underline{a})_i = x_i$. This shows that $\underline{a}$ is invertible and $\underline{u}$, $\underline{w}$ are equivalent, which completes the proof.

4. Relations for subsemigroups in End F

We introduce here the simplest vectors which are surjections but not the automorphisms.

(1) $\underline{p} = (1, x_1, x_2, x_3, \ldots)$,

(2) $\underline{p}^\infty = (x_1, 1, x_2, 1, x_3, 1, \ldots)$.

The k-th power of $\underline{p}$ contains k units $\underline{p}^k = (1, 1, \ldots, 1, x_1, x_2, x_3, \ldots)$ and $\underline{p}^0 = \underline{x}$. We shall show that every vector which is invertible from the left side is equivalent to one of $\underline{p}^k$, $0 \leq k \leq \infty$.

4.1. Every standard vector of the form (1) or (2) can be written as $\underline{u} = \underline{p}^k \circ \underline{u}'$, where $\underline{u}'$ is a free vector containing non-unit components of $\underline{u}$ in the same order. This gives immediately for the set of End_∞ of endomorphisms with infinite images the following result.

4.2. Theorem. $End_\infty = Sur \circ Inj$.

Proof. If $\underline{v} \in End_\infty$, then, by 3.4 and 4.1, $\underline{v} = \underline{a} \circ \underline{u} = \underline{a} \circ (\underline{p}^k \circ \underline{u}') = (\underline{a} \circ \underline{p}^k) \circ \underline{u}' \in Sur \circ Inj$. Conversely, if $\underline{w} = \underline{u} \circ \underline{v}$ where $\underline{u} \in Sur = L$, $\underline{v} \in Inj = Mono$, then there exists a vector $\underline{s}$ which is a left inverse for $\underline{u}$, which gives $\underline{v} = \underline{s} \circ \underline{w}$. This implies $gp(\underline{w}) = gp(\underline{u} \circ \underline{v}) \subseteq gp(\underline{v}) = gp(\underline{s} \circ \underline{w}) \subseteq gp(\underline{w})$, that is, $gp(\underline{w}) = gp(\underline{v})$ is infinitely generated and hence $\underline{w} \in End_\infty$.

4.3. Theorem. $L = \bigcup_k Aut \circ p^k$, $0 \leq k \leq \infty$, where the union is disjoint.

Proof. By 3.3, L is a disjoint union of classes of equivalent vectors. By 3.4, every class contains a standard vector with a Nielsen-reduced set of non-unit components which generate F, since by 2.6, elements of L are surjections. By ([8], p.122) the Nielsen-reduced set of generators for F consists of $x_i^{\delta_i}$, $\delta_i = \pm 1$, $i = 1, 2, \ldots$. This implies that every class contains a vector $\underline{p}^k$, $0 \leqslant k \leqslant \infty$. Since, by 3.7, $\underline{p}^k$ and $\underline{p}^i$ are not equivalent for $k \neq i$, the result follows.

4.4. The set of left inverse vectors for $\underline{p}^k$ consists of the vectors of the form $\underline{s}(k) = (x_{k+1}c_1, x_{k+2}c_2, \ldots)$, where $c_i \in \overline{gp(x_1, x_2, \ldots, x_k)} = \operatorname{Ker} \underline{p}^k$ for $k < \infty$, and $\underline{s}(\infty) = (x_1c_1, x_3c_3, \ldots)$, where $c_i \in \overline{gp(x_2, x_4, \ldots)}$.

Proof. We consider the case $k < \infty$, then $F = N \operatorname{Ker} \underline{p}^k$, where $N = gp(x_{k+1}, x_{k+2} \ldots)$. Let now $\underline{u} \circ \underline{p}^k = \underline{x}$, then we express u_i as n_ic_i, where $n_i \in N$, $c_i \in \operatorname{Ker} \underline{p}^k$. Because $x_n\underline{p}^k = x_{n-k}$, and $u_i\underline{p}^k = n_i\underline{p}^k = \underline{x}_i$, we conclude that $n_i = x_{k+i}$ and $\underline{u} = (u_1, u_2, \ldots) = (x_{k+1}c_1, x_{k+2}c_2, \ldots)$, which completes the proof. In the case of $k = \infty$ the proof is similar.

4.5. $R = \bigcup s(k) \circ \mathrm{Aut}$, $0 \leqslant k \leqslant \infty$.

Proof. If $\underline{u} \in R$, then there exists $\underline{v} \in L$ such that $\underline{u} \circ \underline{v} = \underline{x}$. By 4.3, $\underline{v} = \underline{a} \circ \underline{p}^k$ for an invertible $\underline{a}$, $0 \leqslant k \leqslant \infty$. We get then $\underline{x} = \underline{u} \circ \underline{v} = \underline{u} \circ (\underline{a} \circ \underline{p}^k) = (\underline{u} \circ \underline{a}) \circ \underline{p}^k$ and, by 4.4, $\underline{u} \circ \underline{a} = \underline{s}(k)$ which gives $\underline{u} = \underline{s}(k) \circ \underline{a}^{-1}$ completing the proof. The union above is not disjoint; we can take $\underline{s}(2) = (x_3x_1x_2, x_4, x_5, \ldots)$ and $\underline{s}'(2) = (x_3x_2x_1, x_4, x_5, \ldots)$ which are different but $\underline{s} = \underline{s}' \circ \underline{a}$, where $\underline{a}$ permutes x_1 and x_2. So, $\underline{s} \circ \underline{x}$ and $\underline{s}' \circ \underline{x}$ give two presentations of the same endomorphism.

4.6. We note here that the set of components of a $\underline{s}(k)$ is independent modulo F' and hence $R \subseteq M$.

4.7. Theorem. The subgroup H of F is a retract if and only if there exists a free base $\underline{a}$ of F and a free base $\langle h_j, j \in J, |J| \leqslant \infty \rangle$ of H such that for the partition of the set of natural numbers $N = J \cup K$, we get $h_j = a_j c_j$, $c_j \in gp(a_k,\ k \in K)$, $j \in J$.

Proof. If H is finitely generated, then we can restrict ourselves to a F_n and the result follows from ([8], p. 149). If H is infinitely generated and $\underline{u}$ is any base of H, then $\underline{u}$ defines also a monomorphism of F onto H. By 2.8, H is a retract if and only if $\underline{u} \in R$ and, by 4.5, if and only if $\underline{u} = \underline{s}(k) \circ \underline{a}$, $0 \leqslant k \leqslant \infty$ which implies the result.

In the set of vectors $\underline{s}(k)$ which are left inverses for $\underline{p}^k$ we fix $\underline{s}_k = (x_{k+1}, x_{k+2}, x_{k+3}, \dots)$ for $k < \infty$ and $\underline{s}_\infty = (x_1, x_3, x_5, \dots)$.

4.8. We have the inclusions: (a) $\underline{v} \circ \mathrm{Mono} \subseteq \mathrm{Mono} \circ \underline{v}$, (b) $\underline{v} \circ M \subseteq M \circ \underline{v}$, (c) $\underline{v} \circ L \subseteq L \circ \underline{v}$, (d) $\underline{v} \circ R \subseteq R \circ \underline{v}$ for any vector $\underline{v} \in L$. In particular for $k > 0$:

$$(a')\quad \underline{p}^k \circ \mathrm{Mono} \subset \mathrm{Mono} \circ \underline{p}^k, \qquad (b')\quad \underline{p}^k \circ M \subset M \circ \underline{p}^k,$$

$$(c')\quad \underline{p}^k \circ L \subset L \circ \underline{p}^k, \qquad (d')\quad \underline{p}^k \circ R \subset R \circ \underline{p}^k.$$

Proof. We shall prove first (a′)-(d′). For a given vector $\underline{u}$ we define $\underline{u}(k) = (x_1, x_2, \dots, x_k, u_1(\underline{s}_k), u_2(\underline{s}_k), \dots)$ for $k < \infty$ and $\underline{u}_\infty = (u_1(\underline{s}_\infty), x_2, u_2(\underline{s}_\infty), x_4, u_3(\underline{s}_\infty), x_6, \dots)$. It is easy to see that $\underline{p}^k \circ \underline{u} = (1, \dots, 1, u_1, u_2, \dots) = \underline{u}(k) \circ \underline{p}^k$, for $k < \infty$ and $\underline{p}^\infty \circ \underline{u} = (u_1, 1, u_2, 1, u_3, 1, \dots) = \underline{u}_\infty \circ \underline{p}^\infty$. Now, if $\underline{u} \in \mathrm{Mono}$, that is $\underline{u}$ is free without units, or if $\underline{u} \in M$, then obviously the same is true for $\underline{u}(k)$ and $\underline{u}_\infty$ which proves (a′) and (b′). If $\underline{u} \in L$ ($\underline{u} \in R$) and $\underline{v} \circ \underline{u} = \underline{x}$ ($\underline{u} \circ \underline{v} = \underline{x}$), then the left (right) inverse for $\underline{u}(k)$ is $(x_1, \dots, x_k, v_1(\underline{s}_k), v_2(\underline{s}_k), v_3(\underline{s}_k), \dots)$ and the left (right) inverse for $\underline{u}_\infty$ is $(v_1(\underline{s}_\infty), x_2, v_2(\underline{s}_\infty), x_4, v_3(\underline{s}_\infty), x_6, \dots)$. This means that $\underline{u}(k)$ and $\underline{u}_\infty$ are in L (in R) which proves (c′) and (d′). The unequalities (a′)-(d′) are proper, because every vector $\underline{p}^k \circ \underline{u}$ from the left side has the first component equal to 1 for $k < \infty$, and the second component equal to 1 for $\underline{p}_\infty \circ \underline{u}$, while the right sides contain vectors without this property. The unequalities (a)-(d) follow from (a′)-(d′), because, by 4.3, any vector $\underline{v}$ from L is equal to $\underline{a} \circ \underline{p}^k$ for an invertible $\underline{a}$ and $0 \leqslant k \leqslant \infty$.

4.9. It follows from 4.8 that $(\tilde{a})$ $L\circ Mono \subseteq Mono\circ L$, $(\tilde{b})$ $L\circ M \subseteq M\circ L$, $(\tilde{c})$ $L\circ R \subseteq R\circ L$. The unequalities $(\tilde{a})$, $(\tilde{b})$ are proper, because, by 4.2, $L\circ Mono = End_\infty$ while $Mono\circ L = End$ which is shown below and the similar situation for $(\tilde{b})$ will follow from 5.3 and 5.4.

4.10. Theorem. $End = Inj\circ Sur$ and hence $End = Mono\circ Epi$.

Proof. By 2.3 and 2.5, $Inj = Mono$, $Sur = L$ so, since $End_\infty = L\circ Mono \subseteq Mono\circ L$, we shall only consider a vector $\underline{u}$ with a finitely generated image. Because of 3.3 and 3.4 we take $\underline{u} = (u_1, u_2, \ldots, u_k, 1, 1, \ldots)$ as a standard vector. One can check that $\underline{u} = (u_1(\underline{s}_\infty), \ldots, u_k(\underline{s}_\infty), x_2, x_4, x_6, \ldots)\circ \underline{p}^\infty$. Since $\langle u_1, u_2, \ldots, u_k\rangle$ is a free set and $\underline{s}_\infty$ is a monomorphism onto $gp(x_1, x_3, x_5, \ldots)$, we come to the conclusion that $\underline{u} \in Mono\circ \underline{p}^\infty \subseteq Mono\circ L$ completing the proof.

4.11. Corollary. The endomorphisms which have infinitely generated kernels form the semigroup $Mono\circ \underline{p}^\infty$. The endomorphisms which have finitely generated kernels form the semigroup $\bigcup_k Mono\circ \underline{p}^k$, $k < \infty$.

5. Endomorphisms of a free abelian group

We shall consider a free abelian group of a countably infinite rank as the quotient group F/F' with the abelian base $\underline{x}F' = (x_1F', x_2F', \ldots)$. Every endomorphism of F/F' is uniquely defined by the set of images of elements from the base, so we denote the endomorphism as $\underline{u}F' = (u_1F', u_2F', \ldots)$, where the vector $\underline{u}$ is not fixed. The natural homomorphism $\alpha' : F \to F/F'$ induces the homomorphism $\alpha : End\ F \to End\ F/F'$ which is obviously onto and maps $\underline{u}$ into $\underline{u}F'$. The kernel of α consists of all vectors $\underline{u} = (x_1c_1, x_2c_2, \ldots)$, where $c_i \in F'$, that is $Ker\ \alpha \in M$. We denote $Ker\ \alpha \cap Aut$ by I.

5.1. $Ker\ \alpha \cap R = Ker\ \alpha \cap L = I$, that is every element from $Ker\ \alpha$ which is invertible from one side must be invertible from the other.

Proof. By 2.6, $\mathrm{Ker}\,\alpha \cap L \subseteq M \cap L \subseteq \mathrm{Aut}$, which gives $\mathrm{Ker}\,\alpha \cap L = I$. Now, if $\underline{u} \in \mathrm{Ker}\,\alpha \cap R$, then there exists $\underline{v} \in L$ such that $\underline{u} \circ \underline{v} = \underline{x}$. This implies $\underline{v} \in \mathrm{Ker}\,\alpha \cap L = I$, and hence $\underline{u} \in I$, which was required.

5.2. The homomorphism α maps R, L and Aut onto the corresponding semigroups in $\mathrm{End}(F/F')$, α maps M onto $\mathrm{Mono}(F/F')$ and maps $R \setminus \mathrm{Aut}$, $L \setminus \mathrm{Aut}$ isomorphically into $\mathrm{End}(F/F')$.

We shall prove now a theorem analogous to 4.2 for endomorphisms of F/F''. It is not ah immediate consequence of 4.2, because the image of a monomorphism under α is not necessarily a monomorphism.

5.3. Theorem. $\mathrm{End}_{\infty}(F/F') = \mathrm{Sur}(F/F') \subseteq \mathrm{Inj}(F/F') \subseteq \mathrm{Inj}(F/F') \circ \mathrm{Sur}(F/F')$ and hence, by 2.5, $\mathrm{End}_{\infty}(F/F') = \mathrm{Epi}(F/F') \circ \mathrm{Mono}(F/F') \subseteq \mathrm{Mono}(F/F') \circ \mathrm{Epi}(F/F')$.

Proof. Let $\underline{u}F'$ be an endomorphism of F/F'. To prove the theorem we find a vector $\underline{u}$ such that $\underline{u}\alpha = \underline{u}F'$ and $\underline{u} \in L \circ M \subseteq M \circ L$; then, using α we get the result because of 5.2. So, in the subgroup $\mathrm{gp}(uF')$ we choose an abelian base $\underline{w}F' = (w_1F', w_2F', \dots)$. Since $\mathrm{gp}(\underline{u}F') = \mathrm{gp}(\underline{w}F')$, every u_iF' is a word in generators $\underline{w}F'$, let us say $u_iF' = s_i(w_1F', w_2F', \dots)$. We fix now any vector $\underline{w}$ such that $\underline{w}\alpha = \underline{w}F'$ and introduce the vector $\underline{u}$, where $u_i = s_i(w_1, w_2, \dots)$. Obviously then $\underline{u}\alpha = \underline{u}F'$ and $\mathrm{gp}(\underline{u}) \subseteq \mathrm{gp}(\underline{w})$. Because α maps the components of $\underline{w}$ onto a set of free abelian generators, we get $\underline{w} \in M$. In fact $\mathrm{gp}(\underline{u}) = \mathrm{gp}(\underline{w})$, because $\mathrm{gp}(\underline{u}F') = \mathrm{gp}(\underline{w}F')$ implies that there exist such elements $c_j \in F'$ that every w_i is a word in $u_1c_1, u_2c_2, \dots$, that is $w_i = t_i(u_1c_1, u_2c_2, \dots) \equiv t_i(u_1, u_2, \dots) \bmod F' \equiv t_i(s_1(\underline{w}), s_2(\underline{w}), \dots) \bmod F'$ and, since components of $\underline{w}$ are independent modulo F', we conclude that $w_i = t_i(s_1(\underline{w}), s_2(\underline{w}), \dots) = t_i(u_1, u_2, \dots) \in \mathrm{gp}(\underline{u})$. Now, by theorem 3.4, there exists an invertible vector $\underline{a}$ such that $\underline{a} \circ \underline{u}$ is a free standard vector with infinite number of non-unit components and hence, if we denote by $\underline{r}$ the vector of these non-unit components, we get, by 4.1, $\underline{a} \circ \underline{u} = \underline{p}^k \circ \underline{r}$, $0 \leqslant k \leqslant \infty$. We have then $\underline{u} = (\underline{a}^{-1} \circ \underline{p}^k) \circ \underline{r}$ and $\underline{r} = (\underline{s}_k \circ \underline{a}) \circ \underline{u}$. Since $\underline{w}$ and $\underline{r}$ are two free bases in $\mathrm{gp}(\underline{u})$

there exists an invertible vector $\underline{b}$ such that $\underline{r} = \underline{b} \circ \underline{w}$ and then $\underline{u} = (\underline{a}^{-1} \circ \underline{p}^{k}) \circ \underline{r} = (\underline{a}^{-1} \circ \underline{p}^{k} \circ \underline{b}) \circ \underline{w} = \underline{v} \circ \underline{w} \in L \circ M$. By 4.9.($\tilde{b}$), we have $\underline{u} \in L \circ M \subseteq M \circ L$ which completes the proof.

5.4. Theorem. $\mathrm{End}(F/F') = \mathrm{Inj}(F/F') \circ \mathrm{Sur}(F/F') = \mathrm{Mono}(F/F') \circ \mathrm{Epi}(F/F')$.

Proof. Because of the previous theorem we need to consider only an endomorphism $\underline{u}F'$ with a finitely generated image $gp(\underline{u}F')$. As in the previous proof we take an abelian base $\underline{w}F'$ in $gp(\underline{u}F')$ and fix a vector $\underline{w}$. Then we fix a vector $\underline{u}$ with components in $gp(\underline{w})$. By 3.4, $\underline{u}$ is equivalent to a standard vector $\underline{u}' = \underline{a} \circ \underline{u} = (u_1', u_2', \ldots, u_k', 1, 1, \ldots)$. Since $\underline{w}$ and $\underline{u}'$ are two free vectors generating the same finitely generated subgroup, they are of the same type and, by 3.7, are equivalent. Thus, for an invertible $\underline{a}$, $\underline{u} = \underline{a} \circ \underline{w} = \underline{a} \circ (w_1(\underline{s}_\infty), \ldots, w_k(\underline{s}_\infty), x_2, x_4, x_6, \ldots) \circ \underline{p}^\infty \in M \circ \underline{p}^\infty \subseteq M \circ L$ and using α gives the result.

5.5. Definition. ([2], 1.3). An infinite matrix over Z is called row-finite if every row contains only a finite number of non-zero components. Row-finite matrices form a semigroup. The matrix is called row-bounded if there exists an n such that all columns with indices greater than n consist of zeros.

5.6. The semigroup $\mathrm{End}(F/F')$ is isomorphic to the semigroup of row-finite matrices over Z. The set $\mathrm{End}_\infty(F/F')$ is in one-to-one correspondence with the set of row-bounded matrices.

Proof. Let $\underline{u}F' \in \mathrm{End}(F/F')$, $\underline{u}F' = (u_1F', u_2F', \ldots)$, where $u_i = \prod x_j^{k_{ij}} \bmod F'$. We introduce the exponent matrix $U = (k_{ij})$ to represent $\underline{u}F'$. This correspondence defines the required isomorphism.

The reformulation of 5.3 and 5.4 gives the following theorem.

5.7. Theorem. Every row-finite matrix over Z is a product of a matrix with linearly independent rows and a left invertible matrix. Every row-finite matrix over Z which is not row-bounded is also a product of a left invertible matrix and a matrix with linearly independent rows.

REFERENCES

[1] R. Baer: Groups with proper isomorphic quotient groups. Bull. Amer. Math. Soc. 50 (1944) 267-278.

[2] R.G. Cooke: Infinite matrices and sequence spaces. London, 1950. MacMillan and Co., London, 1950.

[3] H. Federer, B. Jonsson: Some properties of free groups. Trans. Amer. Math. Soc. 68 (1950) 1-27.

[4] R.C. Lyndon: The equation $a^2b^2 = c^2$ in free groups. Michigan Math. J. 6 (1959) 89-95.

[5] R.C. Lyndon, P.E. Schupp: Combinatorial Group Theory (Ergebnisse der Mathematik und ihrer Grenzgebiete 89), Springer-Verlag, Berlin-Heidelberg--New York, 1977.

[6] E.S. Liapin: Semigroups, (Russian), Fiz-Mat. Lit., Moskva, 1960.

[7] O. Macedońska: On Infinite Nielsen Transformations. Math. Scand. 51 (1982) 63-86.

[8] W. Magnus, A. Karrass, D. Solitar: Combinatorial Group Theory. (Pure and Applied Mathematics 13), Interscience Publishers, New York-London-Sydney, 1966.

[9] H. Neumann: On varieties of groups and their associated near-rings. Math. Zeitschr., 65 (1956) 36-69.

[10] H. Neumann: Varieties of Groups (Ergebnisse der Mathematik und ihrer Grenzgebiete 37), Springer-Verlag, Berlin-Heidelberg-New York, 1967.

[11] Z. Semadeni, A. Wiweger: Wstęp do teorii kategorii i funktorów. Biblioteka Matematyczna 55, PWN, Warszawa, 1978.

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CONSTRUCTION OF ORTHOMODULAR LATTICES WITH GIVEN STATE SPACES

1. Introduction

An axiomatic system of quantum logics [5, 8, 15] identifies an "event structure" with an orthomodular poset (logic) or an orthomodular lattice (lattice logic, abbr. OML). States of the system are unified with probability measures on logics. Existence of a state space with preassigned properties is often included in the definition of logics. Logics possesing states of given characteristics have been frequently investigated [1, 3, 4, 6, 12, 13, 14 and others], but such logics are (even in the finite case) quite complicated and their construction is uneasy. One of the authors has recently shown [11] that in a special case, finite hypergraphs can be used for construction of orthomodular posets with desired state space properties. Many examples of application of that method to various problems are presented there.

In this paper we bring a method which enables to "transfer" states from finite hypergraphs onto orthomodular lattices. The proof of the main theorem is constructive so that the resulting logics may be fully described (although being rather complex). The construction may be considered a generalisation of the methods proposed by Greechie [4] and Dichtl [2].

Let us recall a few notions that will be needed in the sequel.

Let L be a logic. An element $b \in L$ is called an atom if $0 < a \leqslant b$ implies $a = b$. If $a, b \in L$ and $a \leqslant b$, we define an interval $[a,b]_L$ to be the set $\{c \in L \mid a \leqslant c \leqslant b\}$.

By a hypergraph we mean a couple $H = (P,X)$, where P is a non-empty set (of vertices) and X is a covering of P by non-empty subsets (edges). The state on a hypergraph H is a mapping $s: P \to [0,1]$ such that $\sum_{u \in U} s(u) = 1$ for any edge U [7]. A hypergraph (Q,Y) is a subhypergraph of (P,X) if $Q \subset P$ and $Y = \{U \cap Q \mid U \in X,\ U \cap Q \neq \emptyset\}$. An edge of a hypergraph is called isolated if it is disjoint to any other edge.

Throughout this paper we assume all hypergraphs and all logics to be finite.

Every logic is a union of all its maximal Boolean subalgebras (called blocks). Thus any logic L can be studied as a union of a family $\mathcal{B}$ of Boolean algebras. In our case of finite logics we can describe these Boolean algebras by the sets of their atoms (which are also atoms of L). Let $a(B)$ denote the set of all atoms of $B \in \mathcal{B}$. Then (P,X), where $P = \bigcup_{B \in \mathcal{B}} a(B)$ and $X = \{a(B) \mid B \in \mathcal{B}\}$, is a hypergraph which completely determines the structure of the finite logic L. The hypergraph (P,X) is called the diagram of the logic L. We can naturally transfer some notions (e.g. isolated edge) from hypergraphs to families of Boolean algebras and to logics.

Each element of a logic L can be described as a subset of some edge of the diagram of L. States on a logic and states on its diagram are in a one-to-one correspondence; a state on a logic is obtained by a unique extension of a state on its diagram. We denote by S(L) the set of all states of a logic L and we use the same notation also for state spaces of hypergraphs.

Definition 1.1. Let $H = (P,X)$ be a hypergraph. We say that vertices $u, v \in P$ are state-equivalent if $s(u) = s(v)$ for any state $s \in S(H)$.

Throughout this paper r(u) will denote the class of all vertices state-equivalent to a vertex u of some hypergraph (or a set of such classes in case the argument is a set of vertices). The state space $S(H)$ of a hypergraph $H = (P,X)$ is fully determined by the partition $r(P)$ of P within the state equivalence and by the values of all states on these classes.

Definition 1.2. Let $H_1 = (P_1,X_1)$, $H_2 = (P_2,X_2)$ be hypergraphs. Let f be a one-to-one mapping of $r(P_1)$ onto $r(P_2)$. For any $s_2 \in S(H_2)$, the mapping $s_2 \circ f$ is a non-negative function on $r(P_1)$. We call f a state isomorphism (and hypergraphs H_1, H_2 state-isomorphic) if it satisfies:

(i) for any $s_2 \in S(H_2)$ the mapping $s_2 \circ f \circ r$ is a state on H_1;

(ii) for any $s_1 \in S(H_1)$ the mapping $s_1 \circ f^{-1} \circ r$ is a state on H_2.

The notion of the state isomorphism can be naturally extended to logics and also to state isomorphism of a logic and a hypergraph - instead of a logic we consider its diagram.

2. Greechie and Dichtl logics

Besides traditional examples - Boolean algebras and lattices of closed subspaces of a separable Hilbert space - logics constructed as a union of a family of Boolean algebras are often used in the theory of orthomodular posets. Such a procedure is limited by (rather strict) conditions. Not every hypergraph is a diagram of a logic. In this paragraph we shall recall sufficient conditions given by Greechie [4] and (a more general one) by Dichtl [2].

Throughout this paragraph $\mathcal{B}$ will denote a family of Boolean algebras and $\mathcal{B}(L)$ the family of all blocks of a logic L.

Definition 2.1. (i) The n-cycle is a sequence $((B_0,b_0), (B_1,b_1),\dots,(B_{n-1},b_{n-1}))$ of (not necessarily distinct) algebras $B_i \in \mathcal{B}$ and (not necessarily distinct) elements $b_i \in B_i \cap B_{i+1}$, $b_i \neq 0$, such that $b_i \leq b'_{i+1}$, $[0,b_i]_{B_i} = [0,b_i]_{B_{i+1}}$ (indices mod n).

(ii) The loop of order n is an n-cycle $((B_0,b_0),\dots$ $\dots,(B_{n-1},b_{n-1}))$ such that $B_i \neq B_j$ for $i \neq j$ and $B_i \cap B_{i+1} = \{0,1,b_i,b_u'\}$ (hence b_i is an atom in both B_i and B_{i+1}).

Definition 2.2. We say that $\mathcal{B}$ is nearly disjoint if for any $A,B \in \mathcal{B}$ the operations of A and B coincide on $A \cap B$ and the intersection $A \cap B$ is either $\{0,1\}$ or $\{0,1,a,a'\}$ or the whole algebra $A = B$.

Theorem 2.3. (Greechie). Let $\mathcal{B}$ be nearly disjoint. On $L = \bigcup_{B \in \mathcal{B}} B$ we define a partial ordering and an orthocomplementation as follows: $a \leq b$ ($a = b'$ resp.) if there is $B \in \mathcal{B}$ such that $a \leq_B b$ ($a = b'^B$ resp.). Then

(i) L is a logic iff there is no loop of order 3 in $\mathcal{B}$,

(ii) a logic L is a lattice iff there is no loop of order 4 in $\mathcal{B}$.

Proof. See [4].

Definition 2.4. $\mathcal{B}$ is pasted if for any $A,B \in \mathcal{B}$, $A \neq B$ the following conditions hold:

(i) A is not contained in B,

(ii) $A \cap B$ is a subalgebra of A and of B on which the operations of A and B coincide,

(iii) for each $a \in A \cap B$, $a \notin \{0,1\}$, there exists a 4-cycle $((A,a), (C_1,a'), (B,a), (C_3,a''))$ (C_1, C_3 arbitrary).

Let us notice that there are no single-atom algebras and no non-isolated two-atom algebras in a pasted family, as a consequence of (i) and (ii) (with the only exception of one trivial Boolean algebra $\{\{0,1\}\}$).

Theorem 2.5 (Dichtl). Let $\mathcal{B}$ be pasted. On $L = \bigcup_{B \in \mathcal{B}} B$ we define a partial ordering and an orthocomplementation in the same way as in Theorem 2.3. Then

(i) L is a logic iff for any 3-cycle $((B_i,b_i))_{i=0}^{2}$ in $\mathcal{B}$ there is a member $B \in \mathcal{B}$ such that $[0,b_i]_{B_i} \subset B$ for $i = 0,1,2$,

(ii) the logic L is a lattice iff for any 4-cycle $((B_i,b_i))_{i=0}^{3}$ in $\mathcal{B}$ there is a 4-cycle $((C_0,a),(C_1,a'),(C_2,a),(C_3,a'))$ in $\mathcal{B}$ such that $b_0,b_2 \leq a \leq b_1',b_3'$.

Proof. See [2].

We call the logics constructed in the way described in Theorem 2.3 Greechie logics, those constructed according to Theorem 2.5 Dichtl logics. Both constructions are described in detail in [9], where the reader may find a number of examples. Let us notice that if $\mathcal{B}$ is nearly disjoint and if $A \not\subset B$ for any $A,B \in \mathcal{B}$, then $\mathcal{B}$ is a pasted family. Hence Theorem 2.3 is a special case of Theorem 2.5 and any Greechie logic is also a Dichtl logic.

Definition 2.6. Let L_1, L_2 be a logics. Let in $L = L_1 \times L_2$ the partial ordering and the orthocomplementation be defined "pointwise", that is, $(a_1,a_2) \leq (b_1,b_2)$ (or $(a_1,a_2) = (b_1,b_2)'$) iff $a_i \leq b_i$ (or $a_i = b_i'$, resp.) for $i = 1,2$. Then L is a logic called the product of logics L_1, L_2.

Any state s on L is of the form $\alpha_1 s_1 + \alpha_2 s_2$, where $s_i \in S(L_i)$ and $\alpha_1, \alpha_2 \geq 0$, $\alpha_1 + \alpha_2 = 1$ [10].

Lemma 2.7. Let L_1 be a Dichtl logic and B a Boolean algebra. Then the product $L = L_1 \times B$ is again a Dichtl logic. If L_1 is a lattice then L is also a lattice.

Proof. The set $\{B_1 \times B \mid B_1 \in \mathcal{B}(L_1)\}$ is the family $\mathcal{B}(L)$ of all blocks of L. We will prove that $\mathcal{B}(L)$ is pasted. Validity of the conditions (i) and (ii) in Definition 2.4 is straightforward. To prove (iii), let $a \in (A_1 \times B) \cap (A_2 \times B)$, $A_1, A_2 \in \mathcal{B}(L_1)$. We may write $a = a_1 \vee a_2$ for $a_1 \in A_1 \cap A_2$ and $a_2 \in B$. According to the assumption and Theorem 2.5 there exists a 4-cycle $((C_0,a_1),(C_1,a_1'),(C_2,a_1),(C_3,a_1'))$ in L_1. Hence there is a 4-cycle $((C_0 \times B,a),(C_1 \times B,a'),(C_2 \times B,a),(C_3 \times B,a'))$ in L. Thus $\mathcal{B}(L)$ is a pasted family. Due to Theorem 2.5, L is a Dichtl logic. The rest of Lemma 2.7 is obvious.

3. Logics and lattices state-isomorphic to hypergraphs

In paragraphs 3 and 4 we investigate the questions which hypergraphs are state-isomorphic to an OML or to a logic. We show, that these two questions are equivalent (Corollary 3.5) and we give a necessary and sufficient condition for this to hold (Theorem 4.1).

Definition 3.1. Let u,v be two vertices of a hypergraph $H = (P,X)$. We define the distance $d(u,v)$ as the minimal n for which there exists a sequence $\{U_1,\dots,U_n\} \subset X$ such that $u \in U_1$, $v \in U_n$ and $U_i \cap U_{i+1} \neq \emptyset$ for $i = 1,\dots,n-1$. If no such sequence exists we define $d(u,v) = \infty$. We put $d(u,u) = 0$. We define the distance of two sets $U,V \subset P$ as $d(U,V) = \min\{d(u,v) \mid u \in U,\ v \in V\}$.

Due to the connection between a logic and its diagram, we can also speak about the distance of atoms of a logic.

Lemma 3.2. There is a non-Boolean Greechie OML L_3 state-isomorphic to the Boolean algebra 2^3.

Proof. Let L_3 be a logic with atoms $\{a_0,a_1,\dots,a_{155}\}$ and blocks $\exp\{a_{2k},a_{2k+1},a_{2k+2}\}$ and $\exp\{a_{2k+1},a_{2k+8},a_{2k+39}\}$, $k = 0,1,\dots,77$ (throughout this example indices are evaluated mod 156). Due to Theorem 2.3, L_3 is a (Greechie) OML. We will prove that any state s on L_3 satisfies

(S1) $s(a_{3k+i}) = s(a_i)$ for $i = 0,1,2$ and $k = 1,2,\dots,51$, and $s(a_0) + s(a_1) + s(a_2) = 1$.

Then the logic L_3 will be state-isomorphic to the block $\exp\{a_0,a_1,a_2\}$.

Every state s on L_3 must satisfy the following system of 156 linear equations:

(S2) $s(a_{2k}) + s(a_{2k+1}) + s(a_{2k+2}) = 1$,
$s(a_{2k+1}) + s(a_{2k+8}) + s(a_{2k+39}) = 1$, $k = 0,1,\dots,77$.

It is clear that the system (S1) solves the system (S2) and that the range of (S1) is 154. Since the range of (S2) is 154 (computer-checked), the two systems are equivalent and hence the solutions of (S1) are the only solutions of (S2). We have proved that L_3 is state-isomorphic to 2^3.

Lemma 3.3. For any $n \geq 3$ there is a Dichtl OML L_n state-isomorphic to the Boolean algebra 2^n and containing a set E_n of atoms satisfying:

(1) each class within the state equivalence in L_n has a single-element intersection with E_n,

(2) the distance of any two elements of E_n is at least 3.

Proof. We shall prove the lemma by an induction on n. Due to technical reasons we shall construct a set D_n in L_n such that each class within the state equivalence has a two-element intersection with D_n and the distance of any two elements of D_n is at least 3. It is obvious that E_n as a subset of D_n exists in L_n.

For n = 3 we take the logic L_3 constructed in Lemma 3.2 and the set $D_3 = \{a_0, a_{14}, a_{28}, a_{42}, a_{56}, a_{70}\}$.

Suppose that there is a logic L_{n-1} and the set D_{n-1} in L_{n-1} satisfying the assumptions of the induction. Let (P,X) be the diagram of L_{n-1} and denote by $b_1, \dots, b_{n-1}, c_1, \dots, c_{n-1}$ the elements of D_{n-1} so that $r(b_i) = r(c_j)$ iff $i = j$. Due to Lemma 2.7 and the remark preceeding it, the product of L_{n-1} with the trivial Boolean algebra $\{0,1\}$ is a Dichtl OML state-isomorphic to 2^n (but it does not possess the required set of atoms). Denote $d = (0,1) \in L_{n-1} \times \{0,1\}$. The diagram of $L_{n-1} \times \{0,1\}$ is isomorphic to the hypergraph $H = (Q,Y)$, where $Q = P \cup \{d\}$, $Y = \{U \cup \{d\} \mid U \in X\}$. The distance (in H) of any two atoms of D_{n-1} equals 2 and the elements of D_{n-1} represent n-1 of the whole of n classes within the state equivalence in H (d being the only member of the remaining class).

Take p+1 disjoint copies $H^0 = Q^0, Y^0)$, $H^1 = (Q^1, Y^1), \dots H^p = (Q^p, Y^p)$ of $H = (Q,Y)$, where p is a common multiple of $\{2,3,\dots,n\}$ greater or equal to $4n^2-1$. Denote by $b_1^k, \dots, b_{n-1}^k, c_1^k, \dots, c_{n-1}^k, d^k \in Q^k$ $(k = 0,1,\dots,p)$ the vertices corresponding to $b_1, \dots, b_{n-1}, c_1, \dots, c_{n-1}, d \in Q$. We identify b_j^k with c_j^{k+j} for all $j = 1,2,\dots,n-1$ and $k = 0,1,\dots,p$ (upper indices evaluated mod p+1). Thus $Q^j \cap Q^k$ contains at most one vertex for $j \neq k$. Let us denote the resulting diagram by G. It is the diagram of a pasted family of Boolean algebras. We use Theorem 2.5 to prove that G is the diagram of a Dichtl OML L_n. No new 3-cycles appeared in the construction (any 3-cycle in G is contained in some H^k). Any 4-cycle in G is either contained in some H^k or it is of the form ((A,a),(B,b), (B,a),(A,c)) where a is an atom and the 4-cycle ((A,a),(B,a′), (B,a),(A,a′)) satisfies the assumptions of Theorem 2.5(ii).

In the logic L_n (in its diagram G) we obtain the following state equivalence relations:

$$r\left(b_i^1\right) = r\left(b_i^{1+i}\right) = r\left(b_i^{1+2i}\right) = \ldots = r\left(b_i^{1+p-i}\right) = r\left(b_i^0\right)$$

for any $i = 1,2,\ldots,n-1$.
Due to the symmetry of our construction, we obtain $r\,b_i^j = r\left(b_i^k\right)$ for any $i = 1,2,\ldots,n-1$ and $j,k = 0,1,\ldots,p$. This implies also $r(d^j) = r(d^k)$ for any $j,k = 0,1,\ldots,p$. Thus L_n is state-isomorphic to 2^n.

The sets Q^{2nj}, Q^{2nk}, where $j,k = 0,1,\ldots,2n-1$ and $j \neq k$, have distance at least 4 and we can select the members of the set D_n with the desired properties from these sets. The proof is finished.

An edge U of a hypergraph is called regular if either of the following conditions is fulfiled:

(i) U is isolated and contains at least 2 vertices,

(ii) U is non-isolated and contains at least 3 vertices.

Observe that in diagrams of logics different from the trivial Boolean algebra $\{0,1\}$ all edges are regular.

Lemma 3.4. Let H be a hypergraph and let all edges of H be regular. Then there is an OML state-isomorphic to H.

Proof. Let $H = (P,X)$ and let $U = \{u_1,u_2,\ldots,u_m\} \in X$, $m \geq 3$. We take the logic L_m with the subset $E_m = \{e_1,e_2,\ldots,e_m\}$ constructed in Lemma 3.3. For any $i = 1,2,\ldots,m$, we identify e_i with u_i. We replace the edge U by the diagram (Q,Y) of L_m. More exactly, we construct the hypergraph $H' = (P \cup Q,\ X \cup Y \setminus \{U\})$. Hypergraphs H and H′ are state-isomorphic. Indeed, any state on H′ restricted to P becomes a state on H and any state on H has an extension to $P \cup Q$ such that it is a state on H′.

We repeat the procedure described above for all edges in X having at least 3 vertices, and obtain a hypergraph G. According to Theorem 2.5, G is the diagram of a Dichtl OML (we use exactly the same arguments as in the proof of Lemma 3.3).

Corollary 3.5 (Pták principle). Any logic is state-isomorphic to an OML.

4. Generalizations

The nullset of a hypergraph H is the set of all vertices u such that $s(u) = 0$ for any state $s \in S(H)$ [7]. The component of a hypergraph is its maximal subhypergraph whose vertices have finite distances. An even graph is a hypergraph that has only two-element edges and contains no loop of an odd order.

Theorem 4.1. Let $H = (P,X)$ be a hypergraph where not all edges of X are single-element. Then H is state-isomorphic to some OML L iff for any vertex $u \in P$ one of the following conditions holds:

(1) the component of H containing u is an even graph,

(2) there are vertices $u_1, \ldots, u_m \in P$, $m \geq 2$, (not necessarily distinct from u and from each other) such that each state $s \in S(H)$ satisfies $s(u) + \sum_{i=1}^{m} s(u_i) = 1$.

Remarks. 1. The condition (2) is trivially satisfied if b is contained in an edge with at least 3 elements.

2. If H admits no states, all our assertions remain valid and an OML state-isomorphic to H exists.

3. If X contains only single-element edges, the hypergraph H is state-isomorphic to the Boolean algebra $\{0,1\}$.

Proof. First we shall prove the necessity. Take an atom $c \in f(r(u))$, where f is the state-isomorphism in question. Vertex c is contained in a regular edge $C = \{c, c_1, \ldots, c_n\}$ of the diagram of L. For any state $t \in S(L)$ we have $t(c) + \sum_{i=1}^{n} t(c_i) = 1$. If $n \geq 2$ then we can choose $m = n$, $u_i \in f^{-1}(r(c_i))$ for $i = 1, \ldots, n$ and (2) is satisfied. Otherwise, $C = \{c, c_1\}$ must be isolated. The hypergraphs state-isomorphic to a single two-element edge are even graphs, so (1) is satisfied in this case.

Now we shall prove the sufficiency. We divide the proof into several steps in which we shall construct hypergraphs $H_1, \ldots, H_4$ state-isomorphic to H, and then apply Lemma 3.4

to H_4. First we describe a tool that will be used in the sequel.

Let (Q,Z) be a hypergraph, let $\{z_1,\dots,z_n\} \in Z$. Consider the hypergraph (Q',Z'), where $Q' = Q \cup \{t\}$, $Z' = Z \cup \cup \{\{t,z_2,\dots,z_n\}\}$. Vertices z_1 and t are state-equivalent in (Q',Z') and (Q,Z), (Q',Z') are state-isomorphic. Whenever we use this construction, we say that (Q',Z') originated by adding a copy t of z_1 to (Q,Z). This procedure works also for $n = 1$. If $n \geq 3$, the added edge $\{t,z_2,\dots,z_n\}$ is regular and we say that we added a regular copy of z_1. When we say that we added copies $t_1,\dots,t_p$ of $y_1,\dots,y_p$, we mean that we added consequently a copy t_i of y_i for $i = 1,\dots,p$ in such a way that each of the vertices $t_1,\dots,t_p$ is contained only in one edge of the resulting hypergraph. This is always possible.

1. We replace all such components of H that are even graphs by isolated two-element edges. We obtain the hypergraph $H_1 = (P_1,X_1)$. All vertices of H_1 not satisfying (2) are contained in regular edges.

2. Let $u \in P_1$ is not contained in a regular edge and $u_1,\dots,u_m$ are the vertices satisfying condition (2). Notice that adding the edge $\{u,u_1,\dots,u_m\}$ would not change the state space provided that all these vertices are distinct. Let H_1' originate from H_1 by adding (distinct) copies $v_1,\dots,v_m$ of $u_1,\dots,u_m$ and by adding the edge $V = \{u,v_1,\dots,v_m\}$. Then H_1' is state-isomorphic to H_1 and u, as well as all added vertices $v_1,\dots,v_m$, is covered by the regular edge V. Repeating this procedure for all vertices not contained in regular edges, we obtain the hypergraph $H_2 = (P_2,X_2)$, where P_2 is covered by regular edges of X_2.

3. Let $Y = \{\{u_i\} \mid 1 \leq i \leq n\}$ be the set of all single-element edges in H_2. If Y is non-empty, then each state on H_2 attains 1 at u_i and u_i is in a (non-isolated) regular edge, hence the nullset of H_2 contains at least two distinct vertices v, w. Let $H_2' = (P_2',X_2')$ be the hypergraph originated from H_2 by adding regular copies v_1, w_1 of v, w. We define the hypergraph $H_3 = (P_3,X_3)$ as follows: $P_3 = P_2'$, $X_3 =$

$= (X_2'-Y) \cup \{\{u_i, v, w\} \mid 1 \leq i \leq n\} \cup \{\{u_1, v, v_1, w, w_1\}\}$. All added edges are regular. If Y is empty, we put $H_3 = H_2$. There are no single-element edges in H_3.

4. The only irregular edges in H_3 are non-isolated two-element edges covered by regular edges. Suppose that $T = \{u, v\}$ is such an edge. There are regular edges $U = \{u, u_1, \ldots, u_m\}$, $V = \{v, v_1, \ldots, v_n\} \in X_3$. Notice that $\sum_{i=1}^{m} s(u_i) + \sum_{i=1}^{n} s(v_i) = 1$ for all $s \in S(H_3)$. We construct the hypergraph $H_3' = (P_3', X_3')$ by adding regular copies $w_1, \ldots, w_n$ of $v_1, \ldots, v_n$. The hypergraph $H_3' = (P_3', (X_3' - \{T\}) \cup \{\{u_1, \ldots, u_m, w_1, \ldots, w_n\}\})$ is state-isomorphic to H_3' and the added edge is regular. Repeating this procedure for all non-isolated two-element edges we obtain the hypergraph H_4. All edges of H_4 are regular.

5. We apply Lemma 3.4 to H_4 and find an OML state-isomorphic to H_4 and hence also to H. The proof is finished.

The condition (2) in Theorem 4.1 is not always easy to verify, so we add two simplier corollaries.

Corollary 4.2. A hypergraph that is covered by regular edges is state-isomorphic to an OML.

Corollary 4.3. A hypergraph with a non-void nullset is state-isomorphic to an OML.

Proof. Let a vertex v belong to the nullset. Any vertex u_1 belongs to an edge $\{u_1, \ldots, u_m\}$, $m \geq 1$, and we have $\sum_{i=1}^{m} s(u_i) + s(v) + s(v) = 1$ for each state s, hence (2) is satisfied.

Let us exhibit the simplest example of a hypergraph not state-isomorphic to any logic.

Example 4.4. Let $G = (\{a, b, c, d\}, \{\{a, d\}, \{a, b, c\}\})$. There is a state s on G such that $s(a) = s(b) = s(c) = 1/3$, $s(d) = 2/3$. Vertex d satisfies neither (1) nor (2), so there is no logic state-isomorphic to G.

Finally, we give a generalization which may simplify an application of our results.

Definition 4.5. The hypergraph with pre-assigned state equivalence is a triple $H = (P, X, E)$, where

(P,X) is a hypergraph and $E \subset P \times P$. A state on H is a mapping $s: P \to [0,1]$ such that $\sum_{u \in U} s(u) = 1$ for all $U \in X$ and $s(u) = s(v)$ for all $(u,v) \in E$.

The definitions of state equivalence and state isomorphism remain the same for hypergraphs with pre-assigned state equivalence.

Theorem 4.6. Let $H = (P,X,E)$ be a hypergraph with pre-assigned state equivalence. Suppose that any vertex $u \in P$ satisfies the condition (2) of Theorem 4.1 or

(1′) the component (Q,Y) of (P,X) containing u is an even graph and $E \cap (Q \times P) = E \cap (P \times Q) = \emptyset$.

Then there is an OML state-isomorphic to H.

Proof. Let us use the notation $R(E) = \{v \mid (v,w) \in E$ or $(w,v) \in E$ for some $w\}$. Let (P_1,X_1) be the hypergraph obtained from (P,X) by adding copies of all elements of $R(E)$. We define the relation $E_1 \subset P_1 \times P_1$ such that $(t_1,t_2) \in E_1$ iff t_1, t_2 are copies of $z_1, z_2 \in P$ and $(z_1,z_2) \in E$. Then $H_1 = (P_1,X_1,E_1)$ is state-isomorphic to H and no edge of X_1 contains more than one element of $R(E_1)$. In H_1 we unify all vertices equivalent within the least equivalence relation containing E_1 and we obtain a hypergraph $H_2 = (P_2,X_2,\emptyset) = (P_2,X_2)$, state-isomorphic to H_1. The assumption of Theorem 4.6 guarantees that we may apply Theorem 4.1 to H_2. The proof is finished.

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REFERENCES

[1] L.J. Bunce, M. Navara, P. Pták, and J.D.M. Wright: Quantum logics with Jauch-Piron states, Quart. J. Math. Oxford (2) 36 (1985), 261-271.

[2] M. Dichtl: Astroids and pastings, Algebra Universalis 18 (1984), 380-385.

[3] R. Godowski: States on orthomodular lattices, Demonstratio Math. 15 (1982), 817-822.

[4] R.J. Greechie: Orthomodular lattices admitting no states, J. Combinatorial Theory 10 (1971), 119-132.

[5] S.P. Gudder: Stochastic Methods in Quantum Mechanics, North Holland, New York, 1979.

[6] S.P. Gudder: Expectation and transition probability, Int. J. Theor. Phys. 20 (1981), 383-395.

[7] S.P. Gudder, M.P. Klay and G.T. Rüttimann: States on hypergraphs, Demonstratio Math. 19 (1986) 503-526.

[8] M. Jauch: Foundations of Quantum Mechanics. Addison-Wesley, Reading, Mass., 1968.

[9] G. Kalmbach: Orthomodular Lattices. Academic Press, London, 1984.

[10] V. Maňasová, P. Pták: On states on the product of logics, Int. J. Theor. Phys. 20 (1981), 451-456.

[11] M. Navara: State space properties of finite logics, to appear in Czechoslovak Math. J., 1987.

[12] P. Pták, V. Rogalewicz: Measures on orthomodular partially ordered sets, J. Pure Appl. Algebra 28 (1983), 75-80.

[13] V. Rogalewicz: Remarks about measures on orthomodular posets, Čas. pěst. mat. 109 (1984), 93-99.

[14] G.T. Rüttimann: Jauch-Piron states, J. Math. Phys. 18 (1977), 189-193.

[15] V. Varadarajan: Geometry of Quantum Theory. Vol.1, Van Nostrand, Princeton, 1968.

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ROOTS LOCALIZATION FOR NEUTRAL EQUATIONS AND THEIR SENSITIVITY FOR A CHANGE OF DELAY

1. Introduction

In this paper quasi-polynomials of the following form

$$H(z) = \sum_{i=0}^{k} z^{i} h_{i}(z), \tag{1.1}$$

where

$$h_{i}(z) = \sum_{\mu \in M_{i}} a_{\mu i} \exp(\mu z), \quad a_{\mu i} \neq 0, \quad \sum_{\mu \in M_{i}} \left|a_{\mu i}\right| < \infty \ \forall i, \tag{1.2}$$

where $M_{i} \subset [0,m] \subset R$ are countable sets, such that

$$\begin{cases} \forall i \quad \exists j \geq i \quad \text{that } \sup M_{j} = \max M_{j} \geq \sup M_{i}, \\ \forall i \quad \exists j \geq i \quad \text{that } \inf M_{j} = \min M_{j} \leq \inf M_{i}, \end{cases} \tag{1.3}$$

are considered.

The localization of roots will be considered for such quasi-polynomials and particularly the limit points of their real parts will be determined. The sensitivity of these points for a change of different retardation will be examined.

2. Roots localization

Let us begin with the following

Theorem 1. Let H(z) be a function in the form described by (1.1), (1.2), (1.3) and let $\tau^0 = \max M_k$, $\tau_0 = \min M_k$. Then:

(i) $\tau^0 < \tau^1 = \max \sum_{i=0}^{k} M_i \Leftrightarrow \exists \{z_i\}_1^\infty$, $\operatorname{re} z_i \to +\infty$ where $H(z_i) = 0$.

(ii) $h_k(z) = \text{const} \exp(\tau^0 z)$ and $\tau^0 = \tau^1 \Leftrightarrow \forall \lambda$ in the half plane $\operatorname{re} z > \lambda$ there exist only a finite number of roots of H(z).

(iii) only $-\infty$, $+\infty$ and real parts of roots of $h_k(z)$ can be the points of accumulation for real parts of roots of H(z).

Proof. Necessity: (i) By contradiction, considering the function $H(z) \exp(-\tau^0 z) \frac{1}{z^k}$.

(ii) There exists x_1 and for any x_0 there exists y_0 such that H(z) has no roots in the following sets $\{(x,y): x \geqslant x_0, y \geqslant y_0\}$, $\{(x,y): x \geqslant x_1\}$, where $z = x + iy$, [2]. Therefore in the half plane $\operatorname{re} z \geqslant x_0$ all roots of H(z) exist only in a bounded set. Thus only a finite number of them could be found therein.

Sufficiency: (i) Due to our assumptions H(z) can be written in the form

$$H(z) = \sum_{i=0}^{k} z^i h_i(z) = \exp(\tau^0 z) \sum_{i=0}^{k} a_i' z^i \exp(\mu_i z) h_i'(z)$$

where $h_i'(z) = 1 + \varepsilon(\operatorname{re} z)$ $(\varepsilon(\operatorname{re} z) \to 0$ if $\operatorname{re} z \to \infty)$.

Now, Rouché's theorem and theorem (12.10) from [2] give the desired conclusion.

(ii) If $\tau^0 < \tau^1$ then from necessity of (i) we obtain a contradiction. If $h_k(z) \neq \text{const} \exp(\tau_0 z)$ then the Hadamard theorem implies that $h_k(z)$ has an infinite number of roots. But $\inf M_k = \min M_k$ and these roots are contained in the strip $\operatorname{re} z \in [\lambda_1, \lambda_2]$ (for some real numbers λ_1, λ_2). From

Rouché's theorem it follows that this strip contains an infinite number of roots of $H(z)$. This contradiction finishes the proof.

(iii) We suppose that there exists a limit point $x_0 = \operatorname{re} z_0 = \lim\limits_{i\to\infty} \operatorname{re} z_i$ and $x_0 \neq -\infty$, $h_k(z_0) \neq 0$. From this it can be deduced that there exist numbers λ_1, λ_2, y_1, M such that $x_0 \in [\lambda_1, \lambda_2]$ and $|H(z)| > M$ for $\{z : \operatorname{re} z \in [\lambda_1, \lambda_2], |\operatorname{im} z| > y_1\}$. Hence it follows that the holomorphic function $H(z)$ has an infinite number of roots in a bounded set. This is the contradiction which proves the theorem.

Example 1. The characteristic quasi-polynomial of the equation

$$\frac{d}{dt}\left[x(t) + cx(t-\tau)\right] = \sum_{i=0}^{k} a_i x(t-\beta_i), \text{ where } c, a_i \in R,\ \beta_i \leq \tau\ \forall i$$

for all $\varepsilon > 0$ and each real number λ has, in the right half plane $\operatorname{re} z > \lambda$, a finite number of zeros outside the strip

$$\left|\frac{1}{\tau} \ln|c| - \operatorname{re} z\right| \leq \varepsilon.$$

Example 2. The characteristic quasi-polynomial for the equation

$$\frac{d}{dt}\left[x(t) + Cx(t-\tau)\right] = \sum_{i=0}^{k} A_i x(t-\beta_i),$$

where C, A_i are real matrices, for all $\varepsilon > 0$ in any right half plane excluding the strips

$$\left|\frac{1}{\tau} \ln|\lambda_i| - \operatorname{re} z\right| \leq \varepsilon$$

has a finite number of roots. The numbers λ_i denote the eigenvalues of matrix C.

Example 3. Now, it will be assumed that C_i, B_j are square real matrices for i = 1,...,r, j = 1,2,..., there exists $\sum_{j=0}^{\infty} B_j$, τ_i are commensurable and $\tau_0 \geq \tau_i \geq \tau_r$, $\tau_0 \geq \beta_j$ $\forall i,j$.

The characteristic quasi-polynomial for the equation

$$\frac{d}{dt}\left[\sum_{i=0}^{r} C_i x(t-\tau_i)\right] = \sum_{j=0}^{\infty} B_j x(t-\beta_j) \tag{2.1}$$

for all $\varepsilon_i > 0$ in any right half plane excluding the strips

$$\left|\frac{1}{\tau}\ln|t_i| - \mathrm{re}\ z\right| \leq \varepsilon_i \quad \text{where} \quad \tau = \frac{\tau_n}{n} \quad \text{and} \quad t_i$$

are the roots of the equation

$$\det\left[\sum_{i=0}^{r} C_i t^{n-i}\right] = 0 \tag{2.2}$$

has a finite numbers of roots.

Corollary 1. On the right-hand side of the vertical line in the complex s-plane there lies a finite number of roots for the quasipolynomial of equation (2.1) if and only if all the roots pf (2.2) lie within the disc $C(0,\exp\gamma) = \{s: |s| < \exp\gamma\}$.

Accordingly to Theorem 1 to characterize the location of zeros of $H(z)$, it is important to designate where the roots of $h_k(z)$ lie and particularly what the least upper bound of real parts of $h_k(z)$ roots is.

The function $h_k(z)$ can be written in the following form

$$h_k(z) = \sum_{i=0}^{n} c_i \exp(-\tau_i z) \quad \text{where} \quad c_0, c_n \neq 0 \ \text{and} \ \tau_0 \geq \dots \geq \tau_n.$$

Theorem 2. If τ_i are pairwise incommensurable then $\inf \mathrm{re}\ z_i = x_1$, $\sup \mathrm{re}\ z_i = x_2$ where z_i denote the roots of $h_k(z)$, x_1 is the unique root of equation $|c_n| =$

$= \sum_{i=0}^{r-1} |c_i| \exp(\tau_n - \tau_i)x$, x_2 is the unique root of equation

$$|c_0| = \sum_{i=1}^{r} |c_i| \exp(\tau_0 - \tau_i)x.$$

Proof. From our assumption it follows that there exists a sequence $\{\Omega_n\}$ such that

$$\forall i \ \Omega_n \tau_i|_{\mathrm{mod}\ 2\pi} \to \varphi_i \quad \text{where} \quad \varphi_i = -(\mathrm{sgn}\ c_i - 1)\frac{\pi}{2}.$$

Let

$$g_n(z) = \sum_{i=1}^{r} c_i \exp(-x\tau_i)\exp(-i(\tau_i + \Omega_n)\tau_i) - |c_0|,$$

$$g_0(z) = \sum_{i=1}^{r} c_i \exp(-x\tau_i)\exp(-i(y\tau_i + \varphi_i)) - |c_0|,$$

$z = x + iy.$

Then the point $z_0 = (x_2, 0)$ is a root of $g_0(z)$ and $g_n(z) \rightrightarrows g_0(z)$ uniformly within some neighborhood F of z_0. This follows from the inequality

$$|g_n(z) - g_0(z)| \leqslant \sum_{i=0}^{r} |c_i| \exp(-x\tau_i) |\exp(-i(\Omega_n \tau_i - \varphi_i)) - 1|.$$

If F is sufficiently small then $g_0(z) \neq 0$ on the boundary of F. Thus, from the Hurwitz theorem it follows that for sufficiently large n $g_n(z)$ has a root in F. Since the neighborhood F can be arbitrarily small, $h_k(z)$ has roots with real parts arbitrarily close to x_2. The second part can be proved by the same method.

Let D denote the set $\{x : x \in [x_1, x_2],\ x \notin \bigcup_{i=2}^{r-1} [x_1^j, x_2^j]\}$, where x_1^j, x_2^j are the roots of the equations

$$|c_j| = \sum_{\substack{i=1 \\ i \neq j}}^{r} |c_i| \exp(\tau_j - \tau_i)x, \quad j = 2, \dots, r-1.$$

Theorem 3. The real parts of roots of $h_k(z)$ are dense in the set D. This theorem can be proved in the same way as the previous one. In this case

$$g_0(z) = \sum_{i \neq j} c_i \exp(-x\tau_i)\exp(-i(y\tau_i - \psi_i)) - |c_j|,$$

where ψ_i are suitably chosen such that $g_0(z) = 0$ for some z, re $z \in D$.

In the case when some of the delays are commensurable the real parts of roots of $h_k(z)$ belong to a certain sum of subintervals contained in D (for instance, in Example 3 the corresponding H(z) will have some commensurable exponents even in case the numbers τ_i are pairwise incommensurable). Then, the function $h_k(z)$ can be written in the form

$$h_k(z) = \sum_{i=1}^{l_1} c_{i1}\exp(-\tau_{i1}z) + \dots + \sum_{i=1}^{l_m} c_{im}\exp(-\tau_{im}z) + c_0 =$$

$$= \quad w_1(z) \quad + \dots + \quad w_m(z) \quad + c_0,$$

where in the function $w_i(z)$ there are only commensurable exponents. Let $W_j(x) = \sup_y |w_j(x+iy)| = w_j(x+iy_0)$ (by commensurability the supremum is found in the compact set). The limit point x_2 can be calculated from the equation

$$|c_0| = \sum_{j=1}^{m} W_j(x). \tag{2.3}$$

In this case it can be noticed that the calculation is more complicated:

1) The functions $W_j(x)$ must be found,

2) The equation (2.3) may have many solutions.

3. Spectrum sensitivity for change of delays

In this section the spectrum and particularly maintenance continuity of the least upper bound of the real part of the spectrum with respect to the change of delays will be considered.

Let $H(z)$ denote a quasi-polynomial in the form (1.1), (1.2), (1.3) with some delays depending on a parameter $\alpha \in [0,r]$, $r > 1$. That is we assume that for some μ in (1.1) we have $\mu = \tau_j = \hat{\tau}_j \alpha$ and the remaining exponents are fixed. The sets M_i, $i = 1,\dots,k$ correspond to $\alpha = 1$ and I is the set of all $\hat{\tau}_j$.

Let $N_\alpha = \{s: H_\alpha(s) = 0,\ \alpha \in [0,r]\}$ for some real number r, $N = \bigcup_{\alpha\in[0,r]} N$. It is obvious that the set N has the following properties $N = \bar{N}$, $N = N_0 \cup N_r$.

Let s_β denote a root of $H_\beta(z)$ and $\gamma \in [0,r]$.

Definition. If there exists a continuous function $f:[\beta,\gamma) \to N$ or $(\gamma,\beta] \to N$ such that $f(\beta) = s_\beta$ and $\lim_{\alpha\to\gamma} |f(\alpha)| = \infty$ then s_β is said to be the p-root in N (parameter root). The set of p-roots will be denoted by pN. If there exists a continuous function $f:[0,r] \to N$ such that $f(\beta) = s_\beta$ then s_β is said to be st-root (structure root). The set of st-roots will be denoted by sN.

Let, for a given α, $g(\alpha)$ denote the least upper bound of real parts of roots of $H_\alpha(z)$, that is,

$$g(\alpha) = \sup\{\operatorname{re} z : H_\alpha(z) = 0\} = \sup \operatorname{re} N_\alpha.$$

Remark. From Theorem 1 it immediately follows that the difference-differential system corresponding to $H_\alpha(z)$ is of retarded or neutral type [1] if and only if $g(\alpha)$ is a finite number.

Let $g_{st}(\alpha)$ denote the least upper bound of real parts of st-roots within N.

Property 1. $N = pN \cup sN$.

This describes the fact that the roots $H_\alpha(z)$ are moving continuously in N with continuous change of parameter α. This can be easily concluded from Rouché's theorem.

Property 2. If for all numbers α belonging to an interval J, closed or not, there exists a number λ that in the right plane re $z \geqslant \lambda$ there is a non-void, finite set of roots of $H_{\alpha}(z)$ then $g(\alpha)$ is continuous on J.

This property can be proved using Property 1 and point (ii) of Theorem 1.

Property 3. If $g(\alpha) \notin \{\mathrm{re}\, z:\ h_k(z)=0\}$ for $\alpha \in J$ then $g(\alpha)$ is continuous on J.

Property 4. The necessary and sufficient condition of continuity of $g(\alpha)$ is the continuity in α of the supremum over the real parts of roots of $h_k(z)$.

Proofs of Property 3 and Property 4 follow from Property 2 and Theorem 1.

Property 5. If $g(\alpha)$ is not continuous at the point α_0 then there exists a sequence $\{(s_n, \alpha_n)\}$ that $H_{\alpha_n}(s_n) = 0$, $\alpha_n \to \alpha_0$, re $s_n \to g(\alpha_0)$ and $|s_n| \to \infty$.

From Property 2 it follows immediately

Corollary 2. The function g is continuous at each point α, for which a corresponding difference-differential system is of retarded type.

In the particular case when $M_k \subset I$ more properties of $g(\alpha)$ could be proved.

Theorem 4. If $M_k \subset I$ then $g(\alpha)$ is continuous on $(0, \infty)$.

Proof of this fact follows from Properties 1 and 4.

Therefore in this case for the continuity verification of $g(\alpha)$ on $[0, \infty)$ we need to check it only at the point $\alpha = 0$.

Now it easy to prove the next theorem.

Theorem 5. If $M_k \subset I$ then

1) If all roots of $h_k(z)$ and their points of accumulation lie on the left hand side of the vertical line re $s = 0$ and $sN \neq \emptyset$ then $g(\alpha)$ is continuous on $[0, \infty)$.

2) If there exists a root or a point of accumulation for real parts of roots of $h_k(z)$ with real part equal zero and all roots of $h_k(z)$ lie on or to the left of the vertical

line re $s = 0$ then necessary and sufficient condition for continuity of $g(\alpha)$ on $[0,\infty)$ is $g_{st}(0) \geqslant 0$.

3) If there exist root of $h_k(z)$ with positive real part then $g(\alpha)$ is discontinuous only at $\alpha = 0$.

Now, the system from Example 3 will be considered. If $I = \bigcup_{i=1}^{k} M_i$ ($\tau_i = \alpha\hat{\tau}_i$ for all $\hat{\tau}_i$ $\bigcup_{i=1}^{k} M_i$) then the necessary and sufficient condition for continuity of $g(\alpha)$ is that all roots of (2.2) lie within the disc $C(0,1)$.

If $M_k \notin I$ and $M_k \cap I \neq \emptyset$ then it is possible that $g(\alpha)$ is not continuous at any point $\alpha \in [0,\infty)$ as the following example demonstrates.

Example 4 (see also [3]). Let $H_\alpha(z)$ correspond to the following differential-difference equation

$$\frac{d}{dt}\left[x(t) + bx(t-\tau_1) + cx(t-\alpha\tau_2)\right] = \varepsilon\left[x(t) + bx(t-\tau_1) + cx(t-\alpha\tau_2)\right].$$

If $\varepsilon = -1$, $b = -\frac{1}{2}$, $c = \frac{1}{2}$ then for τ_1 and $\alpha\tau_2$ incommensurable $g(\alpha) = 0$ but if $l\tau_1 = \alpha\tau_2$ for some natural l then $g(\alpha) = \max_i \operatorname{re} s_i$ where s_i are the roots of equation $\exp(-\tau_1 s) = t$ for t which satisfies $1 - \frac{1}{2}t + \frac{1}{2}t^l = 0$. Thus, the function $g(\alpha)$ is discontinuous at all $\alpha \in [0,\infty)$. If $\varepsilon = 1$, $b = -\frac{1}{2}$, $c = \frac{1}{2}$ then $g(\alpha)$ is continuous at each point $\alpha \geqslant 0$. If $\varepsilon = -1$, $b > 0$, $c = -1$ then $g(\alpha)$ is continuous on $(0,\infty)$ and discontinuous at $\alpha = 0$.

REFERENCES

[1] M. Barszcz: On equivalence relation for linear differential-difference systems, Demonstratio Math. 13 (1980) 951-959.

[2] R. Bellman, K.L. Cooke: Differential-difference equations, Academic Press, New York-London, 1963.

[3] R. Datko: The stabilization of linear functional differential equations, in Calculus of Variations and Control theory, D.L.Russel (editor), Academic Press, New York-London 1976.

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ON DEVIATION FROM AMPLE FLATNESS

0. Introduction

Let M be an n-dimensional $(m > 3)$ Riemannian manifold with a positive-definite metric g_{ij}.

By R_{ijkl}, R_{ij} and R we denote: the covariant curvature tensor, the Ricci tensor and the scalar curvature of the considered manifold, respectively.

Let $f_{i_1 \dots i_p}$ be an skew-symmetric tensor. We say that this tensor is harmonic of type p, if the following conditions are fulfilled (cf. [7])

(0.1)
$$f_{i_1 \dots i_p;j} - f_{j i_2 \dots i_p; i_1} - \dots - f_{i_1 \dots i_{p-1} j; i_p} = 0$$

and

(0.2)
$$f^a{}_{i_2 \dots i_p;a} = 0,$$

where ";" denotes a differentation with respect to the Levi-Civita connection.

By $F\{f_{i_1 \dots i_p}\}$ we denote the quadratic form defined by the formula

(0.3)
$$F\{f_{i_1 \dots i_p}\} = R_{ij} f^{i\, i_2 \dots i_p} f^j{}_{i_2 \dots i_p} + \frac{p-1}{2} R_{ijkl} f^{ij\, i_3 \dots i_p} f^{kl}{}_{i_3 \dots i_p}.$$

For this quadratic form we have the following theorem (cf. [7]).

Theorem 1. For a compact Riemannian manifold M and for a harmonic tensor $f_{i_1\ldots i_p}$ of type p with the property

$$F\left\{f_{i_1\ldots i_p}\right\} \geqslant 0$$

the following equalities are fulfilled

$$f_{i_1\ldots i_p;j} = 0 \quad \text{and} \quad F\left\{f_{i_1\ldots i_p}\right\} = 0.$$

In particular, if the quadratic form $F\left\{f_{i_1\ldots i_p}\right\}$ is positively definite, then there does not exist a nontrivial harmonic tensor of type p. However for the oriented manifold the Betti numbers $B_p = 0$ for $p = 1,2,\ldots,n-1$ (cf. [6]).

A skew-symmetric tensor is called a Killing tensor of type p if the following conditions are satisfied (cf. [7]):

$$(0.4) \qquad p f_{i_1\ldots i_p;j} + f_{j i_2\ldots i_p;i_1} + \ldots + f_{i_1\ldots i_{p-1}j;i_p} = 0$$

in consequence

$$(0.5) \qquad f^{a}{}_{i_2\ldots i_p;a} = 0.$$

For this tensor we have the following theorem:

Theorem 2. Let M be a compact Riemannian manifold. For the corresponding Killing tensor $f_{i_1\ldots i_p}$ of type p with the property

$$F\left\{f_{i_1\ldots i_p}\right\} \leqslant 0$$

the following conditions are fulfilled

$$f_{i_1\ldots i_p:j} = 0 \quad \text{and} \quad F\left\{f_{i_1\ldots i_p}\right\} = 0.$$

In particular, if the quadratic form $F\left\{f_{i_1\ldots i_p}\right\}$ is negatively definite, then there does not exist a nontrivial Killing tensor of type p.

1. Ample manifolds

Let A_{ijkl} be the following tensor

$$A_{ijkl} = R_{ijkl} - a_{ik}g_{jl} + a_{jk}g_{il} - a_{jl}g_{ik} + a_{il}g_{jk}, \tag{1.1}$$

where R_{ijkl} is a covariant curvature tensor and a_{ik} - an arbitrary symmetric tensor.

It is easy to check that the tensor A_{ijkl} fulfills the following conditions:

$$A_{ijkl} = -A_{jikl}, \tag{1.2}$$

$$A_{ijkl} = -A_{ijlk}, \tag{1.3}$$

$$A_{ijkl} = A_{klij}. \tag{1.4}$$

Taking into account the formulae (1.1) and changing indices cyclically we obtain:

$$A_{ijkl} = R_{ijkl} - a_{ik}g_{jl} + a_{jk}g_{il} - a_{jl}g_{ik} + a_{il}g_{jk},$$

$$A_{iklj} = R_{iklj} - a_{il}g_{kj} + a_{kl}g_{ij} - a_{kj}g_{il} + a_{ij}g_{kl},$$

$$A_{iljk} = R_{iljk} - a_{ij}g_{lk} + a_{lj}g_{ik} - a_{lk}g_{ij} + a_{ik}g_{lj}.$$

Adding these equalities side by side we obtain

$$A_{ijkl} + A_{iklj} + A_{iljk} = 0. \tag{1.5}$$

Thus we have showed that A_{ijkl} is a generalized a curvature tensor in the sense of W. Roter's definition introduced in [3].

Definition 1. A covariant curvature tensor A_{ijkl} of the form (1.1) will be called ample tensor.

Definition 2. The Riemannian manifold endowed with the curvature tensor of the form (1.1) will be called the ample manifold.

If $a_{ik} = -Rg_{ik}/2n(n-1)$, then A_{ijkl} is a covariant curvature tensor of the space admitting concircular semi-symmetric metric connection [4].

If $a_{jk} = \frac{-R_{jk}}{n-2} + \frac{Rg_{ik}}{2(n-1)(n-2)}$, then A_{ijkl} is a conformal curvature tensor.

If $a_{jk} = -(\nabla_k p_j - p_j p_k + g_{jk} p^l p_l)$, where p_l is an arbitrary vector, then A_{ijkl} is a covariant curvature tensor of the space admitting a semi-symmetric metric connection (cf. [1], [5]).

Hence, we see that ample spaces comprises a lot of known Riemannian manifolds, and as well as they constitute same subspace of the space with the generalized curvature tensor (cf. [3]).

The ample manifold, with the trivial ample curvature tensor will be called ample flat manifold. As a concrete case of such spaces we can take the concircular-flat spaces as well conformal flat spaces.

2. Deviation from the ample manifold

Let M be the ample manifold. Then

$$R_{ijkl} = A_{ijkl} + a_{jk}g_{il} - a_{ik}g_{jl} + a_{il}g_{jk} - a_{jl}g_{ik}. \tag{2.1}$$

Inserting (2.1) to (0.3) we have

$$\begin{aligned} F\left\{f_{i_1\ldots i_p}\right\} = {} & R_{ij} f^{i i_2\ldots i_p} f^{j}{}_{i_2\ldots i_p} + \\ & + \frac{p-1}{2}(A_{ijkl} - a_{jk}g_{il} + a_{ik}g_{jl} - a_{il}g_{jk} + \\ & + a_{jl}g_{ik}) f^{ij i_3\ldots i_p} f^{kl}{}_{i_3\ldots i_p}. \end{aligned} \tag{2.2}$$

Using the properties of $f^{i_1 i_2\ldots i_p}$ and symmetry of a_{ik} we can rewrite the above equality in the form

(2.3) $$F\{f_{i_1\ldots i_p}\} = R_{ij}f^{ii_2\ldots i_p}f^{j}{}_{i_2\ldots i_p} + \frac{p-1}{2}A_{ijkl}f^{iji_3\ldots i_p}f^{kl}{}_{i_3\ldots i_p} + 2(p-1)a_{jk}f^{ji_2\ldots i_p}f^{k}{}_{i_2\ldots i_p}.$$

Assume that the quadratic form $R_{ij}\xi^i\xi^j$ is positively definite. By L we denote the smallest positive eigenvalue of the metric $\|R_{ij}\|$. Then we have

(2.4) $$R_{ij}\xi^i\xi^j \geqslant L\xi^i\xi_i.$$

To indicate the deformations of ample flat manifolds we introduce some auxiliary notions.

By N we denote $\sup_\xi \dfrac{|a_{ij}\xi^i\xi^j|}{\xi^i\xi_i}$ and by A we denote $\sup_\xi \dfrac{|A_{ijkl}\xi^{ij}\xi^{kl}|}{\xi^{ij}\xi_{ij}}$, where $\xi^{ij} = -\xi^{ji}$.

If we fix up some point of the manifold and consider a coordinate system around it in which $g_{ij} = \delta_{ij}$, then the covariant and contravariant coordinates of the tensor take the same values.

Thus we have

$$R_{ij}f^{ii_2\ldots i_p}f^{j}{}_{i_2\ldots i_p} \geqslant Lf^{i_1\ldots i_p}f_{i_1\ldots i_p},$$

$$a_{ij}f^{ii_2\ldots i_p}f^{j}{}_{i_2\ldots i_p} \geqslant -Nf^{i_1\ldots i_p}f_{i_1\ldots i_p}$$

and

$$A_{ijkl}f^{iji_3\ldots i_p}f^{kl}{}_{i_3\ldots i_p} \geqslant -Af^{i_1\ldots i_p}f_{i_1\ldots i_p}.$$

Hence

$$(2.5)\quad F\{f_{i_1\dots i_p}\} \geq \left[L - 2(p-1)N - \frac{p-1}{2}A\right] f^{i_1\dots i_p} f_{i_1\dots i_p}.$$

In a consequence we have proved

Theorem 1. Let on the compact ample manifold with positive a definite quadratic form $R_{ij}\xi^i\xi^j$ the following inequality hold: $L > \frac{p-1}{2}(4N+A)$. Then there does not exist a nontrivial harmonic tensor of type p.

Similarly, if $R_{ij}\xi^i\xi^j$ is negatively definite quadratic form and by τ we denote maximal eigenvalue (negative) corresponding to the metrix $\|R_{ij}\|$ then

$$(2.6)\quad F\{f_{i_1\dots i_p}\} \leq \left[-\tau + 2(p-1)N + \frac{p-1}{2}A\right] f^{i_1\dots i_p} f_{i_1\dots i_p}.$$

This implies

Theorem 2. Let on the compact ample manifold with negative definite quadratic form $R_{ij}\xi^i\xi^j$ the following inequality hold $L > \frac{p-1}{2}(4N+A)$. Then there does not exist a nontrivial Killing tensor of type p.

BIBLIOGRAPHY

[1] K. Amur, S.S. Pujar: On submanifolds of a Riemannian manifold admitting a metric semi-symmetric connection, Tensor NS 32 (1978) 35-38.

[2] S. Gołąb: Rachunek tensorowy. Warszawa 1966.

[3] W. Roter: On generalized curvature tensors on some Riemannian manifolds, Colloq. Math. 37 (1977) 233-240.

[4] W. Słósarska: On some invariants of Riemannian manifolds admitting a concircularly semi-symmetric metric connection. Demonstratio Math. 17 (1984) 1-7.

[5] K. Yano: On semi-symmetric metric connection, Rev. Roum. Math. Pur. Appl. 15 (1970) 1579-1586.

[6] K. Yano, S. Bochner: Curvature and Betti numbers, Ann. Math. 49 (1948) 379-390.

[7] K. Yano, S. Bochner: Curvature and Betti numbers. Princeton-New Yersey 1953.

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UEBER EINE KLASSE SINGULAERER INTEGRALGLEICHUNGEN MIT ZWEI FESTSTEHENDEN SINGULARITAETEN

Die Untersuchung einer bestimmten Klasse von Rand-Kontaktaufgaben der ebenen Elastizitaetstheorie, sogenannter Bimetallprobleme, kann auf die Betrachtung von singulaeren Integralgleichungen zurueckgefuehrt werden, bei denen neben singulaeren Kernen vom Cauchyschem Typ Kerne mit feststehenden Singularitaeten auftreten [2]. Letztere bestimmen keine vollstetigen Operatoren, sondern haben Einfluss auf die noetherschen Eigenschaften der Gleichung und gehen in das Symbol der Gleichung ein. Als Integraloperatoren mit derartigen Kernen kommen vor

$$(0.1)\qquad \int_{-1}^{1} \frac{(1+x)^{n}(2+x+y)^{2m-1-n}\; u(y)}{\left[(-a(1+x)+b(1+y))^{2} + (2+x+y)^{2}\right]^{m}}\, dy$$

($n \leqslant 2m-1$; $n, m \in N$; a, b...const.) [3], [7].

Fuer den Fall $a = b = 0$ wurden singulaere Integralgleichungen mit Integraloperatoren der Gestalt (0.1) von Dudučava [1] betrachtet. Die vorliegende Arbeit ist der Untersuchung der noetherschen Eigenschaften von Gleichungen mit zwei feststehenden Singularitaeten gewidmet, welche im speziellen Fall Integraloperatoren der Gestalt (0.1) enthalten koennen. Bei der Struktur der Aussagen und der Beweise wird analog zur Methode in [1] vorgegangen. Im ersten Abschnitt der Arbeit wird das Hauptresultat, Aussagen ueber die noetherschen Eigenschaften der betrachteten Gleichungen formuliert, welches im Abschnitt 4 bewiesen wird. In den Abschnitten 2 und 3 werden singulaere Integraloperatoren mit einer feststehenden Singularitaet in verschiedenen Gewichtsraeumen betrachtet. Eine Methode zur expliziten Berechnung des Symbols in gewissen Faellen wird im Abschnitt 5 gegeben.

1. Gegeben sei im Raum $L_p(J^o,(1+x)^{\alpha}(1-y)^{\beta})$ ($J^o = [-1,1]$, $1<p<\infty$; $-1<\alpha,\ \beta<p-1$) der Operator A durch die Gleichung

$$(1.1)\quad (Af)(x) := c_o f(x)+c_1(S_{J^o}f)(x)+(N_{-1}f)(x)+(N_{+1}f)(x)$$

mit

$$(S_{J^o}f)(x) := \frac{1}{\pi i}\int_{-1}^{1}\frac{f(y)}{y-x}\,dy,$$

$$(1.2)\quad (N_{-1}f)(x) :=$$

$$:= \sum_{k=1}^{m}\sum_{n_k^-=0}^{2k-1}\frac{c_k^{(n_k^-)}}{\pi i}\int_{-1}^{1}\frac{(1+x)^{n_k^-}(1+y)^{2k-1-n_k^-}\,f(y)dy}{\left[(-a(1+x)+b(1+y))^2+(2+x+y)^2\right]^k},$$

$$(N_{+1}f)(x) :=$$

$$:= \sum_{k=1}^{m}\sum_{n_k^+=0}^{2k-1}\frac{c_k^{(n_k^+)}}{\pi i}\int_{-1}^{1}\frac{(1-x)^{n_k^+}(1-y)^{2k-1-n_k^+}\,f(y)dy}{\left[(-a(1-x)+b(1-y))^2+(2-x-y)^2\right]^k}$$

$$(c_o, c_1, c_k^{(n_k^-)}, c_k^{(n_k^+)} \ \ldots \ \text{const.}).$$

Bemerkung:

1) Die Kerne der Operatoren (0.1) koennen leicht in die Gestalt (1.2) umgeformt werden.

2) Der Kern des Operators N_{-1} weist eine feststehende Sungularitaet fuer $x = y = -1$ und der Kern des Operators N_{+1} fuer $x = y = 1$ auf.

Dem Operator A wird folgendes Symbol zugeordnet:

$$(1.3)\quad A_{\gamma,\delta}(\lambda,\xi) := \begin{cases} c_0+c_1 s_\gamma(\lambda)+\eta_-(\lambda), & \xi=-\infty\ ,\ -\infty \leq \lambda < +\infty, \\ c_0-c_1 s_\delta(\xi)+\eta_+(\xi), & \lambda=+\infty\ ,\ -\infty < \xi \leq +\infty, \end{cases}$$

wobei $\gamma := (\alpha+1)/p$, $\delta := (\beta+1)/p$, $s_\gamma(\lambda) := \coth \pi(i\gamma+\lambda)$ und $\eta_\mp(\lambda) = (\mathfrak{F}K_\mp)(\lambda)$ mit

$$(K_-f)(t) := \sum_{k=1}^{m} \sum_{n_k^-=0}^{2k-1} \frac{c_k^{(n_k^-)}}{\pi i} \cdot \frac{e^{-(\gamma+n_k^-)t}}{\left[(1+e^{-t})^2+(b-ae^{-t})^2\right]^k},$$

$$(K_+f)(t) := \sum_{k=1}^{m} \sum_{n_k^+=0}^{2k-1} \frac{c_k^{(n_k^+)}}{\pi i} \cdot \frac{e^{-(\delta+n_k^+)t}}{\left[(1+e^{-t})^2+(b-ae^{-t})^2\right]^k}$$

sind. Hierbei bezeichnet $\mathfrak{F}$ die Fouriertransformation, d.h.

$$(\mathfrak{F}K)(\lambda) := \int_{-\infty}^{+\infty} e^{i\lambda t}\, K(t)\,dt.$$

Da $K_\pm(t) \in L_1(-\infty,+\infty)$ gilt, sind die entsprechenden Fouriertransformierten beschraenkte stetige Funktionen mit $(\mathfrak{F}K_\pm)(\pm\infty) = 0$. Wegen $\lim\limits_{\lambda\to\infty} s_\gamma(\lambda) = -\lim\limits_{\xi\to\infty} s_\delta(\xi) = 1$ und

$\lim_{\lambda\to-\infty} s_\gamma(\lambda) = -\lim_{\xi\to+\infty} s_\delta(\xi) = -1$, beschreibt $A_{\gamma,\delta}(\lambda,\xi)$ eine geschlossene Kurve, welche entsprechend wachsendem Argument von λ bzw. ξ orientiert und mit $\mathfrak{m}$ bezeichnet wird. Wir koennen deshalb den funktionentheoretischen Index der Funktion $A_{\gamma,\delta}(\lambda,\xi)$ aufschreiben:

$$\operatorname{ind} A_{\gamma,\delta}(\lambda,\xi) := \frac{1}{2\pi}\left[\arg A_{\gamma,\delta}(\lambda,\xi)\right]\mathfrak{m}.$$

Satz 1.1: Der Operator A (vgl. (1.1)) ist stetig im Raum $L_p(J^o,(1+x)^\alpha(1-x)^\beta)$. Fuer die Gueltigkeit der noetherschen Eigenschaften ist hinreichend und notwendig, dass fuer sein Symbol $A_{\gamma,\delta}(\lambda,\xi)$ gilt:

$$A_{\gamma,\delta}(\lambda,\xi) \neq 0. \tag{1.4}$$

Ist das erfuellt, so gilt $\operatorname{Ind} A = -\operatorname{ind} A_{\gamma,\delta}(\lambda,\xi)$.

Der Beweis dieses Satzes wird im Abschnitt 4 erbracht.

2. Wir wenden uns in diesem Abschnitt dem Fall zu, in dem der Operator A (vgl. (1.1)) nur eine feststehende Singularitaet hat. Zum Beispiel sei der Operator N_{+1} gleich dem Nulloperator.

Ein noch einfacherer Fall liegt vor, wenn N_{-1} und N_{+1} Nulloperatoren sind, d.h.

$$A = c_oI + c_1S_{J^o} \tag{2.1}$$

mit dem Symbol (vgl. (1.3))

$$A_{\gamma,\delta}(\lambda.\xi) = \begin{cases} c_o+c_1s_\gamma(\lambda), & \xi=-\infty,\ -\infty\leq\lambda<+\infty, \\ c_o-c_1s_\delta(\xi), & \lambda=+\infty,\ -\infty<\xi\leq+\infty, \end{cases} \tag{2.2}$$

Der nachfolgende Satz liefert in diesem Fall die Aussage des Satzes 1.1.

Satz 2.1. ([1], Theorem 1.18): Seien $c_o(x)$ und $c_1(x)$ stetige Funktionen auf dem Intervall [a,b]. Der Operator

$B := c_0(x)I+c_1(x)S_{[a,b]}$ ist genau dann noethersch in $L_p([a,b],(a+x)^{\alpha}(b-x)^{\beta})$, wenn fuer sein Symbol

$$B_{\gamma,\delta}(x,\xi) = \begin{cases} c_0(x)+c_1(x), & \xi=+\infty\ ,\ a\leqslant x\leqslant b,\\ c_0(b)-c_1(b)\coth\pi(\xi+i\delta), & x=b,\ -\infty<\xi<+\infty,\\ c_0(x)-c_1(x), & \xi=-\infty\ ,\ b\geqslant x\geqslant a,\\ c_0(a)+c_1(a)\coth\pi(\xi+i\gamma), & x=a,\ -\infty<\xi<+\infty \end{cases}$$

gilt $B_{\gamma,\delta}(x,\xi)\neq 0$ ($x\in[a,b]$, $\xi\in R$). Die Symbolkurve sei so orientiert, dass die Ungleichungen fuer die Argumente auf der rechten Seite von links nach rechts durchlaufen werden. Gilt $B_{\gamma,\delta}(x,\xi)\neq 0$, so ist Ind B = $-$ind $B_{\gamma,\delta}$ und die Invertierbarkeit von B haengt vom Index Ind B ab, d.h. im Falle Ind B $\leqslant 0$ ist B von rechts invertierbar und im Falle Ind B $\geqslant 0$ ist B von links invertierbar. Im Falle Ind B = 0 ergibt sich der inverse Operator durch die Formel

$$B^{-1}f = \frac{c_0}{c_0^2-c_1^2}\,f - \frac{g_-c_1}{c_0+c_1}\,S_{[a,b]}\,\frac{f}{(c_0-c_1)g_-}$$

mit $g_- := \exp\left[\frac{1}{2}\,(I-S_{[a,b]})g\right]$ und $g = \ln\frac{c_0-c_1}{c_0+c_1}$.

Betrachten wir nunmehr den Operator

(2,3)
$$A := c_0I + c_1S_{J^0} + N_{-1}$$

im Raume $L_p(J^0,(1+x)^{\alpha})$. Entsprechend der Beziehung (1.3) ist ihm das Symbol

$$A_{\gamma,1/p}(\lambda,\xi) = \begin{cases} c_0+c_1s_{\gamma}(\lambda)+\eta_-(\lambda), & \xi=-\infty\ ,\ -\infty\leqslant\lambda<+\infty,\\ c_0-c_1s_{1/p}(\xi), & \lambda=+\infty\ ,\ -\infty<\xi\leqslant+\infty \end{cases}$$

zugeordnet.

Satz 2.2. Der Operator A (vgl.(2.3)) ist genau dann noethersch in $L_p(J^o,(1+x)^\alpha)$, wenn fuer sein Symbol $A_{\gamma,1/p}(\lambda,\xi)\neq 0$ gilt. Ist das erfuellt, so gilt Ind A = $-\mathrm{ind}\, A_{\gamma,1/p}$ und die Invertierbarkeit von A haengt vom Index ab.

Beweis. Durch einfache Koordinatentransformation kann der Operator A aequivalent in den Operator $A^o = c_oI+c_1S_J+N^o_{-1}$ im Raume $L_p(J,x^\alpha)$ $(J = [0,1], 1 < p < \infty)$ mit

$$(N^o_{-1}f)(x) := \sum_{k=1}^{m}\sum_{n_k=0}^{2k-1} \frac{c_k^{(n_k)}}{\pi i}\int_0^1 \frac{x^{n_k}y^{2k-1-n_k}f(y)}{[(-ax+by)^2+(x+y)^2]^k}dy$$

ueberfuehrt werden. Dabei wurde $n_k = \bar{n_k}$ gesetzt. Mit Hilfe des Isomorphismus $V:(Vf)(t) := e^{-\gamma t}f(e^{-t}):L_p(J,x^\alpha) \to L_p(R^+)$ kann der Operator A^o in der Form $A^o = V^{-1}\tilde{A}V$ mit $(\tilde{A}u)(t) = c_ou(t) + \int_0^\infty K(t-\tau)u(\tau)d\tau$ geschrieben werden, wobei

$$K(t) = K_1(t) + \sum_{k=1}^{m}\sum_{n_k=0}^{2k-1} K_k^{(n_k)}(t) =$$

$$= \frac{c_1}{\pi i}\frac{e^{-\gamma t}}{1-e^{-t}} + \sum_{k=1}^{m}\sum_{n_k=0}^{2k-1}\frac{c_k^{(n_k)}}{\pi i}\frac{e^{-(\gamma+n_k)t}}{[(1+e^{-t})^2+(b-ae^{-t})^2]^k}$$

ist. Der Operator $\tilde{A}$ ist ein Wiener-Hopf-Operator auf der Halbachse mit unstetigem Subsymbol. Die Theorie solcher Operatoren ist ausfuehrlich in [1] behandelt. Wegen $K_k^{(n_k)}(t)\in L_1(R)$ kann der Operator $\tilde{A}$ in der Form $\tilde{A} = W_{A_\gamma(\lambda)}$ mit $A_\gamma(\lambda) = c_o + \mathcal{F}K$ geschrieben werden. Dabei bezeichnet $W_{a(\lambda)}$ die Einschraenkung des Operators $\mathcal{F}^{-1}a(\lambda)\mathcal{F}$ im Raum $L_p(R)$ auf den Raum $L_p(R^+)$. Aus der Beziehung $A_\gamma(\lambda) = c_o+c_1s_\gamma(\lambda)+\eta_-(\lambda)$ und dem Theorem 4.2 in [1] folgen die Behauptungen des Satzes.

3. Wir betrachten den Operator A (vgl. (2.3)) im Raume $L_p(J^0,(1+x)^{\alpha}(1-x)^{\beta})$ $(1 < p < \infty\,;\ -1 < \alpha,\ \beta < p-1)$. Entsprechend der Beziehung (1.3) ist ihm folgendes Symbol zugeordnet:

$$A_{\gamma,\delta}(\lambda,\xi) = \begin{cases} c_0+c_1 s_{\gamma}(\lambda)+\eta_-(\lambda), & \xi=-\infty\,,\ -\infty \leq \lambda < +\infty, \\ c_0-c_1 s_{\delta}(\xi), & \lambda=+\infty\,,\ -\infty < \xi \leq +\infty\,, \end{cases}$$

Satz '3,1. Der Operator $A^0 = c_0 I+c_1 S_J+N^0_{-1}$ ist genau dann im Raume $L_p(J,x^{\alpha}(1-x)^{\beta})$ noethersch, wenn fuer sein Symbol $A_{\gamma,\delta}(\lambda,\xi) \neq 0$ gilt. Ist das erfuellt, so gilt Ind A^0 = = -ind $A_{\gamma,\delta}(\lambda,\xi)$.

Zum Beweis dieses Satzes werden die Aussagen der nachfolgenden Hilfssaetze benoetigt.

Hilfssatz 3.2 ([5],S.32): Fuer die Zahlen $p, \beta_1,\ldots,\beta_n$ gelte $1 < p < \infty$, $-1 < \beta_k < p-1$ $(k=1,\ldots,n)$ und es sei $g(t) := \prod_{k=1}^{n} |t-t_k|^{\beta_k}$. Wenn $K(\tau,t)$ schwach singulaer ist, dann ist der Operator $(Tf)(t) := \int_{\Gamma} K(\tau,t)f(\tau)d\tau$ vollstetig im Raume $L_p(\Gamma,g(t))$.

Hilfssatz 3.3. Sei N folgender Operator im Raume $L_p(J,g(t))$ (g(t) wie bei Hilfssatz 3.2)

$$(Nf)(x) := \int_0^1 \frac{x^n\, y^{2k-1-n}\quad f(y)}{\left[(-ax+by)^2+(x+y)^2\right]^k}\, dy\,.$$

Weiterhin sei $v_1(x) \in C[0,1]$ eine nichtnegative Funktion, die gleich Null ist in einer Umgebung von x = 0 und gleich 1 ist in einer Umgebung von x = 1. Dann sind folgende Operatoren vollstetig im Raume $L_p(J,g(t))$:

a) $T_1 := v_1(x)N$ und $T_2 := Nv_1(x)I$,

b) $T_3 := a(x)N-Na(x)I$, $a(x) \in C[0,1]$,

c) $T_4 := (1-x)^{\beta/p}\, N(1-x)^{-\beta/p}\, I - N$ $(-1 < \beta < p-1)$.

Beweis. Wir werden in diesem Beweis auftretende verschiedene Konstanten mit dem Buchstaben C bezeichnen. Der Operator T_1 hat folgenden Kern:

$$K_1(x,y) = v_1(x) \frac{x^n y^{2k-1-n}}{[(-ax+by)^2+(x+y)^2]^k}.$$

Da $v_1(x)$ gleich Null ist in einer Umgebung von $x = 0$, so gilt $|K_1(x,y)| < C$ und nach Hilfssatz 3.2 ist T_1 vollstetig in $L_p(J,g(x))$. Analog wird die Vollstetigkeit des Operators T_2 gezeigt.

Wir weisen vorerst nach, dass der Operator T_3 fuer alle Potenzfunktionen x^s ($s \in N$) und damit fuer alle Polynome $a_s(x)$ von hoechtens s-tem Grade in x vollstetig im Raume $L_p(J,g(x))$ ist. Als Kern des Operators T_3 erhalten wir in solchen Faellen:

$$K_3(x,y) = \frac{x^n y^{2k-1-n}(x^s-y^s)}{[(-ax+by)^2+(x+y)^2]^k}.$$

Offensichtlich gilt:

$$\frac{|x^s-y^s|}{((-ax+by)^2+(x+y)^2)^{1/2}} \leqslant \frac{|x^s-y^s|}{x+y} \leqslant \frac{x^s}{x} + \frac{y^s}{y} < C$$

und deshalb ist:

$$|K_3(x,y)| \leqslant C \frac{x^n y^{2k-1-n}}{[(-ax+by)^2+(x+y)^2]^{k-1/2}} \leqslant \frac{x^n y^{2k-1-n}}{(x+y)^{2k-1}} C =$$

$$= C \left(\frac{x}{x+y}\right)^n \left(\frac{y}{x+y}\right)^{2k-1-n} < C.$$

Aus Hilfssatz 3.2 ergibt sich die Vollstetigkeit fuer den Operator T_3, falls $a_s(x)$ ein Polynom in x ist. Wenn $a(x)$

eine beliebige Funktion aus $C[0,1]$ ist, so approximieren wir diese durch Polynome $a_s(x)$ von hoechstens s-tem Grade in x. Fuer solche Polynome sind die Operatoren $T_3^{(s)} := a_s(x)N-Na_s(x)I$ vollstetig und konvergieren ausserdem mit $s \rightarrow \infty$ in der Operatornorm gegen den Operator $T_3 = a(x)N-Na(x)I$, woraus sich die Vollstetigkeit des Operators T_3 ergibt.

Der Operator T_4 hat den Kern

$$K_4(x,y) = \frac{x^n y^{2k-1-n}((1-x)^{\beta/p}(1-y)^{-\beta/p}-1)}{\left[(-ax+by)^2+(x+y)^2\right]^k} .$$

Sei $\beta > 0$ und seien $x,y \geqslant 0$ und hinreichend klein. Dann gilt

$$\frac{|1-(1-x)^{\beta/p}(1-y)^{-\beta/p}|}{\sqrt{(-ax+by)^2+(x+y)^2}} = \frac{|(1-y)^{\beta/p}-(1-x)^{\beta/p}|}{(1-y)^{\beta/p}\sqrt{(-ax+by)^2+(x+y)^2}} \leqslant$$

$$\leqslant C\,\frac{|x-y|}{x+y} \leqslant C.$$

Die weitern Betrachtungen verlaufen wie oben. Die Untersuchung des Falles $\beta < 0$ wird analog durchgefuehrt.

Hilfssatz 3.4. Seien $v_o(x)$ und $v_1(x)$ Funktionen aus $C[0,1]$, die in einer Umgebung des Punktes $x = 1$ bzw. $x = 0$ gleich Null sind. Dann sind die Operatoren

$T_1 := v_1(x)(g_o(x)S_Jg_o(x)^{-1}I-g(x)S_Jg(x)^{-1}I)$ und

$T_2 := v_o(x)(g(x)S_Jg(x)^{-1}I-S_J)$, $(g_o(x) = x^r(1-x)^{\beta/p}$, $g(x) = (1-x)^{\beta/p}$, $-\gamma < r < 1-\gamma)$ vollstetig in $L_p(J,x^\alpha)$.

Beweis. Der Operator T_1 ist genau dann im Raum $L_p(J,x^\alpha)$ vollstetig, wenn der Operator $v_1(x)(x^rS_Jx^{-r}I-S_J)$ im Raum $L_p(J,x^\alpha(1-x)^\beta)$ vollstetig ist. Der Operator $x^rS_Jx^{-r}I$ ist korrekt definiert, wenn wir den Operator $x^{-r}I$ als Operator von $L_p(J,x^\alpha(1-x)^\beta)$ in den Raum $L_p(J,x^{\alpha+rp}(1-x)^\beta)$ und x^rI als Operator in umgekehrter Richtung auffassen. Die Vollstetigkeit kann jetzt wie in Hilfssatz 3.3 gezeigt werden.

Bemerkung. Analog zu Hilfssatz 3.4 kann die Vollstetigkeit folgender Operatoren nachgewiesen werden:
$\hat{T}_1 := (g_o(x)S_Jg_o(x)^{-1}I-g(x)S_Jg(x)^{-1}I)v_1(x)$ und
$\hat{T}_2 := (g(x)S_Jg(x)^{-1}I-S_J)v_o(x)$.

Hilfssatz 3.5. Sei $\tilde{N}_k^{(n)}$ folgender Operator im Raum $L_p(R^+,x^\alpha)$:

$$(\tilde{N}_k^{(n)}f)(x) = \frac{x^n}{\pi i}\int_0^\infty \frac{y^{2k-1-n}\,f(y)dy}{\left[(-ax+by)^2+(x+y)^2\right]^k} \qquad (n \leqslant 2k-1).$$

Dann ist der Operator $T := a(x)\tilde{N}_k^{(n)}-\tilde{N}_k^{(n)}a(x)I$ vollstetig in $L_p(R^+,x^\alpha)$ fuer alle Funktionen $a(x) \in C(R^+)$.

Beweis. Das Funktionensystem $\left\{(x-i)^{j-2+2/p}(x+i)^{-j}\right\}$ $(-\infty < x < +\infty)$ ist vollstaendig in $L_p(R)$. Folglich kann jede Funktion $a(x) \in C(R^+)$ im Raum $L_p(R^+)$ durch Funktionen der Gestalt $a_s(x) = \sum_{j=-s}^{s} d_j(x-i)^{j-2+2/p}(x+i)^{-j}$ approximiert werden. Wie im Beweis des Hilfssatzes 3.3 zeigen wir die Vollstetigkeit des Operators $T_s := a_s(x)\tilde{N}_k^{(n)}-\tilde{N}_k^{(n)}a_s(x)I$. Offensichtlich gilt $\|T-T_s\| \to 0$, woraus die Vollstetigkeit des Operators T folgt.

Weiterhin benoetigen wir zum Beweis des Satzes 3.1 ein bekanntes lokales Prinzip [4]. Sei $\mathcal{A}$ eine Algebra mit Einselement. Die Menge $M \subset \mathcal{A}$ heisst lokalisierende Klasse, wenn gilt: 1. Das Nullelement gehoert nich zu M. 2. Fuer beliebige zwei Elemente f_1 und f_2 aus M existiert ein drittes Element f aus M mit der Eigenschaft $ff_j = f_jf = f$ $(j=1,2)$. Wir nennen die Elemente a und b aus $\mathcal{A}$ M-aequivalent ($a\overset{M}{\sim}b$), wenn $\inf_{f\in M}\|(a-b)f\| = \inf_{f\in M}\|f(a-b)\| = 0$. Das Element a aus $\mathcal{A}$ wird M-invertierbar genannt, wenn Elemente $b \in \mathcal{A}$ und $f \in M$ existieren mit $baf = fab = f$. Das System $\{M_t\}_{t\in T}$ (T...Indexmenge) lokalisierender Klassen heisst ueberdeckend, wenn aus jeder Menge $\{f_t\}_{t\in T}$ $(f_t \in M_t)$ eine endliche Anzahl $f_{t_1},\dots,f_{t_n}$ ausgewaehlt werden kann, deren Summe in $\mathcal{A}$ invertierbar ist.

Hilfssatz 3.6 [5]. Sei $\{M_t\}_{t \in T}$ ein uerberdeckendes System lokalisierender Klassen, $b \in \mathcal{A}$ sei M_t-aequivalent zu $b_t \in \mathcal{A}$ $(t \in T)$ und kommutiere mit allen Elementen von $\bigcup_{t\in T} M_t$. Dann ist fuer die Invertierbarkeit von $b \in \mathcal{A}$ hinreichend und notwendig, dass b_t, fuer alle $t \in T$, M_t-invertierbar ist.

Beweis des Satzes 3.1. Hinlaenglichkeit. Offensichtlich ist der Operator A^o im Raume $L_p(J, x^\alpha(1-x)^\beta)$ dem Operator $B := (1-x)^{\beta/p}A^o(1-x)^{-\beta/p}I$ im Raume $L_p(J,x^\alpha)$ isometrisch aequivalent. Sei B_1 der Operator $B_1 := c_oI+c_1(1-x)^{\beta/p} \times S_J(1-x)^{-\beta/p}I+N^o_{-1}$ im Raume $L_p(J,x^\alpha)$. Nach Hilfssatz 3.3 ist der Operator $B-B_1 = (1-x)^{\beta/p}N^o_{-1}(1-x)^{-\beta/p}I-N^o_{-1}$ vollstetig in $L_p(J,x^\alpha)$, sodass es genuegt, B_1 auf noethersche Eigenschaften zu untersuchen. Dazu benutzen wir das oben angegebene lokale Prinzip. Mit $[L](L_p(J,x^\alpha))$ bezeichnen wir die Faktoralgebra $\mathcal{L}(L_p(J,x^\alpha))/\mathcal{C}_o(L_p(J,x^\alpha))$ und mit $[C]$ die Restklasse, die den Operator $C \in \mathcal{L}(L_p(J,x^\alpha))$ enthaelt. Dabei sind $\mathcal{L}(L_p(J,x^\alpha))$ der Banach-Raum der linearen stetigen Operatoren in $L_p(J,x^\alpha)$ und $\mathcal{C}_o(L_p(J,x^\alpha))$ das Ideal der vollstetigen Operatoren in $\mathcal{L}(L_p(J,x^\alpha))$.

Sei N_t $(t \in [0,1])$ die Menge aller Funktionen $v_t(x) \in C[0,1]$ mit $0 \leqslant v_t(x) \leqslant 1$ und $v_t(x) = 1$ in einer Umgebung von $x = t$. Mit M_t bezeichnen wir die Teilmenge von $[L](L_p(J,x^\alpha))$, die von allen Restklassen der Form $[v_tI]$, $v_t \in N_t$, gebildet werde. M_t ist eine lokalisierende Klasse und das System $\{M_t\}_{t\in[0,1]}$ ist ueberdeckend. Der Operator B_1 ist genau dann noethersch, wenn $[B_1]$ in $[L](L_p(J,x^\alpha))$ invertierbar ist. Nach Hilfssatz 3.6 muss dafuer gezeigt werden, dass eine Schar von M_t-invertierbaren Operatoren $[B_{1,t}]$ $(0 \leqslant t \leqslant 1)$ existiert, fuer die $[B_{1,t}]$ mit allen Elementen aus $\bigcup_{t\in[0,1]} M_t$ kommutiert und $[B_1]$ M_t-aequivalent ist zu allen Klassen $[B_{1,t}]$ $(0 \leqslant t \leqslant 1)$. Wir setzen $B_{1,t} = D = c_oI+c_1g_oS_Jg_o^{-1}I$ fuer $0 < t \leqslant 1$ mit $g_o(x) = x^r(1-x)^{\beta/p}$ und $r = 1-\gamma-\delta$. Wegen der Hilfssaetze 3.3 und 3.4 ist $v_t(x)I(B_1-D)$ fuer $0 < t \leqslant 1$ vollstetig, wenn die Funktion $v_t(x)$

in einer Umgebung von $x = 0$ gleich Null ist. Das gleiche gilt fuer den Operator $(B_1-D)v_t(x)I$. Die Vollstetigkeit der angegebenen Operatoren bedeutet $\|[v_t(x)I(B_1-D)]\|_{[L]} = \|[(B_1-D)v_t(x)I]\|_{[L]} = 0$, was die M_t-Aequivalenz der Operatoren B_1 und D fuer $0 < t \leqslant 1$ nach sich zieht. Offensichtlich ist der Kommutator von D und dem Operator $v_t(x)I$ vollstetig fuer beliebige Funktionen $v_t \in N_t$, d.h. $[D]$ kommutiert mit allen Elementen aus $\bigcup_{t\in[0,1]} M_t$. Der Operator D im Raum $L_p(J,x^\alpha)$ ist isometrisch aequivalent dem Operator $D_1 := c_0I+c_1S_J$ im Raume $L_p(J,x^{p-2-\beta}(1-x)^\beta)$. Der Operator D_1 hat das Symbol

$$(D_1)_{1-\delta,\delta}(\lambda,\xi) = \begin{cases} c_0+c_1\coth\pi(\lambda+i(1-\delta)), & \xi=-\infty, \ -\infty\leqslant\lambda<+\infty \\ c_0-c_1\coth\pi(\xi+i\delta), & \lambda=+\infty, \ -\infty<\xi\leqslant+\infty \end{cases}$$

$$= \begin{cases} c_0-c_1\coth\pi(i\delta-\lambda), & \xi=-\infty, \ -\infty\leqslant\lambda<+\infty \\ c_0-c_1\coth\pi(\xi+i\delta), & \lambda=-\infty, \ -\infty<\xi<+\infty \end{cases}$$

$$= \begin{cases} A_{\gamma,\delta}(+\infty,-\lambda), & +\infty>\lambda\geqslant-\infty \\ A_{\gamma,\delta}(+\infty,\xi), & -\infty<\xi\leqslant+\infty . \end{cases}$$

Aus der Bedingung $A_{\gamma,\delta}(\lambda,\xi) \neq 0$ folgt deshalb $(D_1)_{1-\delta,\delta}(\lambda,\xi)\neq 0$ und ausserdem ergibt sich unmittelbar $\operatorname{Ind} D_1 = 0$. Nach Satz 2.1 ist der Operator D_1 invertierbar und somit ist die Klasse $[D]$ in $[L](L_p(J,x^\alpha))$ invertierbar.

Als Operator $B_{1,o}$ nehmen wir $B_{1,o} := c_0I+c_1S_J+N^o_{-1}$. Wegen Hilfssatz 3.4 ist die Klasse $[B_{1,o}]$ M_o-aequivalent zu $[B_1]$ und kommutiert mit allen Elementen aus $\bigcup_{t\in[0,1]} M_t$. Fuer den Nachweis der M_o-Invertierbarkeit ist die Existenz eines Operators $F \in \mathcal{L}(L_p(J,x^\alpha))$ zu zeigen, fuer den die Operatoren $S_1 := (FB_{1,o}-I)v_o(x)I$ und $S_2 := v_o(x)(B_{1,o}F-I)$ vollstetig sind in $L_p(J,x^\alpha)$. Wir setzen den Operator $B_{1,o}$ durch Null auf die positive Halbachse R^+ fort und erhalten $B_{1,o} = \varkappa_o B_o \varkappa_o$ mit $B_o := c_0I+c_1S_{R^+}+\tilde{N}$ und

$$(\tilde{N}f)(x) := \sum_{k=1}^{m} \sum_{n_k=0}^{2k-1} \frac{c_k^{(n_k)}}{\pi i} \int_0^\infty \frac{x^{n_k} y^{2k-1-n_k} f(y)}{\left[(-ax+by)^2+(x+y)^2\right]^k}\, dy.$$

Dabei bezeichnet $\varkappa_0(x)$ die charakteristische Funktion des Intervalles $[0,1]$. Der Operator B_0 ist invertierbar in $L_p(R^+,x^\alpha)$. Das ergibt sich aus der Beziehung $B_0 = i_0^{-1} W^0_{A_\gamma} i_0$ mit $i_0 f(t) := e^{-\gamma t} f(e^{-t})$ und $A_\gamma = A_{\gamma,\delta}(\lambda,-\infty)$ sowie der Invertierbarkeit des Wiener-Hopf-Operators $W^0_{A_\gamma}$ im Raume $L_p(R)$ ([1], Th.2.18). Wir setzen $F := \varkappa_0 B_0^{-1} \varkappa_0$ und erhalten $S_1 = (FB_{1,0}-I)v_0 I = (\varkappa_0 B_0^{-1} \varkappa_0 B_0 \varkappa_0 - \varkappa_0 I)v_0 I$. Dabei kann $v_0(x)$ so gewaehlt werden, dass $(1-\varkappa_0)v_0 = 0$ gilt. Hieraus ergibt sich $S_1 = \varkappa_0 B_0^{-1}(\varkappa_0 - 1)B_0 v_0 I$. Nach den Hilfssaetzen 3.3 und 3.4 ist der Operator $B_0 v_0 I - v_0 B_0$ vollstetig und somit auch der Operator S_1. Unter Benutzung der Vollstetigkeit des Operators $B_0^{-1} v_0 I - v_0 B_0^{-1}$ kann schliesslich die Vollstetigkeit des Operators S_2 analog gezeigt werden. Somit sind alle Voraussetzungen des Hilfssatzes 3.6 erfuellt, d.h. der Operator A ist noethersch.

Die Notwendigkeit der Voraussetzung $A_{\gamma,\delta}(\lambda,\xi) \neq 0$ und die Indexformel koennen wie in [1] gezeigt werden.

4. Beweis des Satzes 1.1: Hinlaenglichkeit.

Seien $A_{+1} := c_0 I + c_1 S_{J^0} + N_{+1}$ und $A_{-1} := c_0 I + c_1 S_{J^0} + N_{-1}$ Operatoren im Raum $L_0(J^0,(1+x)^\alpha(1-x)^\beta)$. Durch entsprechende Koordinatentransformationen koennen diese aequivalent ueberfuehrt werden in die Operatoren $A^0_{+1} := c_0 - c_1 S_J + N^0_{+1}$ im Raum $L_p(J,x^\beta(1-x)^\alpha)$ bzw. $A^0_{-1} := c_0 I + c_1 S_J + N^0_{-1}$ im Raume $L_p(J,x^\alpha(1-x)^\beta)$ (bzgl. der Bezeichnungen vgl. Abschnitt 3). Wir bemerken, dass Operatoren der Gestalt $A^0_{\pm 1}$ im Abschnitt 3 behandelt wurden. Analog zum Beweis des Satzes 3.1 konstruieren wir ein System lokalisierender Klassen $\{M_t\}$ $(-1 \leq t \leq 1)$ in der Algebra $[L](L_p(J^0,(1+x)^\alpha(1-x)^\beta)$ und zeigen die M_t-Aequivalenzen: $[A] \overset{M_t}{\sim} \left[(1-x)^{-\beta/p} A_{-1} (1-x)^{\beta/p_I}\right] =: \left[B_o\right] \quad (-1 \leq t < 1)$ und

$[A] \overset{M_1}{\sim} \left[(1-x)^{-\alpha/p} A_{+1} (1+x)^{\alpha/p_I}\right] =: \left[B_1\right].$

Ebenfalls wie im Beweis des Satzes 3.1 zeigen wir die Noethereigenschaft fuer die Operatoren B_o und B_1.

Notwendigkeit und Indexformel werden wie in [1] gezeigt, wobei die Aussagen der Hilfssaetze 3.3 - 3.6 benutzt werden.

5. In diesem Abschnitt sollen Moeglichkeiten zur Berechnung der Funktionen $\varkappa_-(\lambda)$ und $\varkappa_+(\lambda)$ (vgl. (1.3)) gezeigt werden. Fuer $\varkappa_-(\lambda)$ gilt $\varkappa_-(\lambda) = (\mathcal{F} K_-)(\lambda) = \int\limits_{-\infty}^{+\infty} e^{i\lambda t} K_-(t)dt$ mit

$$(K_-f)(t) := \sum_{k=1}^{m} \sum_{n_k^-=0}^{2k-1} \frac{c_k^{(n_k^-)}}{\pi i} \frac{e^{-(\gamma+n_k^-)t}}{\left[(1+e^{-t})^2+(b-ae^{-t})^2\right]^k}.$$

Im weiteren berechnen wir

$$b(y) := \mathcal{F}\left(\frac{e^{-\mu t}}{\left[(1+e^{-t})^2+(b-ae^{-t})^2\right]^k}\right)(y) =$$

$$= \int\limits_{-\infty}^{+\infty} \frac{e^{-\mu t}}{\left[(1+e^{-t})^2+(b-ae^{-t})^2\right]^k} e^{iyt} dt.$$

Dabei ist $\mu = \gamma+n_k^-$ oder $\mu = \delta+n_k^-$, Wegen $0 \leq n_k < 2k-1$ und $0<\gamma,\ \delta<1$ ist $0 < \mu < 2k$. Mit Hilfe der Substitution $t = \tau+d$ erhalten wir

$$b(y) = e^{(iy-\mu)d} \int\limits_{-\infty}^{+\infty} \frac{e^{-\mu\tau+iy\tau}}{\left[1+b^2+2e^{-d}(1-ab)e^{-\tau}+(1+a^2)e^{-2d}e^{-2\tau}\right]^k} d\tau.$$

Wir machen den Ansatz:

$$(A+e^{-\tau})(B+e^{-\tau}) = 1+b^2+2e^{-d}(1-ab)e^{-\tau}+(1+a^2)e^{-2d}e^{-2\tau}$$

und erhalten durch Koeffizientenvergleich:

$$d = \frac{1}{2}\ln(1+a^2), \qquad A = (1+a^2)^{-1/2}(1-ab+|a+b|i) \text{ und } B = \bar{A}.$$

A ist genau dann reell, wenn $a = -b$ ist. Unter diesen Umstaenden ist $1-ab = 1+a^2 > 0$. Somit liegen A und B nicht auf der negativen reellen Achse, woraus die Existenz einer Zahl ζ mit $0 \leq \operatorname{Im} \zeta < \pi$, $A = e^{\zeta}$ und $B = e^{\bar{\zeta}}$ folgt. Aus der Tafel [4] oder [6] (S. 155 (22)) folgt

$$b(y) = e^{(iy-\mu)d}B^{\mu-iy-k}A^{-k}\,\frac{\Gamma(\mu-iy)\cdot\Gamma(2k-\mu+iy)}{\Gamma(2k)}\,\times$$

$$\times\,{}_2F_1(k,\mu-iy;2k;1-B/A).$$

Im Spezialfall $a = 0$, $b = 0$ sind $d = 0$, $A = 1$, $B = 1$, $\zeta = 0$, ${}_2F_1(k,\mu-iy;2k;0) = 1$, so dass man erhaelt

$$b(y) = \frac{\Gamma(\mu-iy)\cdot\Gamma(2k-\mu+iy)}{\Gamma(2k)} =$$

$$= \binom{\mu-iy-1}{2k-1}(-1)^{2k}\,\Gamma(-\mu+iy)\cdot\Gamma(\mu-iy+1).$$

Wegen $\Gamma(z)\,\Gamma(1-z) = \pi/\sin(\pi z)$ folgt

$$b(y) = \binom{\mu-iy-1}{2k-1}\cdot(-1)^{2k-1}\,\frac{\pi i}{\sin[\pi(y+\mu i)]}.$$

Das ist das Ergebnis aus [1].

LITERATURVERZEICHNIS

[1] R. Duduchava: Integralequations with fixed singularities. Teubner-Texte zur Mathematik 1979.

[2] L. Jentsch: Ueber ein Bimetallproblem in der Ebene, Zeitschrift fuer Analysis und ihre Anwendungen, Bd. 1, H. 5 (1982), 67-92.

[3] L. Jentsch: Bimetallprobleme mit verschiedenen Kontaktbedingungen, 8. TMP Karl-Marx-Stadt 1983, Teubner-Texte zur Mathematik, Bd. 63, 111-121.

[4] I.M. Ryshik, I.S. Gradstein: Summen-, Produkt- und Integraltafeln, 1957, VEB Deutscher Verlag der Wissenschaften, Berlin.

[5] И.Ц. Гохберг, Н.Я. Крупник: Введение в теорию одномерных интегральных операторов, "Штиинца", 1973.

[6] Г. Бейтмен, А. Эрдейн: Таблицы интегральных преобразований, Наука, Москва 1969.

[7] L. Jentsch: On some applications of integral equations with fixed singularities to boundary-contact-problems in thermoelasticity, Demonstratio Math. 18 (1985) 713-724.

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A UNIFORM INEQUALITY FOR THE FINITE HILBERT TRANSFORM IN WEIGHTED LEBESQUE SPACES

1. Introduction

In [4] M. Rosenblum and J. Rovnyak determined the general form of a nonnegative weight for which the norm of a (sufficiently smooth) function in the corresponding weighted L_2 space is not increasing by applying the finite Hilbert transform. They obtained their result reducing this problem to a generalization of a problem by C. Löwner which they solved first.

In this note we derive an analogous uniform inequality for the finite Hilbert transform and a general family of weights including the weights of Rosenblum and Rovnyak as particular case. Our method of proof is elementary and different from the one used by Rosenblum and Rovnyak thus yielding also a direct proof of the sufficiency part of their result with additional information on the difference in the inequality.

We remark that corresponding inequalities for the Hilbert transform in the periodic case were derived by us in [6], [7]. The existence of such an estimation in this case for a fixed weight satisfying the Helson-Szegö condition follows from [2], Lemma 6, for instance, but emphasis here lies in the explicit form of the constant in the estimation holding uniformly for a sufficiently large subclass of weights. Uniform inequalities of this type are needed in the investigation of corresponding quasi-linear Riemann-Hilbert problems for holomorphic functions (cf. [6], for instance).

2. Statement of result

Theorem. Let g be a nonnegative (not identically vanishing) function on $[-a,a]$ of the form

$$(1)\qquad g(x) = (a^2-x^2)^{-\frac{1}{2}}\left[p+qx - \int_{-1/a}^{1/a} \frac{x^2}{1-xt}\, d\beta(t)\right]$$

with real constants $p(\geq 0)$, q and a non-decreasing function β on $[-1/a, 1/a]$ being constant in a neighbourhood of the endpoints $\pm 1/a$, respectively. For any function $\mu \in L_\infty(-a,a)$ with

$$(2)\qquad |\mu(x)| \leq \gamma < \pi/4 \quad \text{a.e. on } [-a,a]$$

define the weight $\rho = \rho[\mu]$ by $\rho = g\exp(2S\mu)$, where S is the finite Hilbert (or Cauchy) operator

$$(S\mu)(x) = \frac{1}{\pi}\int_{-a}^{a} \frac{\mu(\xi)}{\xi - x}\, d\xi .$$

Then for any Hölder continuous function ψ on $[-a,a]$ there holds the inequality

$$(3)\qquad \|S\psi\|^2_{L_{2,\rho}} \leq C_\gamma^2 \|\psi\|^2_{L_{2,\rho}} - \pi C_\gamma A,$$

where $\|\psi\|_{L_{2,\rho}} := \left(\int_{-a}^{a} \rho(x)\psi^2(x)\,dx\right)^{\frac{1}{2}}$. The positive constant C_γ (≥ 1) is defined by

$$(4)\qquad C_\gamma := \frac{1 + \tan\gamma}{1 - \tan\gamma}$$

and the nonnegative constant $A = A[\mu,\psi]$ is given by

$$A = \int_{-1/a}^{1/a} \frac{1/t^2}{\sqrt{1-a^2t^2}}\,\chi^2(1/t)\,d\beta(t), \tag{5}$$

where

$$\chi(z) = \exp(\Omega(z))\cdot\Psi(z), \quad z = x + iy \notin [-a,a], \tag{6}$$

with the complex Cauchy integrals

$$\Omega(z) = \frac{1}{\pi}\int_{-a}^{a} \frac{\mu(\xi)}{\xi - z}\,d\xi, \tag{7}$$

$$\Psi(z) = \frac{1}{\pi}\int_{-a}^{a} \frac{\psi(\xi)}{\xi - z}\,d\xi. \tag{8}$$

In particular, there holds the uniform inequality

$$\|S\psi\|_{L_{2,\rho}} \leqslant C_\gamma \|\psi\|_{L_{2,\rho}} \tag{9}$$

for any Hölder continuous function ψ on $[-a,a]$.

Remark. The weight ρ is integrable, since, due to (2), the function $(a^2-x^2)^{-1/2}\exp(2S\mu)$ belongs to $L_1(-a,a)$ as follows from the lemma in the appendix of [8] (which there is proven for continuous functions, but the proof can easily be extended to bounded measurable functions).

Corollary. Since the Hölder continuous functions are dense in the weighted Lebesgue space $L_{2,\rho}$, the operator S can be continuously extended onto $L_{2,\rho}$ such that the inequality (9) remains valid for any $\psi \in L_{2,\rho}$.

Proof. We put $\varphi = S\psi$ and write (3) in explicit form:

$$\frac{1}{\pi}\int_{-a}^{a} g\, e^{2S\mu}\varphi^2 dx \leqslant C_\gamma^2\,\frac{1}{\pi}\int_{-a}^{a} g\, e^{2S\mu}\psi^2 dx - C_\gamma A. \tag{10}$$

In virtue of the assumption (2), we have the estimation (cf. [7], Appendix 2)

$$\cos 2\mu\cdot\left[\varphi^2-\psi^2\right] - \sin 2\mu \cdot 2\varphi\psi \geqslant C_{\gamma}^{-1}\varphi^2 - C_{\gamma}\,\psi^2.$$

Therefore, (10) follows from the relation

$$(11) \qquad \frac{1}{\pi}\int_{-a}^{a} g\, e^{2S\mu}\left\{\cos 2\mu\cdot\left[\varphi^2-\psi^2\right] - \sin 2\mu\cdot 2\varphi\psi\right\}dx = -A,$$

taking into account that g is assumed nonnegative. In the sequel we prove (11) for any (not necessarily nonnegative) function g of the form (1) and any Hölder continuous ψ on $[-a,a]$.

At first we take μ as a Hölder continuous function on $[-a,a]$ satisfying (2). Moreover, we start with functions g of the special form

$$(12) \qquad g_\lambda(x) = (a^2-x^2)^{-\frac{1}{2}}\left[p+qx-\beta_\lambda\,\frac{x^2}{1-\lambda x}\right], \qquad \lambda\in(-1/a,1/a),$$

with arbitrary real constant β_λ, i.e. with functions g of the form (1) and $\beta(t) = \beta_\lambda\cdot\theta(t-\lambda)$, where θ means the Heaviside step function.

The functions Ω, Ψ defined by (7), (8), respectively, are holomorphic functions outside the slit $L_0 = [-a,a]$ of the real axis and satisfy $\Omega(\infty) = \Psi(\infty) = 0$. By the Plemelj formulae (cf. [3], Chap. 2, § 17) we have

$$\Omega^{\pm}(x) = \pm\, i\mu + S\mu, \quad \Psi^{\pm}(x) = \pm\, i\,\psi + \varphi$$

for their boundary values on L_0 from above and below, respectively. Therefore, the function χ defined by (6) is a holomorphic function outside L_0, satisfies $\chi(\infty) = 0$ and has the boundary values

$$(13) \qquad \chi^{\pm}(x) = e^{S\mu}\, e^{\pm i\mu}\left[\pm\, i\,\psi + \varphi\right].$$

on L_0. More precisely, at infinity there holds the asymptotic relation

$$\chi(z) \sim -\psi_0/z, \quad \psi_0 = \frac{1}{\pi}\int_{-a}^{a}\psi(\xi)\,d\xi. \tag{14}$$

We further introduce the functions G_λ, $\lambda \in (-1/a, 1/a)$, by

$$G_\lambda(z) = -(z^2-a^2)^{-\frac{1}{2}}\left[p+qz-\beta_\lambda\frac{z^2}{1-\lambda z}\right]. \tag{15}$$

Here $(z^2-a^2)^{\frac{1}{2}}$ means the branch of the root with $(z^2-a^2)^{\frac{1}{2}} \sim z$ for large $|z|$. The functions G_λ are holomorphic outside the slit L_0 and satisfy the asymptotic relations

$$G_0(z) \sim \beta_0 z, \quad G_\lambda(z) \sim -\beta_\lambda/\lambda - q \quad \text{if} \quad \lambda \in (-1/a, 0) \cup (0, 1/a) \tag{16}$$

at infinity.

Taking into account (14) and (16) by the residue theorem we have

$$\frac{1}{2\pi i}\int_L \chi^2(\zeta)\,G_0(\zeta)\,d\zeta = -\beta_0\psi_0^2 \tag{17}$$

for any simple closed smooth contour L encircling the slit L_0 in clockwise direction and since $z^2/[1-\lambda z] = -(z^2/\lambda)/[z-(1/\lambda)]$ and $(z^2-a^2)^{\frac{1}{2}} = \pm([1/\lambda^2]-a^2)^{\frac{1}{2}}$ for $\lambda \in (0, 1/a)$ and $\lambda \in (-1/a, 0)$, respectively,

$$\frac{1}{2\pi i}\int_L \chi^2(\zeta)G_\lambda(\zeta)\,d\zeta = \mp\beta_\lambda([1/\lambda^2]-a^2)^{-\frac{1}{2}}(1/\lambda^3)\cdot\chi^2(1/\lambda) \tag{18}$$

if $\lambda \in (0, 1/a)$ and $\lambda \in (-1/a, 0)$, respectively, for any such contour L not including the point $1/\lambda$ of the real axis.

We now contract the contour L to the slit L_0 thereby encircling the points $\mp a$ by circles with sufficiently small radii going to zero. If the function ψ vanishes at the points $\pm a$, near these points we have

(19) $$\chi^2(\zeta)(\zeta^2-a^2)^{-\frac{1}{2}} \sim \Psi^2(\pm a)(\zeta^2-a^2)^{-\frac{1}{2}} \times$$
$$\times \left(\frac{a-\zeta}{a+\zeta}\right)^{\frac{2}{\pi}\mu(\pm a)} \exp\left(\frac{2}{\pi}\int_{-a}^{a}\frac{\mu(\xi)-\mu(\pm a)}{\xi\mp a}\,d\xi\right)$$

so that, in virtue of (2), for x = ±a

(20) $$\left|\chi^2(\zeta)(\zeta^2-a^2)^{-\frac{1}{2}}\right| = O(|\zeta\mp a|^{-\delta})$$

with $\delta = \frac{1}{2} + 2\gamma/\pi < 1$.

If ψ does not vanish at the points $\pm a$, the expression $\Psi(\pm a)$ in (19) has to be replaced by

(21) $$\psi(\pm a)\,\frac{1}{\pi}\ln\frac{|a-z|}{|a+z|} + \frac{1}{\pi}\int_{-a}^{a}\frac{\psi(\xi)-\psi(\pm a)}{\xi - a}\,d\xi,$$

so that (20) holds with arbitrary $\delta > \frac{1}{2} + 2\gamma/\pi$ which can be chosen smaller than 1, too.

Therefore, we obtain the relation

(22) $$\frac{1}{2\pi}\int_{-a}^{a}\left[{\chi^{+}}^{2}(x) + {\chi^{-}}^{2}(x)\right] g_\lambda(x)\,dx = -A_\lambda$$

with the constant A_λ given by

(23) $$A_0 = \beta_0\psi_0^2 = \beta_0 \lim_{u\to\pm\infty}\left[u^2\chi^2(u)\right] =$$
$$= \beta_0 \lim_{t\to 0}\left[\frac{1/t^2}{\sqrt{1-a^2t^2}}\,\chi^2(1/t)\right],$$

where we have used (14), and

(24) $$A_\lambda = \mp\beta_\lambda\left(\left[1/\lambda^2\right]-a^2\right)^{-\frac{1}{2}}(1/\lambda^3)\cdot\chi^2(1/\lambda) =$$
$$= \beta_\lambda(1/\lambda^2)\chi^2(1/\lambda)\Big/\sqrt{1-a^2\lambda^2}$$

for $\lambda\in(0,1/a)$ and $\lambda\in(-1/a,0)$, respectively.

By (13), the relation (22) becomes

$$\frac{1}{\pi}\int_{-a}^{a} g_\lambda e^{2S\mu}\left\{\cos 2\mu\cdot\left[\varphi^2-\psi^2\right] - \sin 2\mu\cdot 2\varphi\psi\right\}dx = -A_\lambda, \tag{25}$$

what, by comparison of (5) for $\beta(t) = \beta_\lambda\cdot\theta(t-\lambda)$ with (23), (24), proves the relation (11) for the functions (12). In particular, we have

$$\frac{1}{\pi}\int_{-a}^{a} \hat{g}\, e^{2S\mu}\left\{\cos 2\mu\cdot\left[\varphi^2-\psi^2\right] - \sin 2\mu\cdot 2\varphi\psi\right\}dx = 0 \tag{26}$$

for functions $\hat{g}$ of the form

$$\hat{g}(x) = (a^2-x^2)^{-\frac{1}{2}}\,[p+qx] \tag{27}$$

with arbitrary real constants p, q (for $\mu \equiv 0$ cf. also [4], Prop. 3). Taking into account (26) and integrating (25) with respect to the measure $d\beta$, we obtain (11) for an arbitrary function g of the form (1).

If μ is merely a bounded measurable function on $[-a,a]$, the preceding derivation also goes through with the following modifications: firstly, by the general Plemelj-Privalov formulae (cf. [1], Chap. 10, § 3) the boundary relation (13) holds almost everywhere on $[-a,a]$; secondly, the estimation (20) is obtained directly from (2) for $x \in [-a,a]$:

$$\left|\chi^2(\zeta)\,(\zeta^2-a^2)^{-\frac{1}{2}}\right| \leqslant$$

$$\leqslant (K_0+K_1\cdot|\ln|\zeta-a||)\,|\zeta^2-a^2|^{-\frac{1}{2}}\cdot|\zeta-a|^{-2\gamma/\pi} \leqslant K_2|\zeta-a|^{-\delta}$$

with $\delta > \frac{1}{2} + 2\gamma/\pi$ and positive constants K_0, K_1, K_2 in the vicinity of $\zeta = a$ and analogously for the point $-a$. Thus, the relation (11) and hence the theorem are completely proved.

3. Particular cases

In the special case $\mu \equiv 0$ with $\gamma = 0$ and $C_\gamma = 1$ the inequality (9) with $\varphi = S\psi$ becomes

$$\int_{-a}^{a} g(x)\varphi^2(x)\,dx \leqslant \int_{-a}^{a} g(x)\psi^2(x)\,dx \tag{28}$$

which is the inequality by Rosenblum and Rovnyak ([4], Prop.4). It is sharpened by the corresponding relation (3) which in this particular case is an identity as follows from (11). I.e., we have the relation

$$\int_{-a}^{a} g(x)\varphi^2(x)\,dx = \int_{-a}^{a} g(x)\psi^2(x) - \pi A, \tag{29}$$

where the constant A is given by (5) with the function $\chi(z) = \Psi(z)$ defined by (8).

We further consider some simple particular cases of (3) with special choices for the function g.

1) $p = 1$, $q = 0$, $\beta(t) \equiv 0$ yields $g(x) = (a^2-x^2)^{-\frac{1}{2}}$ and we obtain the inequality

$$\int_{-a}^{a} \frac{e^{2S\mu}\,\varphi^2(x)}{\sqrt{a^2-x^2}}\,dx \leqslant C_\gamma \int_{-a}^{a} \frac{e^{2S\mu}\,\psi^2(x)}{\sqrt{a^2-x^2}}\,dx. \tag{30}$$

2) $p = a^2$, $q = 0$, $\beta(t) = \theta(t)$, the Heaviside step function, yields $g(x) = (a^2-x^2)^{\frac{1}{2}}$ and we get

$$\int_{-a}^{a} e^{2S\mu}\sqrt{a^2-x^2}\,\varphi^2(x)\,dx \leqslant \tag{31}$$

$$\leqslant C_\gamma \int_{-a}^{a} e^{2S\mu}\sqrt{a^2-x^2}\,\psi^2(x)\,dx - \frac{C_\gamma}{\pi}\left(\int_{-a}^{a}\psi(x)\,dx\right)^2.$$

3). $p = a,\ q = 1,\ \beta(t) \equiv 0$ and $p = a,\ q = -1,\ \beta(t) \equiv 0$ yields $g(x) = (a-x)^{-\frac{1}{2}}(a+x)^{\frac{1}{2}}$ and $g(x) = (a-x)^{\frac{1}{2}}(a+x)^{-\frac{1}{2}}$, respectively, so that we obtain

$$\int_{-a}^{a} e^{2S\mu}\sqrt{\frac{a+x}{a-x}}\,\varphi^2(x)dx \leqslant C_\gamma \int_{-a}^{a} e^{2S\mu}\sqrt{\frac{a+x}{a-x}}\,\psi^2(x)dx, \tag{32}$$

$$\int_{-a}^{a} e^{2S\mu}\sqrt{\frac{a-x}{a+x}}\,\varphi^2(x)dx \leqslant C_\gamma \int_{-a}^{a} e^{2S\mu}\sqrt{\frac{a-x}{a+x}}\,\psi^2(x)dx. \tag{33}$$

The inequalities (30)-(33) hold for any $\psi \in L_{2,\rho}$. Namely, in virtue of the lemma in the appendix of [8], in all three cases one has $\rho^{-1} \in L_r$ for some $r > 1$. Therefore, by Hölder s inequality, $L_{2,\rho}$ is continuously imbedded in L_s, $s = 2r/(1+r) > 1$. Hence, if a sequence of Hölder continuous ψ_n converges in $L_{2,\rho}$ to a prescribed $\psi \in L_{2,\rho}$, there follows that $\varphi_n = S\psi_n$ converges to $\varphi = S\psi$ in L_s and a subsequence of φ_n converges almost everywhere to φ. The integrals over ψ_n on the right sides of (30)-(33) converge to the corresponding integrals over ψ and Fatou´s lemma applied to the integrals on the left sides of (30)-(33) then implies the validity of these inequalities for any $\psi \in L_{2,\rho}$.

The corresponding equalities to (30)-(33) in case $\mu \equiv 0$ have been derived by M. Schleiff [5].

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REFERENCES

[1] G.M. Golusin : Geometrische Funtionentheorie, Berlin 1957.

[2] R. Hunt, B. Muckenhoupt, R. Wheeden: Weighted norm inequalities for the conjugate function and Hilbert transform, Trans. Amer. Math. Soc. 176 (1973) 227-251.

[3] N.I. Muskhelishvili: Singular integral equations, Groningen 1953.

[4] M. Rosenblum, J. Rovnyak: Two theorems on finite Hilbert transforms, J. Math. Anal. Appl. 48 (1974) 708-720.

[5] M. Schleiff: Singulare Integraloperatoren in Hilbert-Räumen mit Gewichtsfunktion, Math. Nachr. 42 (1969) 145-155.

[6] E. Wegert, L.v. Wolfersdorf: On a class of quasi-linear Riemann-Hilbert problems, Z. Anal. Anwendungen (to appear).

[7] L.v. Wolfersdorf: On the theory of the nonlinear Riemann-Hilbert problem for holomorphic functions, Complex Variables 3 (1984) 323-346.

[8] L.v. Wolfersdorf: On the theory of nonlinear singular integral equations of Cauchy type, Math. Meth. in the Appl. Sci. 7 (1985) 493-517.

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ESTIMATORS OF FUNCTIONS IN SOME DIFFERENCE EQUATIONS

In this paper we are concerned with the following estimation problem. Let

$$(1)\qquad \begin{cases} X_{n+1} = X_n + f(X_n) + b(X_n)\cdot Y_{n+1} \\ X_0 = x_0 \end{cases} \qquad n=1,2,3,\ldots$$

be the system of one dimensional difference equations where $Y_1, Y_2, \ldots$ are independent random variables and the functions f, b are unknown. In this paper we are to estimate these functions.

The process (X_n) described by (1) satisfies also the following difference equations

$$(2)\qquad X_n = \sum_{k=0}^{n-1} f(X_k) + \sum_{k=0}^{n-1} b(X_k)\cdot Y_{k+1}$$

T.S. Lee, F. Kozin in [3], Prakasa-Rao in [4], [5] and many others have found some estimators of the drift for diffusion processes which satisfy a stochastic equation

$$(3)\qquad d\xi_t + a(\xi_t)dt + \sigma(\xi_t)dW_t,$$

where (W_t) is the Wiener process. The process (X_n) seems to approach the process ξ_t. Having estimators $\hat{f}, \hat{b}$ of f, b it is easier to find the form of estimators $\hat{a}, \hat{\sigma}$ of a, σ in (3). Devroye [1] in his paper estimates the conditional expectation of random variables. Following his method we shall try to estimate the unknown functions f and b. We assume that the following conditions are satisfied:

(4) f and b are continuous real functions defined on the set of real numbers R.

(5) There exists a constant M such that for every x we have

$$|f(x)| + |b(x)| \leq M.$$

(6) For every real numbers $x_1 < x_2$ we have

$$P\{\omega \in \Omega \text{ there exists } n \text{ such that } x_1 \leq X_n(\omega) \leq x_2\} = 1.$$

(7) $Y_1, Y_2, Y_3, \ldots$ are independent random variables with $EY_i = 0$ and $EY_i^2 = 1$.

The condition (6) means that the process X_n is to be in the neighbourhood of every x to estimate f(x) and b(x).

In other words almost every trajectory $X(\omega)$ must be dense in the set of all real numbers R.

The problem is: how to estimate the functions f and b having the results $X_1, X_2, \ldots, X_n$ only.

Let us define a sequence $h_n = h_n(x)$ as follows

$$(8) \qquad h_n(x) = \begin{cases} 1 & \text{if there is no } i \leq n \text{ such that } x-1 \leq X_i \leq x+1 \\ \dfrac{1}{d_n} & \text{if there are the unique natural numbers } i_1, i_2, i_3, \ldots, i_{d_n} \end{cases}$$

such that $1 \leq i_1 < i_2 < i_3 < \ldots < i_{d_n} \leq n$ and we have $x - \frac{1}{j} \leq X_{i_j} \leq x + \frac{1}{j}$, $j = 1, 2, \ldots, d_n$.

In this case $\lim_{n\to\infty} h_n(x) = 0$ with probability one for all x. We will consider the following estimators

$$(9) \qquad \hat{f}_n(x) = \sum_{i=1}^{n} \frac{K\left(\frac{X_i - x}{h_i}\right) \cdot (X_{i+1} - X_i)}{\sum_{j=1}^{n} K\left(\frac{X_j - x}{h_j}\right)},$$

(10)
$$\hat{b}_n^2(x) = \sum_{i=1}^{n} \frac{K\left(\frac{X_i - x}{h_i}\right) \cdot (X_{i+1} - X_i)^2}{\sum_{j=1}^{n} K\left(\frac{X_j - x}{h_j}\right)} - (\hat{f}_n(x))^2,$$

where

$$K(x) = \begin{cases} 1 & \text{if } |x| \leq 1 \\ 0 & \text{if } |x| > 1. \end{cases}$$

For simplicity we make the following notations

$$W_{n,i}(x) = \frac{K\left(\frac{X_i - x}{h_i}\right)}{\sum_{j=1}^{n} K\left(\frac{X_j - x}{h_j}\right)} \quad \text{with } \frac{0}{0} = 0.$$

Then the estimators $\hat{f}_n(x)$ and $\hat{b}_n(x)$ have the short forms

(11)
$$\hat{f}_n(x) = \sum_{i=1}^{n} W_{n,i}(x)(X_{i+1} - X_i),$$

(12)
$$\hat{b}_n^2(x) = \sum_{i=1}^{n} W_{n,i}(x)(X_{i+1} - X_i)^2 - (\hat{f}_n(x))^2.$$

We formulate now the main theorem of this paper.

Theorem. Let the system (1) be given. Under the assumptions (4), (5), (6), (7) we have

(13)
$$\lim_{n \to \infty} \hat{f}_n(x) = f(x) \quad \text{a.s. (almost surely)}$$

(14)
$$\lim_{n \to \infty} \hat{b}_n^2(x) = b^2(x) \quad \text{a.s.} \quad \text{for every } x \in R.$$

To prove this theorem we shall formulate the following lemma.

Lemma 1 (Szabłowski [2]). Let $\{a_n\}_{n=1}^{\infty}$, $\{x_n\}_{n=1}^{\infty}$ be two sequences with $a_n \geq 0$ such that the series

$$\sum_{n=1}^{\infty} \frac{a_n}{\sum_{i=1}^{n} a_i} x_n \quad \text{converges,} \quad \lim_{n\to\infty} \frac{a_n}{\sum_{i=1}^{n} a_i} = 0$$

and

$$\sum_{n=1}^{\infty} \frac{a_n}{\sum_{i=1}^{n} a_i} = \infty \quad \text{then we have} \quad \lim_{n\to\infty} \frac{\sum_{i=1}^{n} a_i x_i}{\sum_{j=1}^{n} a_j} = 0.$$

Proof. See [2].

Proof of the Theorem. First we shall prove that $\lim_{n\to\infty} \hat{f}_n(x) = f(x)$ a.s. We have

$$\left|\hat{f}_n(x) - f(x)\right| = \left|\sum_{i=1}^{n} W_{n,i}(x)(X_{i+1} - X_i) - \sum_{i=1}^{n} W_{n,i}(x) f(x)\right| =$$

$$= \left|\sum_{i=1}^{n} W_{n,i}(x)\cdot\left[f(X_i) - f(x) + b(X_i)Y_{i+1}\right]\right| \leqslant \left|\sum_{i=1}^{n} W_{n,i}(x)\left(f(X_i) - f(x)\right)\right| +$$

$$+ \left|\sum_{i=1}^{n} W_{n,i}(x) b(X_i) Y_{i+1}\right| = |J_n| + |I_n|.$$

Let (Ω, F, P) be a probability space on which the system (1) is defined. We can find $\Omega_0 \subset \Omega$ with $P(\Omega_0) = 1$ such that for every $\omega \in \Omega_0$ we have $\lim_{n\to\infty} h_n(x) = 0$ for all rational numbers x. It is easy to verify that also $\lim_{n\to\infty} h_n(x) = 0$ for all real numbers $x \in R$ if $\omega \in \Omega_0$.

Now we can estimate J_n. Let $\omega \in \Omega_0$, $\varepsilon > 0$. Then there exists a natural number n_1 such that $|X_i - x| < h_n$ implies $|f(X_i) - f(x)| < \frac{\varepsilon}{2}$ for $n \geqslant n_1$. There exists also $n_0 > n_1$ such that for $n \geqslant n_0$ we have $2M\, d_{n_1}/d_n < \varepsilon/2$ because $\lim_{n\to\infty} d_n = \infty$. For $n \geqslant n_0$ we have

$$|J_n| = \left|\sum_{i=1}^{n} W_{n,i}(x)\big(f(X_i)-f(x)\big)\right| \leqslant \sum_{i=1}^{n_1} W_{n,i}(x)\left|f(X_i)-f(x)\right| +$$

$$+\sum_{i=n_1+1}^{n} W_{n,i}(x)\left|f(X_i)-f(x)\right| \leqslant 2M\, d_{n_1}/d_n + \frac{\varepsilon}{2} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

Therefore $J_n \to 0$ a.s.

Now we are to prove that $I_n \to 0$ where $I_n = \sum_{i=1}^{n} W_{n,i}(x)\cdot b(X_i)\cdot Y_{i+1}$. Let F_n denote the σ-field generated by $X_1, X_2, \ldots, X_n, Y_1, Y_2, \ldots, Y_n$. Let us consider another process

$$A_n = \sum_{m=1}^{n} W_{m,m}(x)\, b(X_m)\, Y_{m+1}.$$

It is easy to see that $\{A_n, F_{n+1}\}$ is a martingale. We will show that the martingale $\{A_n, F_{n+1}\}$ is convergent. It suffices to show that $\sup_n E|A_n| < \infty$. We have

$$\sup_n E|A_n| = \sup_n E\left|\sum_{m=1}^{n} W_{m,m}(x)\cdot b(X_m)\cdot Y_{m+1}\right| \leqslant$$

$$\leqslant \sup_n \left[E\left|\sum_{m=1}^{n} W_{m,m}(x)\cdot b(X_m) Y_{m+1}\right|^2\right]^{\frac{1}{2}} \leqslant$$

$$\leqslant \sup_n \left\{\sum_{m=1}^{n} E\left[W_{m,m}(x)^2\, b(X_m)^2\right] E Y_{m+1}^2\right\}^{\frac{1}{2}} \leqslant$$

$$\leqslant \sup_n M\left(\frac{1}{1} + \frac{1}{4} + \frac{1}{9} + \ldots + \frac{1}{d_n^2}\right)^{\frac{1}{2}} < \infty.$$

Therefore $\lim_{n\to\infty} A_n = A$ a.s. and using Lemma 1 we obtain

$$\lim_{n\to\infty} \sum_{i=1}^{n} W_{n,i}(x) b(X_i) Y_{i+1} = 0 \quad \text{a.s.}$$

Finally $\lim\limits_{n\to\infty} I_n = 0$ a.s. We have proved that $\lim\limits_{n\to\infty}\left|\hat{f}_n(x)-f(x)\right|=0$ a.s.

Now we are to prove that for every $x \in R$ we have

$$\lim_{n\to\infty}\left|\hat{b}_n^2(x) - b^2(x)\right| = 0 \quad \text{a.s.}$$

Let us apply the following estimation

$$\left|\hat{b}_n^2(x)-b^2(x)\right| = \left|\sum_{i=1}^{n} W_{n,i}(x)\cdot(X_{i+1}-X_i)^2 - \hat{f}_n^2(x) - b^2(x)\right| \leqslant$$

$$\leqslant \left|\sum_{i=1}^{n} W_{n,i}(x)\left[\left(f(X_i) + b(X_i)Y_{i+1}\right)^2 - \hat{f}_n^2(x) - b^2(x)\right]\right| \leqslant$$

$$\leqslant \sum_{i=1}^{n} W_{n,i}(x)\left|f^2(X_i) - f^2(x)\right| + \sum_{i=1}^{n} W_{n,i}(x)\left|f^2(x) - \hat{f}_n^2(x)\right| +$$

$$+ \left|\sum_{i=1}^{n} 2W_{n,i}(x)\, f(X_i)\cdot b(X_i)\cdot Y_{i+1}\right| +$$

$$+ \left|\sum_{i=1}^{n} W_{n,i}(x)\left[b^2(X_i)\cdot Y_{i+1}^2 - b^2(X_i)\right]\right| +$$

$$+ \sum_{i=1}^{n} W_{n,i}(x)\cdot\left|b^2(X_i) - b^2(x)\right| = C_n + D_n + \left|M_n\right| + \left|N_n\right| + B_n.$$

According to the part of the prove that $\lim\limits_{n\to\infty} J_n = 0$ a.s. it is easy to see that $\lim\limits_{n\to\infty} C_n = 0$ and $\lim\limits_{n\to\infty} B_n = 0$ a.s. The proof is analogous. Also $\lim\limits_{n\to\infty} D_n = 0$ because $\lim\limits_{n\to\infty}\left|f^2(x)-\hat{f}_n^2(x)\right|=0$.

It is easy to see that $\lim\limits_{n\to\infty} M_n = 0$ a.s. The proof is analogous to that of the fact that $\lim\limits_{n\to\infty} I_n = 0$ a.s.

It remains to prove that $\lim_{n\to\infty} N_n = 0$ a.s. where

$$N_n = \sum_{i=1}^{n} W_{n,i}(x)\cdot b^2(X_i)\cdot\left[Y_{i+1}^2 - 1\right].$$

Let us define a process Z_n as follows

$$Z_n = \sum_{m=1}^{n} W_{m,m}(x)\cdot b^2(X_m)\cdot(Y_{m+1}^2 - 1).$$

The process Z_n is a martingale with respect to F_{n+1} because $EY_i^2 = 1$. To prove that $\lim_{n\to\infty} Z_n = Z$ it suffices to verify that $\sup_n E|Z_n| < \infty$. We have

$$E|Z_n| \leqslant \left[E\sum_{m=1}^{n} W_{m,m}^2(x)\cdot b^4(X_m)\cdot(Y_{m+1}^2 - 1)^2\right]^{\frac{1}{2}} \leqslant$$

$$\leqslant \left(\sum_{m=1}^{n} \frac{1}{m^2} M^4 \; E\left|Y_1^2 - 1\right|^2\right)^{\frac{1}{2}} < \infty.$$

Therefore

$$\sup_n E|Z_n| \leqslant \sup_n M^2\sqrt{E\left|Y_1^2-1\right|^2}\cdot\left(\sum_{m=1}^{n}\frac{1}{m^2}\right)^{\frac{1}{2}} < \infty.$$

Finally we obtain that $\lim_{n\to\infty} Z_n = Z$ does exist and using Lemma 1 we have

$$\lim_{n\to\infty} N_n = 0.$$

The proof of the theorem is complete.

REFERENCES

[1] L. Devroye: On the almost everywhere convergence of nonparametric regression function, Ann. Statist. 9 (1981) 1310-19.

[2] P.J. Szabłowski: Application of the generalized strong laws of large numbers to the proper choice of the application coefficients in stochastic approximation with corelated disturbances. Lecture Notes in Pure and Applied Mathematics Differencial Games and Control Theory III, Volume 44, Marcel Dekker INC (1979).

[3] T.S. Lee, F. Kozin: Almost sure asymptotic likelihood theory for diffusion processes, J. Appl. Probability, 14 (1977) 527-537.

[4] Prakasa-Rao: Estimation of the drift for diffusion processes. Proc. 6-th International Summer School on Problems of Model Choice and Parameter Estimation in Regression Analysis, Selling Rugen, Nor. 1983 ed. Olaf Bunke, Seminarbericht no. 51.

[5] Ishwar V. Basawa, Prakasa Rao: Statistical Inference for Stochastic Processes. London 1980.

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HAMILTONIAN CYCLES THROUGH MATCHINGS

Introduction

We consider only finite graphs without loops and multiple edges. The set of vertices of a graph G is denoted by $V(G)$, and the set of edges by $E(G)$. $|V(G)|$ is the order of G, $e(G) = |E(G)|$ the size of G.

For two vertices x and y of G we define $e(x,y) = 1$ if $xy \in E(G)$, and $e(x,y) = 0$ otherwise. $N(x,G)$ is the neighbourhood of x in G, $N(A,G) = \bigcup_{x \in A} N(x,G)$ for $A \subset V(G)$.

Let $X \subset V(G)$ and $Y \subset E(\bar{G}) \cup E(\bar{G})$, where $\bar{G}$ is the complement of G. Then $G(X)$ is the subgraph of G induced by X, $G - X = G(V(G)-X)$, and $G+Y$ is the graph with the vertex set $V(G)$ and the edge set $E(G) \cup Y$. We write $G-x$ in place of $G-\{x\}$, and $G+xy$ in place of $G+\{xy\}$.

For graphs G and H, $G \cup H$ denotes the union of G and H, while mG is the disjoint union of m copies of G. $G*H$ is the graph with vertex set $V(G) \cup V(H)$ and the edge set $E(G) \cup E(H) \cup \{xy : x \in V(G), y \in V(H)\}$, then define $F*G*H$ to be the graph $(F*G)*(G*H)$.

A matching M of G is a set of independent edges of G. This set we shall consider as a graph with vertex set $V(M)$ an edge set $E(M)$. M is called a perfect matching of G if $V(G) = V(M)$. M is maximal matching of G if M is not proper subset of any matching of G. A vertex x of G is covered by M if $x \in V(M)$.

Let $g(n,k) = (4n^2-4nk-4n+2k+5k^2)/8$ and $h(n,k) = \max(g(n,k), g(n,(2n-2)/3))$. We verify that $h(n,k) = g(n,k)$ for $k \leqslant (2n+4)/15$ and $h(n,k) = g(n,(2n-2)/3)$ for $(2n+4)/15 \leqslant k \leqslant (2n-2)/3$.

We shall now define two families of graphs: $\mathcal{G}_n$ and $\mathcal{H}(n,k)$. If $(n+2)/3$ is an integer, $\mathcal{G}_n$ is the family of graphs $G = ((n+2)/3\ K_1)*H$, where H is any graph of order $(2n-2)/3$ containing a perfect matching, otherwise $\mathcal{G}_n = \emptyset$.

For $k \geqslant 2$ even let $H(n,k) = (k/2\ K_1)*K_k*K_{n-3k/2}$. Then

$$\mathcal{H}(n,k) = \begin{cases} \{H(n,k)\} & \text{if } 2 \leqslant k < (2n+4)/14 \text{ and } k \text{ is even,} \\ \{H(n,k),\ H(n,(2n-2)/3\} & \text{if } k = (2n+4)/15, \\ \{H(n,(2n-2)/3\} & \text{if } (2n+4)/15 < k \leqslant (2n-2)/3 \text{ and } (2n-2)/3 \text{ is integer,} \\ \emptyset & \text{otherwise.} \end{cases}$$

Note that $H(n,(2n-2)/3) \in \mathcal{G}_n$, and every $G \in \mathcal{G}_n$ has order n.

The order of the graph $H(n,k)$ is n, minimum degree k and size $g(n,k)$, while every $G \in \mathcal{H}(n,k)$ has order n, size $h(n,k)$ and minimum degree either k or $(2n-2)/3$.

If $G \in \mathcal{G}_n \cup \mathcal{H}(n,k)$, then G has a matching which is not contained in any Hamiltonian cycle of G.

Haggkvist proved the following

Theorem A ([3] Corollary 1). If in a graph G of order $n \geqslant 3$,

$$d(u,G) + d(v,G) \geqslant 4n/3$$

for every pair of nonadjacent vertices u and v, then every matching of G lies in a Hamiltonian cycle of G.

We shall prove that a slightly stronger version of this theorem (Theorem 1) follows from a result of Berman:

Theorem B ([2], see also [3]). Let G be a graph of order $n \geqslant 3$, such that for every pair of nonadjacent vertices u and v of G

$$d(u,G) + d(v,G) \geqslant n + 1.$$

Then every matching of G lies in a cycle.

We shall also prove that in a graph of order $n \geq 3$, size greater then $h(n,k)$, and minimum degree at least k, every matching lies in a Hamiltonian cycle. Moreover, we prove that if G is a graph of order $n \geq 3$, $\delta(G) \geq k$, $e(G) = h(n,k)$, and G has a matching which is not contained in any Hamiltonian cycle of G, then $G \in \mathcal{H}(n,k)$.

Let us mention finally, that in [1] similar results for graphs containing every matching in a cycle have been proved.

Results

Theorem 1. Let G be a graph of order $n \geq 3$, such that for every pair of distinct nonadjacent vertices u and v

$$d(u,G) + d(v,G) \geq (4n-4)/3.$$

Then every matching of G lies in a Hamiltonian cycle of G, unless $G \in \mathcal{G}_n$.

Proof. Let G be a graph satisfying the assumptions of the theorem, and suppose that M is a matching of G which lies in no Hamiltonian cycle of G. We shall prove that $G \in \mathcal{G}_n$.

Since our theorem is easy to verify for $n = 3$ and for $n = 4$, we shall assume $n \geq 5$. Without loss of generality we may also assume that M is maximal matching of G.

Let us denote by L the set of vertices which are not covered by the matching M. We have $d(u,G) + d(v,G) \geq (4n-4)/3$, and therefore $d(u,G) + d(v,G) \geq n+1$, for every pair of nonadjacent vertices u and v. Applying Theorem B we deduce easily that L is not empty.

By the maximality of M, the set L is independent. Let us put $L = \{a_1, \dots, a_l\}$, where l is the cardinality of L.

Now, let G be the graph obtained from G by addition to G' l new vertices $b_1, b_2, \dots, b_l$, and joining each new vertex b_i, $1 \leq i \leq l$, to the vertex a_i and all its neighbours in G. Consider the matching $M' = M \cup \{a_i b_i : 1 \leq i \leq l\}$. We shall prove that M' is not contained in any cycle of G'. In fact, suppose that C' is a cycle of G', such that M' lies in C'. Since M' is perfect matching of G', C' is Hamiltonian cycle of G'.

Then, for every i, $1 \leqslant i \leqslant l$, the vertex b_i has two neighbours in C': a_i, since $a_ib_i \in M'$, and c_i, say. By the construction of G', $a_ic_i \in E(G)$, and therefore we may replace each segment $a_ib_ic_i$ of C' with the edge a_ic_i. The cycle obtained in such a way from C' is Hamiltonian cycle of G containing M, a contradiction.

Thus, by Theorem B, there are two nonadjacent vertices of G', x_0 and y_0, say, such that $d(x_0,G')+d(y_0,G') \leqslant n+1$.

Since for every new vertex b_i, $1 \leqslant i \leqslant l$, $d(b_i,G') = d(a_i,G')$, and every vertex nonadjacent to b_i is also nonadjacent to a_i, we may assume that x_0 and y_0 are vertices of G. Observe also that if $x \in L \cup N(L,G)$, then $d(x,G') \geqslant d(x,G)+1$.

For $x \in V(G)$ we define $\varepsilon(x) = 1$ if $x \in L \cup N(L,G)$, and $\varepsilon(x) = 0$ otherwise. Then we have $n+1 \geqslant d(x_0,G')+d(y_0,G') \geqslant d(x_0,G)+\varepsilon(x_0)+d(y_0,G)+\varepsilon(y_0) \geqslant (4n-4)/3+\varepsilon(x_0)+\varepsilon(y_0)$, and therefore $l \geqslant (n-4)/3+\varepsilon(x_0)+\varepsilon(y_0)$.

We shall consider three cases:

Case 1. $\varepsilon(x_0) = \varepsilon(y_0) = 0$.

The set $L \cup \left\{x_0,y_0\right\}$ is independent. Thus $d(x,G) \leqslant n-l-2 \leqslant (2n-2)/3$, for every $x \in L \cup \left\{x_0,y_0\right\}$. On the other hand, $d(a,G) + d(b,G) \geqslant (4n-4)/3$ for $a,b \in L \cup \left\{x_0,y_0\right\}$. So, $l = (n-4)/3$ and $d(x,G) = (2n-2)/3$ for every vertex $x \in L \cup \left\{x_0,y_0\right\}$. Hence $uv \in E(G)$ for every $u \in L \cup \left\{x_0,y_0\right\}$, and for every $v \in V(G) - (L \cup \left\{x_0,y_0\right\})$. Moreover, M is perfect matching of G-L. The reader can easily find a Hamiltonian cycle of G, containing M. A contradiction.

Case 2. $\varepsilon(x_0) = 1$, $\varepsilon(y_0) = 0$.

The set $L \cup \left\{y_0\right\}$ is independent and contains at least two vertices. We repeat the arguments used in the previous case to prove that $l = (n-1)/3$. M is perfect matching of G-L, and G-L has $n-l = (2n+1)/3$ vertices. A contradiction, since $(2n+1)/3$ is never an even integer.

Case 3. $\varepsilon(x_0) = \varepsilon(y_0) = 1$.

Then $l \geqslant 2$, and $(4n-4)/3 \leqslant d(a,G) + d(b,G) \leqslant 2(n-l)$, for every pair of distinct vertices a and b of L. We deduce easily that $l = (n+2)/3$ and $d(x,G) = (2n-2)/3$ for every vertex $x \in L$. Thus $G \in \mathcal{G}_n$, and the theorem is proved. ■

Corollary 2. Let G be a graph of order $n \geq 3$, such that $\delta(G) \geq (2n-2)/3$. Then every matching of G lies in a Hamiltonian cycle, unless $G \in \mathcal{G}_n$.

Theorem 3. Let G be a graph of order $n \geq 3$, such that $\delta(G) \geq k$, and the size of G is at least $h(n,k)$. Then every matching of G lies in a Hamiltonian cycle of G, unless $G \in \mathcal{H}(n,k)$.

Proof. By induction on n. It is easy to check the theorem for $n \leq 5$. So, let us suppose that $n \geq 6$ is such an integer that the theorem holds for every graph of order less than n.

Let G be a graph of order n, satisfying the assumptions of Theorem 3, and let M be a matching of G. Without loss of generality we may assume that M is a maximal matching of G. We deduce from Corollary 2 that Theorem 3 holds when $\delta(G) \geq$ $\geq (2n-2)/3$, so we may assume that $k < (2n-2)/3$.

Since the function $h(n,k)$ is decreasing with respect to k, for $k \leq (2n-2)/3$, we may assume that the set of vertices of degree k in G is not empty. Now, one can easily check the theorem for $k \leq 2$, henceforth we may assume $k \geq 3$.

In the proof we shall use the following elementary relations:

(1) $h(n,k) - h(n-1,k-1) > k+1$,

(2) $h(n,k) - h(n-2,k-2) > k+n-2$,

(3) $g(n,k) - g(n-3),k-2) = 2n+k-5$,

(4) $g(n,(2n-2)/3) - g(n-3,(2(n-3)-2)/3) = (8n-17)/3$.

We shall prove first, that if one of propositions (i), (ii) given below is not satisfied, then M lies in a Hamiltonian cycle of G.

(i) Any vertex a of degree k in G is not covered by M.

Proof of (i). Suppose that there exist a vertex a of degree k, and a vertex b such that $ab \in M$.

If there is a neighbour of a, x say, such that x is not covered by M, define the graph $G' = G - a + xb$ and the matching $M' = M - \{ab\} \cup \{bx\}$. We have clearly $\delta(G') \geq k-1$, $e(G') \geq h(n,k)-k$, and by (1), $e(G') \quad h(n-1,k-1)$. By the induction hypothesis, the matching M' lies in a Hamiltonian cycle C' of G'. Then replace in C' the edge bx with the path bax to obtain a Hamiltonian cycle of G containing M.

So, let us suppose that x is a covered neighbour of a such that $x \neq b$, and let y be the neighbour of x in M. Consider the graph $G'' = G - \{a,x\} + by$, and the matching $M'' = M - \{ab, xy\} \cup \{by\}$. Then $\delta(G'') \geq k-2$, $e(G'') > h(n,k) - k - n + 2$, and by (2), $e(G'') > h(n-2,k-2)$. Then, by the induction hypothesis, M'' lies in a Hamiltonian cycle C'' of G''. The cycle obtained from C'' by replacing the edge by with the path $baxy$ is Hamiltonian cycle containing M. □

(ii) Let a be a vertex of degree k, and let x be a neighbour of a. Then $d(x,G) \geq k+2$.

Proof of (ii). Suppose $d(x,G) \leq k+1$. Since a is not covered by M, and M is maximal matching of G, x is covered by M. Let y be the neighbour of x in M.

Consider the graph $G' = G - x + ay$, and the matching $M' = M - \{xy\} \cup \{ay\}$.

We have $\delta(G') \geq k-1$, $e(G') \geq h(n,k) - k - 1$, and by (1), $e(G) > h(n-1,k-1)$. By the induction hypothesis, M' lies in a Hamiltonian cycle C' of G'. Now replace in C' the edge ay with the path axy to obtain a Hamiltonian cycle of G containing M. □

Let a be a vertex of degree k. By (i), we may assume that a is not covered by M, by the maximality of M, all the neighbours of a are covered by M. Since $k > 2$, there exist two neighbours of a, x and y say, such that $xy \notin M$. Let us denote by z and t the vertices for which $xz \in M$ and $yt \in M$.

Consider the graph $G' = G - \{a,x,y\} + zt$. By (ii) we have $\delta(G') \geq k-2$; we have also $e(G') \geq e(G) - k - d(x,G) - d(y,G) + 3 + e(x,y) - e(z,t)$.

We apply the relations (3) and (4) to prove that $e(G') \geq h(n-3,k-2)$, and that if the equality holds, then necessarily $e(G) = h(n,k)$, $d(x,G) = d(y,G) = n-1$, and $e(x,y) = e(z,t) = 1$.

Consider now the matching $M' = M - \{xz, yt\} \cup \{zt\}$. By the induction hypothesis, either M' lies in a Hamiltonian cycle of G' or $G' \in \mathcal{H}(n-3,k-2)$. If M' lies in a Hamiltonian cycle C' of G', then M lies in the cycle obtained from C' by replacing the edge zt with the path $zxayt$. If $G' \in \mathcal{H}(n-3,k-2)$ the reader may check by simple inspection that either $G \in \mathcal{H}(n,k)$ or M lies in a Hamiltonian cycle of G.

REFERENCES

[1] A. Benhocine, A.P. Wojda: Graphs containing every matching in a cycle, (to appear).

[2] K.A. Berman: Proof of a conjecture of Häggkvist on cycles and independent edges, Discrete Math. 46 (1983) 9-13.

[3] R. Haggkvist: On F-Hamiltonian graphs, in: J.A. Bondy and U.S.R. Murthy, eds., Graphs Theory and Related Topics (Academic Press, New York, 1979) 219-231.

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A REMARK ON A CHARACTERIZATION OF THE POISSON PROCESS

A characterization of the Poisson process by properties of conditional moments obtained by Bryc (1987) may be simplified in the following way.

Theorem. Let $X = (X_t)_{t \geqslant 0}$ be a square integrable stochastic process such that for any $0 \leqslant s_1 \leqslant \dots \leqslant s_n < s < t$, $n = 1,2,\dots$

(1)
$$EX_t = t,$$

(2)
$$\mathrm{cov}(X_s, X_t) = s,$$

(3)
$$E(X_s|X_{s_1},\dots,X_{s_n}) = A_1X_{s_n} + A_0,$$

(4)
$$E(X_s|X_{s_1},\dots,X_{s_n}, X_t) = B_1X_{s_n} + B_2X_t + B_0,$$

(5)
$$\mathrm{Var}(X_s|X_{s_1},\dots,X_{s_n}, X_t) = C(X_t - X_{s_n}),$$

where A_0, A_1, B_0, B_1, B_2, C are some constants depending on $s_1,\dots,s_n$, s, t only. Then X is a Poisson process.

In Bryc (1987) it was additionally assumed that

(6)
$$\mathrm{Var}(X_s|X_{s_1},\dots,X_{s_n}) = \mathrm{const}.$$

Proof. We will prove that (6) follows from (1)-(5) and then our Theorem will be a consequence of Bryc characterization. From (1), (2) and properties of conditional expectations we easily compute

$$A_1 = 1,\quad A_0 = s - s_n,\quad B_1 + B_2 = 1,\quad B_0 = 0.$$

Let us denote $Y = (X_{s_1},\dots,X_{s_n})$. Then from (5) we obtain

$$E(X_s^2|Y) = E(E(X_s^2|Y,\ X_t)|Y) = B_1^2X_{s_n}^2 - CX_{s_n} + \tag{7}$$

$$+ (2B_1B_2X_{s_n} + C)E(X_t|Y) + B_2^2E(X_t^2|Y).$$

We observe that (3) implies

$$E(X_tX_s|Y) = E(E(X_t|Y,\ X_s)X_s|Y) = \tag{8}$$

$$= E(X_s^2|Y) + (t - s)E(X_s|Y).$$

On the other hand from (4) we get

$$E(X_tX_s|Y) = E(X_tE(X_s|Y,\ X_t)|Y) = \tag{9}$$

$$= B_1X_{s_n}E(X_t|Y) + B_2E(X_t^2|Y).$$

The formulas (7)-(9) imply

$$B_1E(X_s^2|Y) = B_1^2X_{s_n}^2 - CX_{s_n} + (B_1B_2\ X_{s_n} + C)E(X_t|Y) +$$

$$+ B_2(t - s)E(X_s|Y).$$

Now we apply once again (3). Hence

$$B_1E(X_s^2|Y) = X_{s_n}^2(B_1^2 + B_1B_2) + X_{s_n}(B_1B_2(t - s_n) + B_2(t - s)) +$$

$$+ C(t - s_n) + B_2(t - s)(s - s_n).$$

Consequently

$$E(X_s^2|Y) = X_{s_n}^2 + 2A_0X_{s_n} + \text{const},$$

since $B_2(t - s_n) = s - s_n$. Applying (3) to the above equation we get (6).□

Similar simplification may be also introduced to Proposition 3.1 in Bryc (1987). (The assumption (13) of that Proposition follows from the remained ones and thus may be deleted).

REFERENCE

W. Bryc: A characterization of the Poisson process by conditional moments, Stochastics 20 (1987), 17-26.

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Dietmar Dorninger, Günther Hasibeder

RECONSTRUCTION OF DEFORMED REGULAR POLYGONS: A PROBLEM ARISING FROM CELL-BIOLOGY

In cytogenetics, it is an important question to determine the spatial order of chromosomes in a simple haploid genome during metaphase. For this purpose the coordinates of the centromeres are measured, which are assumed to lie in the corners of a regular n-gon, say $P_1P_2...P_nP_1$, within a plane σ (see e.g. [1], [3], [4]). However, the measuring process and a subsequent statistical analysis usually only result in an irregular arrangement of points $Q_1, Q_2, ..., Q_n$ within the plane σ, where Q_i, for $i = 1,2,...,n$, denotes the point obtained from P_i by displacing P_i in any direction within σ. The question arises how to find the original order. Heslop-Harrison & Bennett ([3], [4]) suggested a method which has the disadvantage that $\frac{1}{2}(n-1)!$ permutations have to be considered and that it does not guarantee to reveal the original order in each case.

In this paper, we formulate an algorithm which only requires polynomial time and which, under the assumption that the displacement of each P_i - even in an arbitrary direction of the three-dimensional space - is smaller than a specific amount $s(n)$, always reconstructs the original order of indices. A lower bound $v(n)$ for $s(n)$ has been given in [2] (there constructed for a different algorithm and displacements only within the plane). As numerical calculations show, $v(n)$ is very close to $s(n)$.

Let d be the side-length of the regular n-gon $P_1P_2\ldots P_nP_1$, where P_{i-1} and P_{i+1} are the neighbours of P_i for $i=1,2,\ldots,n$. Here and from now on all indices shall be understood modulo n. Further let S_i, for $i = 1,2,\ldots,n$, be the closed sphere with center P_i and radius θ, $0 < \theta < d/2$, and correspondingly C_i be the closed circle in σ with center P_i and radius θ (i.e. $C_i = \sigma \cap S_i$). θ shall be the same for all spheres and circles. In the following, we always suppose n to be an integer and $n > 3$.

Q_i shall be an arbitrarily chosen but fixed point of S_i (or C_i, if only the special case of the plane is considered), $\underline{Q} := \{Q_1, Q_2, \ldots, Q_n\}$, and bd S_i (or bd C_i) denotes the boundary of S_i (respectively C_i). We denote by $[XY]$ the straight line connecting two points X and Y, by $\overline{XY}$ the line segment, and by XY the distance between these two points.

We say that the one-sided neighbourhood condition for spheres (or circles) - in short ONCS (respectively ONCC) - is fulfilled, if and only if for any $i \in \{1,2,\ldots,n\}$ and any points $X_i \in S_i$ (respectively C_i) the following is true: $\overline{X_iX_{i+1}} < \overline{X_iX_{i+2}}$. As a consequence, also $\overline{X_iX_{i-1}} < \overline{X_iX_{i-2}}$ for all i.

Remark 1. Let Y and Z be two points, S_Z be a closed sphere with center Z and radius r, and $Y \notin S_Z$. Then there exist two uniquely defined points $R = R(Y,S_Z)$, $T = T(Y,S_Z)$ such that

(a) $\{R,T\} = [XZ] \cap \text{bd}\, S_Z$,

(b) $\overline{YR} < \overline{YX} < \overline{YT}$ for all $X \in S_Z \setminus \{R,T\}$,

(c) $\overline{YR} = \overline{YZ} - r$, $\overline{YT} = \overline{YZ} + r$.

Lemma 2. Let A, B, C be three points of a plane σ, and let S_A, S_B, S_C be three closed and mutually disjoint spheres with centres A, B, C and radii r_A, r_B, r_C, respectively. Then, for any given points $A_1 \in S_A$, $B_1 \in S_B$, $C_1 \in S_C$ there exist points A_0, B_0, C_0 such that

(a) $A_0 \in \sigma \cap \mathrm{bd}\, S_A$, $B_0 \in \sigma \cap \mathrm{bd}\, S_B$, $C_0 \in \sigma \cap \mathrm{bd}\, S_C$,

(b) $B_0 \in [A_0B]$, $C_0 \in [A_0C]$,

(c) $\overline{A_0B_0} = \overline{A_0B} + r_B$, $\overline{A_0C_0} = \overline{A_0C} - r_C$,

(d) $\overline{A_0C_0} - \overline{A_0B_0} \leq \overline{A_1C_1} - \overline{A_1B_1}$.

Proof. Starting with the given points A_1, B_1, C_1 we construct A_0, B_0, C_0 as follows:

(1) $A_2 := A_1$, $B_2 := T(A_2,S_B)$, $C_2 := R(A_2,S_C)$.

(2) A_2 is revolved about the axis $[BC]$ into $\sigma \cap S_A$; this is obviously possible, though the resulting point, which we will denote as A_3, is not always uniquely determined. $B_3 := T(A_3,S_B)$, $C_3 := R(A_3S_C)$.

(3) A_0 is defined by $A_0 \in [A_3C] \cap \mathrm{bd}\, S_A$ and $\overline{A_0C} \leq \overline{A_3C}$. $B_0 := T(A_0,S_B)$, $C_0 := C_3$.

From this construction we observe:

(1) $\overline{A_2B_2} = \overline{A_2B} + r_B \geq \overline{A_1B_1}$ and $\overline{A_2C_2} = \overline{A_2C} - r_C \leq \overline{A_1C_1}$.

(2) $\overline{A_3B} = \overline{A_2B}$, $\overline{A_3C} = \overline{A_2C}$.

(3) By the triangle inequality, $\overline{A_0B} \geq \overline{A_3B} - (\overline{A_3C} - \overline{A_0C})$. $A_0 \in \sigma \cap \mathrm{bd}\, S_A$ and, according to Remark 1, $B_0 \in [A_0B] \cap \mathrm{bd}\, S_B$, $C_0 \in [A_0C] \cap \mathrm{bd}\, S_C$, $\overline{A_0B_0} = \overline{A_0B} + r_B$, $\overline{A_0C_0} = \overline{A_0C} - r_C$.

Now all the assertions of Lemma 2 immediately follow.

Corollary 3. The ONCS is satisfied if and only if the ONCC is fulfilled.

Lemma 4. If, for $n \geq 6$, the ONCS is satisfied, then

$$\max\{\overline{Q_iQ_{i-1}}, \overline{Q_iQ_{i+1}}\} < \min\{\overline{Q_iQ_{i+k}} \mid 3 \leq k \leq n-3\}$$

for $i = 1,\ldots,n$.

Proof. We denote the bisecting plane of a line segment XY by $b[XY]$; and for any arbitrary but fixed index i we write $R_j := R(Q_i,S_j)$, $T_j := T(Q_i,S_j)$, where R and T are defined according to Remark 1.

Considering $b[P_iP_{i+1}]$, we observe that $\overline{Q_iP_{i+3}} > \overline{Q_iP_{i-2}}$. Hence, by Remark 1 and the ONCS, we obtain

$$\overline{Q_iQ_{i+3}} \geqslant \overline{Q_iR_{i+3}} = \overline{Q_iP_{i+3}} - \theta > \overline{Q_iP_{i-2}} - \theta = \overline{Q_iR_{i-2}} > \overline{Q_iQ_{i-1}}. \tag{1}$$

Similarly we see from $b[P_{i+2}P_{i+3}]$ that $\overline{Q_iP_{i+3}} > \overline{Q_iP_{i+2}}$ and further

$$\overline{Q_iQ_{i+3}} \geqslant \overline{Q_iR_{i+3}} > \overline{Q_iR_{i+2}} > \overline{Q_iQ_{i+1}}. \tag{2}$$

Considering $b[P_{i+k}P_{i+k+1}]$ for $k = 3,4,\ldots,[n/2]-1$ shows that $\overline{Q_iP_{i+k+1}} > \overline{Q_iP_{i+k}}$, which implies

$$\overline{Q_iQ_{i+k+1}} \geqslant \overline{Q_iR_{i+k+1}} > \overline{Q_iR_{i+k}}. \tag{3}$$

Putting together (1), (2), (3) yields

$$\max\{\overline{Q_iQ_{i-1}}, \overline{Q_iQ_{i+1}}\} < \min\{\overline{Q_iQ_{i+k}} \mid 3 \leqslant k \leqslant [n/2]\}$$

and, on account of the symmetry in the regular polygon $P_1P_2\ldots P_nP_1$, the assertion of Lemma 4 follows. □

Now we consider the complete graph G whose vertices are the elements of $\underline{Q}$ and whose (undirected) edges Q_iQ_j ($i \neq j$; $i,j \in \{1,2,\ldots,n\}$) are the line segments between any two points of $\underline{Q}$. The edges Q_iQ_j are valued by the corresponding Euklidian distances $\overline{Q_iQ_j}$. Next we construct a series of subgraphs H_k of G, $k = 0,1,2,3$, and a final subgraph H, all subgraphs with vertex set $\underline{Q}$, by the following procedure.

The Polygon Reconstructing Algorithm (PR-Algorithm)

Step 0. Let H_0 be the spanning subgraph of G with empty edge-set.

Step 1. Obtain H_1 from H_0 by connecting by an edge each vertex Q_i to the vertex nearest to it in G. If there are two vertices with the same minimal distance to Q_i, add both corresponding edges.

Step 2. Let I_1 be the subset of those vertices which have degree 1 in H_1. If $I_1 = \emptyset$, put $H = H_1$ and end the algorithm. If $I_1 \neq \emptyset$, connect each vertex of I_1 to the vertex of H_1 that is second-nearest to it in G. If there are two such vertices, add both corresponding edges. The graph resulting this way from H_1 shall be denoted as H_2.

Step 3. Construct H_3 by removing the longest edge from each triangle in H_2.

Step 4. Let I_3 be the subset of those vertices which have degree 1 in H_3. If $I_3 = \emptyset$, put $H = H_3$ and end the procedure. In case $I_3 \neq \emptyset$ connect each vertex of I_3 to the vertex which is third-nearest to it in G. Denote the resulting graph as H and end the algorithm.

Theorem 5. If the ONCS is fulfilled, the graph H obtained by the PR-Algorithm is a Hamiltonian circuit in which Q_{i-1} and Q_{i+1} are the neighbours of Q_i for all $i = 1,2,\ldots,n$, i.e. the PR-Algorithm reproduces the original order of indices as defined by the polygon $P_1P_2\ldots P_nP_1$.

Proof. As a consequence of the ONCS and Lemma 4 we obtain: The graph H_1 only contains edges of the form Q_iQ_{i+1} and, if H_1 is not equal to the Hamiltonian circuit $Q_1Q_2\ldots Q_nQ_1$, all the connected components of H_1 are paths of the form $Q_jQ_{j+1}\ldots Q_{j+k}$.

The second-longest edge in G from the endpoint Q_j of such a path may be Q_jQ_{j-1} or Q_jQ_{j+2}, or both edges may have the same length. If $k = 1$ and Q_jQ_{j+2} becomes an edge of H_2, also $Q_{j+1}Q_{j+2}$ has to become an edge of H_2 at Step 2, because otherwise $\overline{Q_{j-1}Q_{j+1}}$ would be smaller than $\overline{Q_{j+1}Q_{j+2}}$ and this would imply, together with the ONCS, that $\overline{Q_{j-1}Q_j} < \overline{Q_{j-1}Q_{j+1}} < \overline{Q_{j+1}Q_{j+2}} < \overline{Q_jQ_{j+2}} \leqslant \overline{Q_jQ_{j-1}}$, a contradiction. A similar consideration can be performed with the second endpoint Q_{j+k}. Thus we have ensured that Step 3 produces a graph H_3 whose connected components are paths of the form $Q_mQ_{m+1}\ldots Q_{m+r}$, unless H_3 is the Hamiltonian circuit $Q_1Q_2\ldots Q_nQ_1$.

Finally we observe that the third-longest edges incident with the endpoints Q_m and Q_{m+r} of such a path must be Q_mQ_{m-1} and $Q_{m+r}Q_{m+r+1}$, respectively, where Q_{m-1} and Q_{m+r+1} are endpoints of maximal connected paths of H_3 themselves. This ensures that H is the Hamiltonian circuit $Q_1Q_2\dots Q_nQ_1$.

Theorem 6. The ONCS is satisfied if and only if $\Theta < s(n)$, where $s(n) = d\cdot r/2$ and r is defined by the system of equations

$$(4)\qquad r\left(1+\sqrt{1-r^2}\Big/\sqrt{\cosh^2 t-r^2}\right)\cosh t = 1 + 2\cos(2\pi/n)$$

$$(5)\qquad \left(\sqrt{1-r^2} - r^2\Big/\sqrt{\cosh^2 t-r^2}\right)\sinh t = 2\sin(2\pi/n)$$

subject to $(r,t)\ ]0,1[\times]0,\infty[$. Under this restriction, there is a unique solution of system (4), (5), and thus s(n) also is uniquely defined.

Proof. By Corollary 3, the ONCS is equivalent to the ONCC. Therefore it suffices to consider only points within the plane σ. Let A, B, C denote three consecutive corners of the polygon $P_1P_2\dots P_nP_1$ such that $\overline{AC} > \overline{AB} = \overline{BC} = d$. Further let K_A, K_B, K_C be three closed circles in σ with centres A, B, C, respectively, and equal radii r. Then we have to find the maximum value of r such that

$$(6)\qquad \min\left\{\overline{XZ}-\overline{XY} \,\middle|\, X\in K_A \wedge Y\in K_B \wedge Z\in K_C\right\} \geqslant 0.$$

This maximum value of r is equal to the unique value of r for which

$$(7)\qquad \min\left\{\overline{XZ}-\overline{XY} \,\middle|\, X\in K_A \wedge Y\in K_B \wedge Z\in K_C\right\} = 0.$$

As a consequence of Lemma 2, (7) is equivalent to

$$(8)\qquad \min\left\{\overline{XZ}-\overline{XY} \,\middle|\, X\in \mathrm{bd}\,K_A \wedge Y = T(X,K_B) \wedge Z = R(X,K_C)\right\} = 0$$

where $T(X,K_B)$ and $R(X,K_C)$ are defined according to Remark 1 (when analogously formulated for circles). Further, (8) is equivalent to

$$(9)\qquad \min\left\{\overline{XC}-\overline{XB}\,\middle|\, X \in \mathrm{bd}\ K_A\right\} = 2r.$$

In order to solve Eq. (9) we put d = 2 and choose the following coordinates for A, B, C in $\mathbb{R}^2$:

$$A = (-1 - 2\cos\alpha;\ 2\sin\alpha)^T,$$
$$B = (-1;\ 0)^T,$$
$$C = (1;\ 0)^T$$

with $0<\alpha\leq\pi/2$. Geometrical considerations immediately show that the unique value of r satisfying Eq. (9) is equal to the minimum value of r such that $h \cap k \neq \emptyset$, where h and k are defined by

$$(10)\qquad h = \left\{X \middle| \overline{XC}-\overline{XB} = 2r\right\},$$

$$(11)\qquad k = \left\{X \middle| \overline{AX} = r\right\}.$$

h is a branch of a hyperbola and k is a circular line. In case r assumes the minimum value such that $h \cap k \neq \emptyset$, the curves h and k must touch. This fact will enable us to formulate equations for the coordinates of point A. We use the parametric representation of h,

$$\underline{x} = (-r\cosh t;\ \sqrt{1-r^2}\,\sinh t)^T,\quad t \in \mathbb{R}.$$

In each point of the hyperbola branch, the normal unit vector pointing towards the abscissa is given by

$$\underline{n} = (\cosh^2 t - r^2)^{-1/2}\cdot(-\sqrt{1-r^2}\,\cosh t;\ -r\sinh t)^T.$$

Thus in case of tangency the coordinates of A satisfy the equation $A = \underline{x} + r\cdot\underline{n}$, and for $\alpha = 2\pi/n$ (the exterior angle of the regular n-gon) this equation is equivalent to Eqs. (4), (5). Since we look for the minimum value of r such that the hyperbola branch h and the circular line k touch, t must be positive. Taking general d instead of d = 2, r has to be multiplied by d/2 in order to yield s(n).

Theorem 7. If $e < s(n)$, $s(n)$ as defined by Theorem 6, then for any set $\underline{Q}$ the PR-Algorithm reproduces the original order of indices in the polygon $P_1P_2 \ldots P_nP_1$.

Proof. Theorem 5 and Theorem 6.

Theorem 8. $\lim\limits_{n\to\infty} v(n) = \lim\limits_{n\to\infty} s(n) = d\sqrt{2}/3$.

Proof. We consider the hyperbola branch h and the circular line k as defined by (10) and (11), but now we use the representations

$$(12) \qquad h\colon y = \pm\sqrt{(1/r^2-1)(x^2-r^2)}, \quad x \in \,]-\infty, -r];$$

$$(13) \qquad k\colon (x + 1 + 2\cos\alpha)^2 + (y - 2\sin\alpha)^2 = r^2.$$

After inserting (12) into (13) and multiplying with r^2, we obtain the equation

$$(14) \qquad x^2 + 2r^2(1+2c)x + 4r^2(1+c) = \pm\, 4r\sqrt{1-c^2}\sqrt{(1-r^2)(x^2-r^2)}$$

wherein $c = \cos\alpha$. If $n \to \infty$, α becomes 0, c tends to 1, and thus Eq. (14) is reduced to

$$(15) \qquad x^2 + 6r^2x + 8r^2 = 0.$$

Geometrical and continuity considerations show that Eq. (15) must have a real root of multiplicity 2 if r is the minimum value such that $h \cap k \neq \emptyset$. Hence we obtain $r = 2\sqrt{2}/3$.

Since we have chosen $d = 2$ as in the proof of Theorem 6, it follows for general d that $\lim\limits_{n\to\infty} s(n) = r\cdot d/2 = d\sqrt{2}/3$. From the formula for v(n) stated in [2], we see that $\lim\limits_{n\to\infty} v(n)$ also equals $d\sqrt{2}/3$.

Remark 9. For any preassigned value of $n \in [3, \infty[$ Eqs. (4), (5) yield a unique solution $(r,t) \in [0, 2\sqrt{2}/3] \times \,]0, \infty[$; with increasing n, r increases continuously and strictly monotonically. For $n = 3$ we obtain $r = 0$.

By means of Theorem 6 we established the table below. It can be used in the following ways: Firstly, it contains the values of $s(n)/d$, which have been truncated to two significant digits, for preassigned values of n. Secondly, the table shows the smallest integer values of n corresponding to the truncated values of $s(n)/d$.

n	4	5	6	7	8	9	10	11	12
s(n)/d	.14	.23	.28	.31	.34	.36	.37	.38	.39

n	13	15	17	20	25	35	62	471
s(n)/d	.40	.41	.42	.43	.44	.45	.46	.47

Numerical calculations show that the explicit formula $v(n)$ from [2] gives an excellent lower bound for $s(n)$:

(a) $0 < s(5) - v(5) < 8 \cdot 10^{-3} \cdot d$,

(b) $0 < s(n) - v(n) < 2 \cdot 10^{-3} \cdot d$ for all $n \geq 6$.

($v(n)$ is not defined for $n = 4$).

Remark 10. The PR-Algorithm does not yield the correct reproduction of the original order of indices for all possible sets Q in the case $\theta \geq s(n)$. This remains true even if we assume that $Q \leq \delta$. In view of $s(n) < d/2$ one could suspect that $\lim_{n\to\infty} s(n) = 0.5 \cdot d$, but as we have shown, $\lim_{n\to\infty} s(n) = d\sqrt{2}/3 \approx 0.471 \cdot d$.

REFERENCES

[1] D.J. **Coates**, D. **Smith**: The spatial distribution of chromosomes in metaphase neuroblast cells from subspecific F_1 hybrids of the grasshopper Caledia captiva, Chromosoma, 90 (1984), 338-348.

[2] D. **Dorninger**, W. **Timischl**: Geometrical constraints on Bennett´s predictions of chromosome order, Heredity 58 (1987), 321-325.

[3] J.S. Heslop-Harrison, M.D. Bennett: Prediction and analysis of spatial order in haploid chromosome complements, Proc. R. Soc. Lond. B, 218 (1983), 211-223.

[4] J.S. Heslop-Harrison, M.D. Bennett: The spatial order of chromosomes in root-tip metaphases of Aegilops umbelluleta, Proc. R. Soc. Lond. B, 218 (1983), 225-239.

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Włodzimierz Smoleński

ON THE KERNEL OF A PROBABILITY MEASURE ON A BANACH SPACE

1. Introduction

The linear kernel of a probability measure on a separable Banach space E can be defined as the intersection of all pre-supports of this measure. A linear subspace U of E is called a pre-support of a probability measure μ if the following condition holds:

for every $\varepsilon > 0$ there exists a convex and compact subset K such that for every continuous linear functional f if

$$K \subseteq \{e \in E:\ |f(e)| \leqslant 1\} \quad \text{then} \quad \mu(\{e \in E:\ |f(e)| \leqslant 1\}) > 1-\varepsilon$$

A linear functional f on E is called Lusin measurable if for every $\varepsilon > 0$ there exists a convex and compact subset K of E such that $\mu(K) > 1-\varepsilon$ and f restricted to K is continuous. It is well known (cf. [1] or [2]) that if a Lusin measurable functional vanishes on a pre-support then it is almost everywhere equal to zero. A. Wiśniewski ([3]) asked if one could change "pre-support" to "kernel" in the above statement under the additional assumption that the kernel was a dense subspace of E. The aim of this note is to give a counterexample.

2. Counterexample

L e m m a . Let E be an infinite dimensional separable Banach space. Then there exist linear subspaces E_1 and E_2 of E with the following properties:

i) $E_1 \cap E_2 = \{\emptyset\}$;

ii) for $i = 1,2$, $E_i = \operatorname{span} K_i$, where K_i is a symmetric convex compact subset of E;

iii) for $i = 1,2$ E_i is a separable Hilbert space with the topology introduced by the unit ball K_i;

iv) for $i = 1,2$ E_i is a dense subspace of E.

Proof. Assume first that $E = L_2([0,1])$. We put for E_1 the Sobolev space $W_2^1([0,1]) = \{f \in L_2 : \|f'\|_2 < \infty\}$. Let now Z be a totally disconnected compact subset of $[0,1]$ with the Lebesgue measure $\frac{1}{2}$. Let T be the following unitary operator on L_2:

$$Tf(t) = \begin{cases} +f(t) & \text{for } t \in Z \\ -f(t) & \text{otherwise.} \end{cases}$$

We put $E_2 = T(E_1)$. It should be clear that properties i)-iv) are fulfilled. In the general case it is enough to observe that in every separable Banach space we can imbed $L_2([0,1])$ as a dense subspace in a linear and continuous way. This finishes the proof of the lemma.

Theorem. Let E be an infinite dimensional Banach space. Then there exist a probability measure μ on E and a linear functional f on E with the following properties:

I) for every $\varepsilon > 0$ there exists a convex compact subset K of E such that $\mu(K) > 1-\varepsilon$ and f restricted to K is continuous;

II) the linear kernel J_μ of μ is a dense subspace of E;

III) for every $e \in J_\mu$ $f(e) = 0$;

IV) $\mu(\{e \in E : f(e) = 0\}) < 1$.

Proof. Let E_1 and E_2 be linear subspaces of E with properties i)-iv) from the lemma above. Let $\{e_n\}$ be an orthonormal basis of E_2. Let T be the following operator from the open unit interval $(0,1)$ to E_2:

$$T(t) = \sum_{n=1}^{\infty} t^n e_n.$$

Let λ be the Lebesgue measure on $(0,1)$. We define $\nu = \lambda \circ T^{-1}$. Let now γ be any full Gaussian measure on E_1. We put $\mu = (\gamma+\nu)/2$.

We shall prove now that $J_\mu \cap E_2 = \{0\}$. To see this let $x \in E_2$. Since $J_\nu = \{0\}$ (cf. [2] p.758) thus there exists a linear subspace E_3 of E_2 such that $x \notin E_3$, $\nu(E_3) = 1$ and E_3 is standard (i.e. a union of countably many convex compact sets) (cf. [2] p.755). If we put $E_4 = \mathrm{span}(E_1 \cup E_3)$ we obtain a standard pre-support of μ which does not contain x.

At this point we are able to define f :

$$f(x) = \begin{cases} 0 & \text{for } x \in J_\mu \cup E_1 \\ \langle x, e_1 \rangle & \text{for } x \in E_2. \end{cases}$$

Using the Hamel basis argument we extend f in a linear way on the whole E.

Let K_1 and K_2 be the convex compact sets defined in the lemma above. It is clear that $\mu(nK_1 \cup nK_2)$ tends to one as n tends to infinity. Thus to get property I) it is enough to observe that f restricted to K_1+K_2 is continuous (easy exercise).

Next since J_μ contains J_γ, J_γ is dense in E_1 and E_1 is dense in E we obtain property II). III) is trivial and IV) follows from the fact that $\nu(\{e \in E\colon f(e) = 0\}) = 0$. Q.E.D.

REFERENCES

[1] S. Chevet : Kernel associated with a cylindrical measure. Probability in Banach spaces III, Proceedings, Medford 1980, L.N.M. 860, Springer, 1981.

[2] W. Smoleński: Pre-supports and kernels of probability measures in Frechel spaces, Demonstratio Math. 10 (1977), 751-762.

[3] A. Wiśniewski: oral communication.

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[illegible]

A FABRY-TYPE GAP THEOREM FOR TAYLOR SERIES

1. Introduction

Let f(z) be an analytic function which is regular about the origin where it admits the expansion

$$f(z) = \sum_{n=1}^{\infty} a_n z^{\lambda_n}.$$

Gap theorems constitute the results concerning the relation between the lacunary distribution of the exponents $\{\lambda_n\}$ and the impossibility of the analytic continuation of the function f(z) beyond its circle of regularity. Two main [illegible] made in this direction; the first was made by [illegible] (cf. E.C. Titchmarsh ([10]; p. 223)) who showed [illegible]

$$\frac{\lambda_{n+1}}{\lambda_n} > 1 + \theta, \quad (\theta > 0)$$

where θ is a positive number independent of n, then the circle of regularity of f(z) will be its natural boundary. The second [illegible] Fabry [ ] who relaxed the restriction (1.1) to the condition

$$\lim_{n\to\infty} \frac{\lambda_n}{n} = \infty$$

[illegible] discussion [illegible]

[illegible] [ ] showed that if λ_n is a sequence [illegible] then there exists

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Jerome A. Adepoju

A FABRY-TYPE GAP THEOREM FOR FABER SERIES

1. Introduction

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The gap-theorems constitute the results concerning the relation between the lacunary distribution of the exponents (λ_ν) and the impossibility of the analytic continuation of the function f(z) beyond its circle of regularity. Two main attempts were made in this direction; the first was made by Hadamard c.f., e.g. Titchmarsh ([10]; p.223) who showed that if

(1.1) $$\frac{\lambda_{\nu+1}}{\lambda_\nu} \geq 1 + \theta, \quad (\nu \geq 1)$$

where θ is a positive number independent of ν, then the circle of regularity of f(z) will be its natural boundary. The second was by Fabry [5] who relaxed the restriction (1.1) to the condition that

(1.2) $$\lim_{\nu\to\infty} \frac{\lambda_\nu}{\nu} = \infty$$

with the same conclusion obtained.

Erdos [3] and Polya [9] proved that if λ_ν is a sequence of integers for which $\lim_{\nu\to\infty}\inf \frac{\lambda_\nu}{\nu} < \infty$, then there exists a power series $\sum_{\nu=1}^{\infty} a_\nu z^\nu$ whose circle of convergence is the unit circle and for which the unit circle is not the natural boundary, thus showing that in some sense Fabry's result is best possible. This paper extends these results concerning the power series in disks to Faber series in Faber regions.

2. Preliminaries and notation

Our notation is basically that of Newns [8], Erdos [3], Nassif [7] and Turan [12]. A Faber curve C is a simple closed regular curve which is the map in the z-plane of the circle $|t| = \gamma$ by the Faber transformation

$$z \equiv \phi(t) = t + \sum_{n=0}^{\infty} a_n t^{-n} \tag{2.1}$$

which is supposed to be conformal for $|t| > T$ for some $T < \gamma$.

The transformation $\Phi(t)$ has an inverse

$$t = \psi(z) \tag{2.2}$$

which is assumed to be conformal for $|z| > T'$, say, in the z-plane. Studies on Faber transformation and Faber regions can be found in [4], [6], [8], and [13].

It is known [4] that if the function F(z) is regular in the Faber region $D(C_r)$, $r > T$, then it admits in the same region the Faber series

$$F(z) = \sum_{k=0}^{\infty} \beta_k f_k(z) \tag{2.3}$$

where

$$\beta_k = \frac{1}{2\pi i}\int_{|t|=r'} F\{\phi(t)\} t^{-k-1} dt, \quad T < r' < r,\ k \geqslant 0. \tag{2.4}$$

It follows that if F(z) has singularities on the Faber curve C_γ, then

$$\lim_{k\to\infty} \sup |\beta_k|^{1/k} = \frac{1}{\gamma}\ . \tag{2.5}$$

From [8, p.749] we have that

$$f_n(\phi(t)) = t^n + P_n(t); \quad |t| = r > T \tag{2.6}$$

where

$$|P_n(t)| \leqslant \frac{lT_1^n}{2\pi\Delta}, \quad T < T_1 < r, \tag{2.7}$$

where l is the length of the curve C_{T_1}, and Δ is the distance between the curves C_{T_1} and C_r.

We easily establish, by applying [8, p.749], (2.7), and [7, equation (2.9)], that $P_n(t)$ admits the expansion

$$P_n(t) = P_{n,o} + \sum_{k=1}^{\infty} P_{n,k} t^{-k} \tag{2.8}$$

with $P_{n,o} = o$, and thus $P_n(t)$ contains negative powers of t only.

Two familiar results which are closely related to the gap-theory will be applied here. The first is the latest development of the well-known gap-theory inequality due to Littlewood [6] and Turan [12, equation (1.7) p.24] and given as follows.

Suppose that n is an integer $\geqslant 2$, $(b_\nu)_1^n$ are arbitrary complex numbers and that $(\lambda_\nu)_1^n$ are increasing positive integers. Then

$$\max_{o \leqslant x \leqslant 2\pi} \left| \sum_{\nu=1}^{n} b_\nu e^{i\lambda_\nu x} \right| \leqslant \left(\frac{48\pi}{\delta}\right)^n \max_{\alpha \leqslant x \leqslant \alpha + \delta} \left| \sum_{\nu=1}^{n} b_\nu e^{i\lambda_\nu x} \right| \tag{2.9}$$

where α is a real number and $o < \delta < 2\pi$.

This inequality relates the maximum modulus of the finite sum $\sum_{\nu=1}^{n} b_\nu e^{i\lambda_\nu x}$ on the whole unit circle to the maximum modulus of the same sum on an arc of the circle. It is considered the "real" foundation of the gap theory. The second result is also known (c.f. Turan [12, equation (2.9) p.24]), but it will be modified here to suit our argument in Faber region. Our modification is as follows:

Let t_1 be a certain fixed point of the circle $|t| = \gamma$ and let $\arg t_1 = \omega$, and choose positive numbers η_1 and η_2 such that

$$(2.10) \qquad \eta_1 < \frac{1}{10}, \quad \eta_2 < \frac{1}{2}\left(1 - \frac{T}{\gamma}\right), \quad \eta_2 < \frac{11}{120}\eta_1^2.$$

For this choice, the counterparts of (2.3) and (2.5) of [12] are the contours C_1 and C_2 in the t-plane, corresponding to the point t_1 and to the numbers η_1 and η_2, constructed as follows:

$$(2.11) \quad C_1 = \begin{cases} |t| = \gamma(1+2\eta_2) & \text{when } |\arg t - \omega| < 2\eta_1 \\ \gamma(1-2\eta_2) \leqslant |t| \leqslant \gamma(1+2\eta_2) & \text{when } |\arg t - \omega| = 2\eta_1 \\ |t| = \gamma(1-2\eta_2) & \text{when } 2\eta_1 < |\arg t - \omega| \leqslant \pi \end{cases}$$

and

$$(2.12) \quad C_2 = \begin{cases} |t| = \gamma(1 \pm \eta_2) & \text{when } |\arg t - \omega| < \eta_1 \\ \gamma(1-\eta_2) \leqslant |t| \leqslant \gamma(1+\eta_2) & \text{when } |\arg t - \omega| = \eta_1. \end{cases}$$

We easily verify by virtue of (2.10) that if $t_0 \varepsilon \bar{D}(C_2)$ while t describes the arcs of C_1, then

$$(2.13) \quad \begin{cases} |\tau| < 1 - \dfrac{\eta_2}{2} \; ; & |\arg \tau| \leqslant \eta_1 \\ |\tau| < 1 + \dfrac{15}{4}\eta_2 ; & \eta_1 \leqslant |\arg \tau| \leqslant \pi \end{cases} \; ; \quad (\tau = t/t_0).$$

Now, let N be a positive number (presumably large) and consider the polynomial

$$(2.14) \qquad \Pi_n(\tau) = \sum_{k=0}^{n} \frac{(\tau + N)^k}{(1 + N)^{k+1}}$$

which on expansion gives

$$(2.15) \qquad \Pi_n(\tau) = \sum_{k=0}^{n} C_{n,k}\tau^k$$

where

(2.16) $$C_{n,k} = \frac{1}{(1+N)^{k+1}} \sum_{m=0}^{n-k} \binom{m+k}{m} \left(\frac{N}{1+N}\right)^{m}; \qquad 0 \leqslant k \leqslant n.$$

On the other hand, summing the series on the right hand side of (2.14), we obtain

(2.17) $$\frac{1}{1-\tau} - \Pi_n(\tau) = \left(\frac{\tau+N}{1+N}\right)^{n+1} \Big/ 1-\tau.$$

We can deduce from (2.10), (2.11), (2.12) and (2.13) that when $t_0 \in \bar{D}(C_2)$ and t is in C_1, then there is a positive number η_3 such that for sufficiently large values of N,

(2.18) $$\left|\frac{1}{1-\tau} - \Pi_n(\tau)\right| < \frac{1}{\eta_3}(1-\eta_3)^{n+1}; \qquad (\tau = t/t_0).$$

Equation (2.18) constitutes the needed modified version of the result of Turan [12, equation (2.9) p.24] for Faber regions we need.

3. The main theorem

Theorem 1. Suppose that the function $g(z)$, given by the Faber series

(3,1) $$g(z) = \sum_{\nu=1}^{\infty} a_\nu f_{\lambda_\nu}(z)$$

is regular within and has singularities on the Faber curve C. If the exponents $(\lambda_\nu)_1^\infty$ satisfy the restriction (1.2) of Fabry, then each point of the curve C is a singularity of $g(z)$.

Proof. We note from (2.5) that as $g(z)$ is regular within and has singularities on the Faber curve C_γ, we have

(3.2) $$\limsup_{\nu \to \infty} |a_\nu|^{1/\lambda_\nu} = \frac{1}{\gamma}.$$

Suppose, to the contrary that $g(z)$ is regular at a point z on C and that $z_1 = \phi(t_1)$ so that t_1 lies on $|t| = \gamma$ and that arg $t_1 = \omega$. Choose the positive numbers η_1 and η_2 of (2.10) so small that if the contour C_1 is constructed as in (2.11), and corresponding to the point t_1 and numbers η_1 and η_2, then the image Ω_1 in the z-plane of the contour C_1 will be such that the function $g(z)$ is regular in $D(\bar{\Omega}_1)$. Thus there is a finite positive constant K for which

$$(3.3) \qquad \max_{z \in \Omega_1} |g(z)| = \max_{t \in C_1} |g\{\phi(t)\}| < K$$

where

$$(3.4) \qquad I = \frac{1}{2\pi i} \int_{C_1} \frac{g\{\phi(t)\}}{t} \left\{ \frac{1}{1-\tau} - \Pi_{\lambda_n}(\tau) \right\} dt =$$

$$= \frac{1}{2\pi i} \int_{C_1} \frac{g\{\phi(t)\}dt}{t - t_o} - \frac{1}{2\pi i} \int_{C_1} \frac{g\{\phi(t)\}}{t} \Pi_{\lambda_n}(\tau) dt =$$

$$= I_1 - I_2; \qquad (t_o \in \bar{D}(C_2)).$$

To evaluate I_1 we denote by $\mathcal{U}(T, C_1)$ the annular region $\bar{D}(C_1) - \bar{D}(T)$. Since $g(z)$ is regular in $D(\Omega_1)$, it follows that $g\{\phi(t)\}$ is regular in $\mathcal{U}(T, C_1)$, and we observe from (2.6) and (3.1) that, in the annulus $D(\gamma) - \bar{D}(T)$, the function $g\{\phi(t)\}$ can be represented as

$$(3.5) \qquad g\{\phi(t)\} = \sum_{\nu=1}^{\infty} a_\nu t^{\lambda_\nu} + \sum_{\nu=1}^{\infty} a_\nu P_{\lambda_\nu}(t).$$

In view of (2.7) and (3.5) we deduce that in the exterior domain $E(T)$ of T,

$$\limsup_{\nu \to \infty} |a_\nu P_{\lambda_\nu}(t)|^{1/\lambda_\nu} \leq \frac{T_1}{\gamma} < 1 \qquad (T_1 < \gamma).$$

It follows that the function whose restriction in $D(\gamma) - \bar{D}(T)$ is the sum of the series

$$H(t) = \sum_{\nu=1}^{\infty} a_\nu P_{\lambda_\nu}(t); \quad t \,\varepsilon\, (T, C_1) \tag{3.6}$$

will be regular in $E(T)$ and in particular in $\mathcal{U}(T,C_1)$. Since $g\{\phi(t)\}$ and $H(t)$ are both regular in $\mathcal{U}(T,C_1)$, so also is the function $G(t) = g\{\phi(t)\} - H(t)$. Also, in view of (3.2) and (3.5) we may deduce that in $\mathcal{U}(T,C_1)$,

$$g\{\phi(t)\} = G(t) + H(t) \tag{3.7}$$

where $H(T)$ is given by (3.6) and $G(t)$ is regular in $\bar{D}(C_1)$ and in $D(\gamma)$, and is given by

$$G(t) = \sum_{\nu=1}^{\infty} a_\nu t^{\lambda_\nu}. \tag{3.8}$$

Introducing (3.6) and (3.7), in (3.4) with $t_0 \,\varepsilon\, D(C_1)$, we deduce that

$$I_1 = G(t_0) + \sum_{\nu=1}^{\infty} \frac{a}{2\pi i} \int_{C_1} \frac{P_{\lambda_\nu}(t)}{t - t_0}\, dt = G(t_0). \tag{3.9}$$

To evaluate I_2, we observe from (2.3), (2.4), and (3.1) that

$$I_2 = \sum_{k=0}^{\lambda_n} C_{\lambda_n,k}\, t_0^k \,\frac{1}{2\pi i} \int_{C_1} g\{\phi(t)\}\, t^{-k-1} dt = S_{\lambda_n}(t_0) \tag{3.10}$$

and since

$$\frac{1}{2\pi i} \int_{C_1} g\{\phi(t)\}\, t^{-k-1} dt = \begin{cases} a_\nu & \text{when } k = \lambda_\nu \\ 0 & \text{otherwise,} \end{cases}$$

we have

$$S_{\lambda_n}(t_0) = \sum_{\nu=1}^{n} C_{\lambda_n,k}\, a_\nu t_0^{\lambda_\nu} \tag{3.11}$$

so that

$$I = G(t_o) - S_{\lambda_n}(t_o). \tag{3.12}$$

Applying (2.18), (3.11) and the inequality (3.3) to the integral given by (3.4) and by the construction of the contour C_1 and the inequalities (2.10) we can deduce that

$$\max_{t_o \varepsilon \bar{D}(C_2)} \left| G(t_o) - S_{\lambda_n}(t_o) \right| < \frac{2K}{\eta_3}(1-\eta_3)^{\lambda_n+1}. \tag{3.13}$$

Putting $t_o = r_o e^{ix}$ and in view of the construction of C_2, we note that

$$\gamma(1-\eta_2) \leqslant r_o \leqslant \gamma(1+\eta_2); \quad \omega - \eta_1 \leqslant x \leqslant \omega + \eta_1$$

so that (3.12) may be written as

$$\max_{\substack{\gamma(1-\eta_2) \leqslant r_o \leqslant \gamma(1+\eta_2) \\ \omega-\eta_1 \leqslant x \leqslant \omega+\eta_1}} \left| G(r_o e^{ix}) - S_{\lambda_n}(r_o e^{ix}) \right| < \frac{2K}{\eta_3}(1-\eta_3)^{\lambda_n+1}. \tag{3.14}$$

Since

$$S_{\lambda_{n+1}}(r_o e^{ix}) - S_{\lambda_n}(r_o e^{ix}) = \sum_{\nu=1}^{n+1} b_\nu e^{i\lambda_\nu x},$$

we have from (3.11)

$$\begin{cases} b_\nu = (C_{\lambda_{n+1},\lambda_\nu} - C_{\lambda_n,\lambda_\nu}) a_\nu r_o^{\lambda_\nu}, & \nu = 1,2,\ldots,n \\ b_{n+1} = C_{\lambda_{n+1},\lambda_{n+1}} a_{n+1} r_o^{\lambda_{n+1}}. \end{cases} \tag{3.15}$$

Applying the inequality (2.9) of Littlewood-Turan, we put n+1 for n, $2\eta_1$ for δ, $\omega - \eta_1$ for α and take $\{b_\nu\}_1^{n+1}$, as given by (3.15), to obtain, in view of (3.14), the inequality

(3.16) $$E_n = \max_{0\le x\le 2\pi} \left| S_{\lambda_{n+1}}(r_o e^{ix}) - S_{\lambda_n}(r_o e^{ix}) \right| <$$

$$< \frac{96\pi K}{\eta_1\eta_3} \left(\frac{24\pi}{\eta_1}\right)^{n/\lambda_n} (1-\eta_3)^{\lambda_n}.$$

Applying the gap condition (1.2) of Fabry, we have that there is an integer $n_o = n_o(\eta_1, \eta_2)$ such that for $n > n_o$,

(3.17) $$E_n < \frac{96\pi K}{\eta_1\eta_3}\left(1 - \frac{\eta_3}{2}\right)^{\lambda_n}, \text{ when } \gamma(1-\eta_2) \le r_o \le \gamma(1+\eta_2).$$

Furthermore, by virtue of (3.17), the strictly increasing property of the exponents (λ_ν) and the maximum modulus principle, we can deduce that

(3.18) $$\max_{|t|\le\gamma(1+\eta_2)} \left| S_{\lambda_{n+l}}(t) - S_{\lambda_n}(t) \right| < \frac{192\pi K}{\eta_1\eta_3^2}\left(1 - \frac{\eta_3}{2}\right)^{\lambda_n};$$

$(n > n_o$, l positive integer $\geq 1)$.

It follows that the set $\{S_{\lambda_n}(t)\}$ of polynomials converges uniformly in $|t| \le \gamma(1+\eta_2)$ to a function $p(t)$ which is therefore regular there. On the other hand we note from (3.13) that the same sequence $\{S_{\lambda_n}(t)\}$ converges uniformly to $G(t)$ in the annulus sector $D(C_2)$. Hence $p(t) \equiv G(t)$ in $\bar{D}(C_2)$ and therefore the function $p(t)$ will be equal to $G(t)$ in the region where both functions are regular. But we have seen that $G(t)$ is regular in $D(\bar{C}_1)$, therefore, we may infer that $p(t) = G(t)$ in $D(\bar{C}_1)$. In particular, in view of (3.8) we may write

(3.19) $$p(t) = \sum_{\nu=0}^{\infty} a_\nu t^{\lambda_\nu}; \quad |t| = \gamma(1-2\eta_2).$$

Since $p(t)$ is regular in $|t| \le \gamma(1+\eta_2)$ and by the uniqueness property of Taylor's expansion, we may infer that the expression (3.19) is valid in $|t| \le \gamma(1+\eta_2)$. Thus

$$\limsup_{\nu\to\infty} |a_\nu|^{1/\lambda_\nu} \leqslant \frac{1}{\gamma(1+\eta_2)}$$

and this contradicts (3.2), and the theorem is proved.

4. Construction of Faber series

Before stating our second result in which we shall show that the condition (1.2) is best possible, we first construct the Faber series

$$g(z) = \sum_{k=1}^{\infty} a_k f_{n_k}(z) \tag{4.1}$$

where

$$a_k = \alpha_k \gamma^{-k} \tag{4.2}$$

and the coefficients $\{\alpha_k\}_1^\infty$ are determined in such a way that function g(z) is regular inside, has singularities on C and possesses at least one regular point on C.

To do this we argue as in [3]. We observe that if $(\lambda_\nu)_1^\infty$ is a sequence of strictly increasing positive integers such that

$$\liminf_{\nu\to\infty} \frac{\lambda_\nu}{\nu} < \infty$$

then there is a constant number a such that

$$1 \leqslant \liminf_{\nu\to\infty} \frac{\lambda_\nu}{\nu} = a < \infty$$

and hence for a finite number b > a we have

$$\liminf_{\nu\to\infty} \frac{\lambda_\nu}{\nu} < b, \quad (b > 1). \tag{4.3}$$

Having fixed the number b, the following Lemma is to be established.

Lemma 1. There exists two sequences $\{a_i\}_1$ and $\{b_i\}_1$, of positive integers for which

$$b_i = \left[(1 + 3b)a_i\right], \quad a_{i+1} > b_i^2; \quad (i \geq 1) \tag{4.4}$$

such that the number τ_i of indices $\{\lambda_\nu\}$ in the invervals $I_i = (a_i, b_i)$ is $\tau_i \geq 2a_i$, and such that if n_k denote the indices lying in I_i, then

$$\liminf_{k\to\infty} \frac{n_k}{k} \leq \infty \tag{4.5}$$

where square brackets denote integral part throughout.

P r o o f . By (4.3) there exists a sequence $\{m_j\}_1^\infty$ of strictly increasing integers such that

$$m_1 > 9(b + 1) \quad \text{and} \quad \lambda_{m_j} < b_{m_j} \tag{4.6}$$

writing $\mu_j = \left[b_{m_j}\right]$, we easily see that

$$\lambda_{m_j} \leq \mu_j. \tag{4.7}$$

If r_j denote the number of indices $\{\lambda_\nu\}$ which are less than or equal to μ_j, then

$$r_j \geq \frac{\mu_j}{b}, \quad (j \geq 1).$$

Also, if we put

$$\nu_j = -\left[\frac{\mu_j}{3b}\right]; \quad (j \geq 1)$$

then

$$s_j \leq \nu_j; \quad (j \geq 1)$$

where s_j is the number of the indices $\{\lambda_\nu\}$ which are less than or equal to ν_j.

Considering the intervals (ν_j, μ_j), we easily deduce that

$$t_j = r_j - s_j \geq \frac{2}{3b}\mu_j \geq 2\nu_j; \quad (j \geq 1) \tag{4.8}$$

where t_j is the number of indices $\{\lambda_\nu\}$ lying in (ν_j, μ_j). The required sequences are constructed as follows:

$$(4.9)\quad \begin{cases} a_1 = \nu_1; \quad b_1 = \left[(1+3b)a_1\right], \\ a_2 = \min\limits_j \left\{\nu_j : \nu_j > b_1^2\right\}, \quad b_2 = \left[(1+3b)a_2\right] \\ \text{and in the i-th step,} \\ a_i = \min\limits_j \left\{\nu_j : \nu_j > \nu_{i-1}^2\right\}, \quad b_i = \left[(1+3b)a_i\right]. \end{cases}$$

If we now denote by $\{n_k\}$ the indices $\{\lambda_\nu\}$ lying in the intervals

$$I_i = (a_i, b_i), \quad i = 1,2,\ldots;$$

we can easily deduce that

$$(4.10)\qquad \liminf_{k\to\infty} \frac{n_k}{k} \leqslant \liminf_{i\to\infty} \frac{n_{\rho_i}}{\rho_i} \leqslant \frac{1+3b}{2} < \infty$$

where

$$\rho_i = \max_k \,(a_i < n_k < b_i); \quad (i \geqslant 1)$$

and the lemma is established.

In the construction of $g(z)$, only the n_k's shall be used but with the restriction that the terms corresponding to the indices $\{n_{\rho_i}\}$, $i = 1,2,\ldots$ are not zeros (cf. [3]).

The coefficients $\{\alpha_k\}$ of (4.2) are evaluated as follows: We fix the integer r by

$$(4.11)\qquad r > \max(12,\ 1 + 13b)$$

and to each interval $I_i = (a_i, b_i)$ we associate the interval J_i defined by

$$(4.12)\qquad J_i = \left(\frac{a_i}{r}\left(1 - \frac{1}{r}\right),\ \ \frac{b_i}{r}\left(1 + \frac{3}{r}\right)\right).$$

The coefficients $\{\alpha_k\}$ for which the corresponding indices $\{n_k\}$ in (4.1) lie in the interval I_i, $(i \geqslant 1)$ are determined by the system of the homogeneous linear equations

$$(4.13)\qquad \sum_{n_k \in I_i} x_k \binom{nk}{n} \left(\frac{r-1}{r}\right)^{n_k - m} = 0, \quad m \in J_i.$$

Clearly, (4.13) has a solution (non-unique) for $\{x_k\}$ which are not all zeros (see [3]). If we write

$$\xi_i = |x_k|; \quad x_k = \xi_i \alpha_k \tag{4.14}$$

in (4.13) we easily note that there is at least one value α_{k_i} of the set $\{\alpha_k\}$ for which

$$|\alpha_k| = 1; \quad \limsup_{k \to \infty} |\alpha_k|^{1/k} = 1 \tag{4.15}$$

and hence

$$\limsup_{n_k \to \infty} |\alpha_k|^{1/n_k} = \frac{1}{\gamma} \tag{4.16}$$

so that $g(z)$ is regular in and has singularities on the Faber curve C. Since $x_{\rho_i} \neq 0$, the coefficients $a_{\rho_i} = 0$, $i = 1, 2, \ldots$, and hence the Faber series $g(z)$ will always contain the non-zero terms $a_{\rho_i} f_{n_k}(z)$; $i = 1, 2, \ldots$.

5. The converse theorem

Theorem 2. Let $(\lambda_\nu)_1^\infty$ be a sequence of strictly increasing positive integers for which $\liminf_{\nu \to \infty} \frac{\lambda_\nu}{\nu} < \infty$. Then there exists a Faber series $\sum_{\nu=1}^{\infty} a_\nu f_{\lambda_\nu}(z)$ which is regular inside, has singularities on C, and is regular at least one point of C.

Proof. We first show that the power series

$$G(t) = \sum_{k=1}^{\infty} a_k t^{n_k} \tag{5.1}$$

where $\{a_k\}$ are evaluated as in (4.2) and (4.13), is regular in $D(\gamma)$ and also at the point $t = \gamma$. That $G(t)$ is regular in $D(\gamma)$ follows easily from (4.16). To prove that $G(t)$ is regular at $t = \gamma$, write

(5.2) $$t = \gamma\left(\xi + \frac{r-1}{r}\right)^{n_k}$$

so that $G(t) = \sum_{k=1}^{\infty} \alpha_k \left(\xi + \frac{r-1}{r}\right)^{n_k} = h(\xi)$, say, where

(5.3) $$h(\xi) = \sum_{k=1}^{\infty} \alpha_k \sum_{m=0}^{n_k} \binom{n_k}{m} \xi^m \left(\frac{r-1}{r}\right)^{n_k-m} .$$

That is $h(\xi) = \sum_{m=0}^{\infty} b_m \xi^m$, where

(5.4) $$b_m = \sum_{n_k \geqslant m} \alpha_k \binom{n_k}{m} \left(\frac{r-1}{r}\right)^{n_k-m}$$

and argue as in [3]. In view of (5.2) $G(t)$ is regular at $t =$ if and only if $h(\xi)$ is regular at $\xi = \frac{1}{r}$ and this would be the case if the radius of regularity of (5.3) is greater than $\frac{1}{r}$. As in [3], we can show by considering the cases where m lies or m does nor lie in any of the intervals $\{J_i\}_i^{\infty}$, that

$$|b_m| < 2Kr^2\left(r - \frac{1}{3r}\right)^m ; \quad (m > 1)$$

in both cases, so that

(5.5) $$\limsup_{m\to\infty} |b_m|^{1/m} \leqslant r - \frac{1}{3r} ,$$

as required.

We now consider the Faber series

(5.6) $$g(z) = \sum_{k=1}^{\infty} a_k f_{n_k}(z).$$

In view of [8, p.749], we have as in (2.6) and (2.7) that

(5.7) $$f_{n_k}(z) = \{\psi(z)\}^{n_k} + q_{n_k}(z)$$

where

$$\left|q_{n_k}(z)\right| \leqslant \frac{lT_1^{n_k}}{2\pi\Delta}, \tag{5.8}$$

l, T_1 and Δ are as given in (2.7). Hence the Faber series (4.1) may be written in the form

$$g(z) = \sum_{k=1}^{\infty} a_k\{\psi(z)\}^{n_k} + \sum_{k=1}^{\infty} a_k q_{n_k}(z). \tag{5.9}$$

Write $z_0 = \phi(\gamma)$ so that $z_0 \varepsilon C$. Since $\sum_{k=1}^{\infty} a_k t^{n_k}$ of (5.1) is regular at $t = \gamma$, then the series

$$\sum_{k=1}^{\infty} a_k\{\psi(z)\}^{n_k}; \quad (t = \psi(z)) \tag{5.10}$$

will be regular at the point $z_0 \varepsilon C$. Moreover, in view of (4.2), (4.16) and (5.8) we can deduce that the series $\sum_{k=1}^{\infty} a_k q_{n_k}(z)$ is regular in $E(C_{T_1})$ for $T_1 > T$ and arbitrarily close to T.

Hence the Faber series (4.1) is regular at the point $z_0 \varepsilon C$. Moreover, recalling (2.5) and (4.15), we conclude that the same Faber series is regular within, has singularities on C, and is regular at the point $z_0 \varepsilon C$. The required properties of the Faber series stated in Theorem 2 hold, and the theorem is, therefore, established.

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REFERENCES

[1] J.A. Adepoju : Basic sets of Goncarov polynomials and Faber Regions. Ph.D. Thesis, University of Lagos, 1979.

[2] S.Ya. Alper : On the overconvergence of polynomials. Acad. Nauk SSSR (NS) 59(1948), 625-627.

[3] P. Erdos : On the converse of Fabry´s gap theorem, Trans. Amer. Math. Soc. 57 (1945), 102-104.

[4] G. Faber : Uber die polynomische Entwickelung, Math. Annalen, 57 (1903), 389-408.

[5] E. Fabry : Sur les series de Taylor qui ont une infinite de points singuliers. Acta Math. (1899), 65-87.

[6] J.E. Littlewood : Mathematical Notes (12): An inequality for a sum of cosines. J. Lond. Math. Soc. 12, (1937), 217-222.

[7] M. Nassif : On the transpose of simple sets of polynomials effective in Faber regions, Proc. Amer. Math. Soc. 25 (1970), 208-219.

[8] W.F. Newns : The range of effectiveness of basic sets, Proc. Lond. Math. Soc. Ser. 3, 18, (1968), 745-767.

[9] G. Polya : On converse gap theorems, Trans. Amer. Math. Soc. 52, (1942), 65-71.

[10] E.C. Titchmarsh : The theory of functions. Oxford, 1939, Second Ed.

[11] P. Turan : On a theorem of Littlewood, J. Lond. Math. Soc. 21, (1946), 268-275.

[12] P. Turan : On the gap theorem of Fabry, Acta Math. Acad. Sci. Hungar. (Budapest), 1 (1947) 21-29.

[13] J.I. Ullman : Studies on Faber polynomials I, Trans. Amer. Math. Soc. 94, (1960), 515-528.

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MULTIPLICITIES OF EIGENVALUES

A relationship between the algebraic multiplicity, the geometric multiplicity and the multiplicity of λ as root of the minimal polynomial of a complex matrix with eigenvalue λ is established.

For any complex (n,n)-matrix with eigenvalue λ let $m = m(\lambda)$ be the algebraic multiplicity, $\hat{m} = \hat{m}(\lambda)$ be the geometric multiplicity, and $m^* = m^*(\lambda)$ be the multiplicity of λ as root of the minimal polynomial of the matrix. It is well known that $\hat{m} \leqslant m$, see e.g. Gantmacher [2]. The following theorem gives a relationship between all three multiplicities.

Theorem. $\hat{m} + m^* - 1 \leqslant m \leqslant \hat{m}m^*$.

Proof. Take any complex (n,n)-matrix A with eigenvalue $\lambda \in C$. We consider the powers $A_\lambda^0 = I$, A_λ, $A_\lambda^2, \ldots$ of $A_\lambda = A - \lambda I$. The kernels of the linear transformations $A_\lambda^i\colon C^n \to C^n$ $(i = 0,1,\ldots)$ are denoted by K_i and their dimensions as d_i. It follows that $\{0\} = K_0 \subset K_1 \subset K_2 \subset \ldots$ and hence that $0 = d_0 \leqslant d_1 \leqslant d_2 \leqslant \ldots$. As $d_1 = \hat{m} > 0$, it follows that $d_0 < d_1$. Note that $d_i \leqslant n$ for each $i \in \{0,1,\ldots,n\}$. The smallest index i for which d_i is greatest, is denoted by $\underline{i}$. Then $\underline{i} \geqslant 1$ and $d_{\underline{i}-1} < d_{\underline{i}} = d_{\underline{i}+1} = d_{\underline{i}+2} = \ldots$. We now show that

$$0 = d_0 < d_1 < \ldots < d_{\underline{i}-1} < d_{\underline{i}} = d_{\underline{i}+1} = \ldots .$$

For $\underline{i} \leqslant 2$ this is clear. So we may assume that $\underline{i} > 2$. As $d_{\underline{i}-1} < d_{\underline{i}}$ it follows that $K_{\underline{i}-1}$ is a proper subspace of $K_{\underline{i}}$.

Let $u \in K_{\underline{i}} \setminus K_{\underline{i}-1}$. Hence, $A_{\lambda}^{\underline{i}-1}u \neq 0 = A_{\lambda}^{\underline{i}}u$ and therefore $A_{\lambda}^{\underline{i}-2}(A_{\lambda}u) \neq 0 = A_{\lambda}^{\underline{i}-1}(A_{\lambda}u)$. This means that $A_{\lambda}u \in K_{\underline{i}-1} \setminus K_{\underline{i}-2}$. So $K_{\underline{i}-2}$ is a proper subspace of $K_{\underline{i}-1}$, and therefore $d_{\underline{i}-2} < d_{\underline{i}-1}$. Similarly, one can show that $d_{\underline{i}-3} < d_{\underline{i}-2}, \dots, d_1 < d_2$. The "dimension-jumps" $d_i - d_{i-1}$ $(i = 1, 2, \dots)$ form a non-increasing sequence, i.e. $d_1 - d_0 \geqslant d_2 - d_1 \geqslant d_3 - d_2 \geqslant \dots$ with $d_i - d_{i-1} = 0$ for $i > \underline{i}$. So $d_1 - d_0$ is the "maximal jump".

As $(d_1 - d_0) + (d_2 - d_1) + \dots + (d_{\underline{i}} - d_{\underline{i}-1}) = d_{\underline{i}}$, it follows that $d_{\underline{i}} \leqslant \underline{i}(d_1 - d_0)$. On the other hand, all "dimension-jumps" are $\geqslant 1$. Hence, $(d_1 - d_0) + \underline{i} - 1 \leqslant d_{\underline{i}}$.

From the theory of Jordan canonical form, see e.g. [1], we know that $\underline{i} = m^*$ and $d_{\underline{i}} = m$. Taking also into account that $d_0 = 0$, we find that $\hat{m} + m^* - 1 \leqslant m \leqslant m^* \hat{m}$, which is the desired result.

Corollary. $\hat{m} = m \Leftrightarrow m^* = 1$, and $m^* = m \Leftrightarrow \hat{m} = 1$.

Proof. Direct consequence of Theorem 1.

The above corollary asserts that A is similar with a diagonal matrix iff $m^* = 1$ for all eigenvalues of A (i.e. the minimal polynomial of A has only single roots). It also follows that the characteristic and the minimal polynomial are the same (except for a factor -1) iff $\hat{m} = 1$ for all eigenvalues of A.

REFERENCES

[1] A. Galperin, Z. Waksman: An elementary approach to Jordan theory, Amer. Math. Monthly, 87 (1980) 728-732.

[2] F.R. Gantmacher: Applications to the theory of matrices, Interscience Publ., Inc., New York, 1959.

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OSCILLATION PROPERTIES OF SOLUTIONS OF A SYSTEM OF SECOND ORDER DIFFERENTIAL EQUATIONS ON THE PLANE

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1. Introduction

In this paper we will consider systems of two ordinary differential equations of the second order each of the general form

$$\begin{cases} \ddot{\underset{\sim}{x}}_1 = \underset{\sim}{f}(\underset{\sim}{j},\underset{\sim}{x}_1,\underset{\sim}{x}_2,\dot{\underset{\sim}{x}}_1,\dot{\underset{\sim}{x}}_2), \\ \ddot{\underset{\sim}{x}}_2 = \underset{\sim}{g}(\underset{\sim}{j},\underset{\sim}{x}_1,\underset{\sim}{x}_2,\dot{\underset{\sim}{x}}_1,\dot{\underset{\sim}{x}}_2), \end{cases} \tag{1.1}$$

(where $\underset{\sim}{j}$ is the identity function, that is a function for which $\underset{\sim}{j}(x) = x$ for all $x \in R$ - see Menger [1]), or the apparently more special system

$$\begin{cases} \ddot{\underset{\sim}{x}}_1 = \underset{\sim}{p}(\underset{\sim}{j},\underset{\sim}{x}_1,\underset{\sim}{x}_2,\dot{\underset{\sim}{x}}_1,\dot{\underset{\sim}{x}}_2) - \dot{\underset{\sim}{x}}_1\underset{\sim}{k}(\underset{\sim}{j},\underset{\sim}{x}_1,\underset{\sim}{x}_2,\dot{\underset{\sim}{x}}_1,\dot{\underset{\sim}{x}}_2), \\ \ddot{\underset{\sim}{x}}_2 = \underset{\sim}{g}(\underset{\sim}{j},\underset{\sim}{x}_1,\underset{\sim}{x}_2,\dot{\underset{\sim}{x}}_1,\dot{\underset{\sim}{x}}_2) - \dot{\underset{\sim}{x}}_2\underset{\sim}{k}(\underset{\sim}{j},\underset{\sim}{x}_1,\underset{\sim}{x}_2,\dot{\underset{\sim}{x}}_1,\dot{\underset{\sim}{x}}_2). \end{cases} \tag{1.2}$$

They can be considered as equations of movement in the x_1x_2-plane of a material point of unit mass moving under the influence of the plane force $[\underset{\sim}{f},\underset{\sim}{g}]$ or the influence of the plane force $[\underset{\sim}{p},\underset{\sim}{q}]$ and the resistance $-\underset{\sim}{k}[\dot{\underset{\sim}{x}}_1,\dot{\underset{\sim}{x}}_2]$ (parallel to the velocity).

We will show that under a very weak assumption (see assumptions F_2 or H_2 in §4) all the characteristics (that is all projections of solutions on the x_1x_2-plane) of the system (1.1) (or of the system (1.2)) will assume, locally, a very simple shape (see Theorem A of the n° 4.1 and Theorem B of the n° 4.2 and also the figures 3a, 3b and 3c.

Under a more strong assumption (see the assumption H_3 in §5) all the characteristics of the system (1.2) will assume a very simple global shape (see Theorem C of the n° 5.5 and also the Figures 5a,b,c,d ,d and e).

The method of proofs used here (especially of the proof of the fundamental Lemma 4.1) are similar to the method used in Tatarkiewicz´s paper [2] (especially to the method used in the proof of Lemma 2.1 of that paper).

2. Preliminaries

We admit some assumptions:

Assumption F_1. We have $C^0 \ni \underset{\sim}{f},\underset{\sim}{g}: R_5 \rightarrow R$ and any initial condition

$$(2.1) \qquad \underset{\sim}{x}_i(t^0) = x_i^0, \quad \dot{\underset{\sim}{x}}_i(t^0) = v_i^0, \quad i = 1,2$$

(where $(t^0,x_1^0,x_2^0,v_1^0,v_2^0) \in R_5$) determines uniquely the corresponding saturated solution (x_1,x_2) of the system (1.1).

Assumption F_2. For any point $(t,x_1,x_2,v_1,v_2) \in R_5$ such that

$$(2.2) \qquad x_1^2 + x_2^2 > 0$$

and

$$(2.3) \qquad -x_2v_1 + x_1v_2 = 0$$

we have

$$(2.4) \qquad -x_2\underset{\sim}{f}(t,x_1,x_2,v_1,v_2) + x_1\underset{\sim}{g}(t,x_1,x_2,v_1,v_2) > 0.$$

Assumption H_1. We have $C^0 \ni \underset{\sim}{p},\underset{\sim}{q},\underset{\sim}{k} : R_5 \rightarrow R$ and any initial condition (2.1) determines uniquely the corresponding saturated solution $(\underset{\sim}{x}_1,\underset{\sim}{x}_2)$ of the system (1.2).

Assumption H_2. For any point $(t,x_1,x_2,v_1,v_2) \in R_5$ verifying the conditions (2.2) and (2.3) we have

$$(2.5)\quad -x_2\underset{\sim}{p}(t,x_1,x_2,v_1,v_2) + x_1\underset{\sim}{q}(t,x_1,x_2,v_1,v_2) > 0.$$

Under the assumption F_1 (under the assumption H_1) all the solutions of the system (1.1) (of the system (1.2) respectively) are of class C^2.

Now we have

Lemma 2.1. If the functions $\underset{\sim}{p},\underset{\sim}{q},\underset{\sim}{k}$ satisfy the assumptions H_1 and H_2, then the functions $\underset{\sim}{f},\underset{\sim}{g}$ defined by the formulae

$$(2.6)\begin{cases}\underset{\sim}{f}(t,x_1,x_2,v_1,v_2) := \underset{\sim}{p}(t,x_1,x_2,v_1,v_2)-v_1\underset{\sim}{k}(t,x_1,x_2,v_1,v_2),\\ \underset{\sim}{g}(t,x_1,x_2,v_1,v_2) := \underset{\sim}{q}(t,x_1,x_2,v_1,v_2)-v_2\underset{\sim}{k}(t,x_1,x_2,v_1,v_2).\end{cases}$$

verify the assumptions F_1 and F_2.

Proof. Let the point $(t,x_1,x_2,v_1,v_2)\in R_5$ satisfy the condition (2.2), (2.3) and (2.4). Then by H_2 and the definition (2.6) we have

$$-x_2\underset{\sim}{f}(t,x_1,x_2,v_1,v_2) + x_1\underset{\sim}{g}(t,x_1,x_2,v_1,v_2) =$$

$$= -x_2\underset{\sim}{p}(t,x_1,x_2,v_1,v_2) + x_1\underset{\sim}{q}(t,x_1,x_2,v_1,v_2) +$$

$$- (x_2v_1 + x_1v_2)\underset{\sim}{k}(t,x_1,x_2,v_1,v_2) > 0$$

and the condition (2.4) is verified.

We admit one assumption more

Assumption H_3. For any point $(t,x_1,x_2,v_1,v_2)\in R_5$ such that (2.2) we have (2.5).

The assumption H_3 implies - of course - the assumption H_2.

Lemma 2.2. If the functions $\underset{\sim}{p},\underset{\sim}{q}$ satisfy the assumptions H_1 and H_3, then

$$(2.7)\quad \underset{\sim}{p}(t,0,0,v_1,v_2) = 0 = \underset{\sim}{q}(t,0,0,v_1,v_2)$$

for any triple $(t,v_1,v_2)\in R_3$.

Proof. Let $(t^o,v_1^o,v_2^o)\in R_3$ be a triple such that $q(t^o,0,0,v_1^o,v_2^o)\neq 0$. Suppose that $\underset{\sim}{q}(t^o,0,0,v_1^o,v_2^o)>0$ (if

$\underset{\sim}{q}(t^0,0,0,v_1^0,v_2^0) < 0$, then we consider the function $-\underset{\sim}{q}$). From the continuity of $\underset{\sim}{q}$ there exists an $\varepsilon > 0$ such that if $|x_1| < \varepsilon$ then $\underset{\sim}{q}(t^0,x_1,0,v_1^0,v_2^0) > 0$. It follows that if $-\varepsilon < x_1 < 0$ then

(2.8) $$x_1\underset{\sim}{q}(t^0,x_1,0,v_1^0,v_2^0) < 0.$$

If H_3 is verified and if $x_1 \neq 0$, then we have (2.5) if and only if for all $(t,x_1,x_2,v_1,v_2) \in R_5$. In particular for $(t^0,x_1,0,v_1^0,v_2^0) \in R_5$ and $-\varepsilon < x_1 < 0$ we have

$$x_1\underset{\sim}{q}(t^0,x_1,0,v_1^0,v_2^0) > 0$$

which contradicts (2.8) and it must be $\underset{\sim}{q}(t,0,0,v_1,v_2) = 0$.

An analogous proof gives us $\underset{\sim}{p}(t,0,0,v_1,v_2) = 0$ and we have proved the formula (2.7).

From Lemma 2.1 it follows that under the assumptions H_1 and H_3 the system (1.2) admits the trivial solution $\underset{\sim}{x}_1 = 0$, $\underset{\sim}{x}_2 = 0$.

As under our assumptions the initial conditions determine the corresponding solutions uniquely, we have

Lemma 2.3. Under assumption H_1 and H_3, if $(\underset{\sim}{x}_1,\underset{\sim}{x}_2)$ is a non trivial solution of the system (1.2) and $\underset{\sim}{x}_1(t_0) = 0 = \underset{\sim}{x}_2(t_0)$, then

(2.9) $$\dot{\underset{\sim}{x}}_1^2(t_0) + \dot{\underset{\sim}{x}}_2^2(t_0) > 0.$$

The condition H_3 has a simple mechanical (and geometrical) interpretation. The condition (2.5) is verified for all $(t,x_1,x_2,v_1,v_2) \in R_5$, which means that the vector $[\underset{\sim}{p},\underset{\sim}{q}]$ makes - if only (2.2) is verified - with the vector $[-x_2,x_1]$ an angle smaller than $\frac{\pi}{2}$. That is the vector $[\underset{\sim}{p},\underset{\sim}{q}]$ of the acting forces lies always on one side of the half straight line beginning at the pole (0,0) and going through the point (x_1,x_2) - see fig.1. And the vector of "resistance" $-\underset{\sim}{k}[\dot{x}_1,\dot{x}_2]$ parallels to the velocity.

The interpretation of the conditions F_2 and H_2 are similar and can be left to the reader.

3. Polar coordinates

Let $C^k \ni \underset{\sim}{x}_1, \underset{\sim}{x}_2 : I \to R$, where I is a connected subset of R and $k \geqslant 0$, be a couple of functions. We introduce a couple (r,u) of functions given for all $t \in I$ by the formulae

$$(3.1) \qquad \underset{\sim}{x}_1(t) = \underset{\sim}{r}(t)\cos \underset{\sim}{u}(t), \quad \underset{\sim}{x}_2(t) = \underset{\sim}{r}(t)\sin \underset{\sim}{u}(t).$$

Such functions always exist. If $\underset{\sim}{r}(t) > 0$ and $\underset{\sim}{u} \in C^0$ then it is known that the functions $\underset{\sim}{r}, \underset{\sim}{u}$ are determined uniquely (the function $\underset{\sim}{u}$ can be added by a constant $2\Pi k$) and $\underset{\sim}{r} \in C^0$. If $\underset{\sim}{r}(t) \geqslant 0$ and there exists a $t_0 \in I$ such that $\underset{\sim}{r}(t_0) = 0$ then it can happen that $\underset{\sim}{u}$ is not continuous - and this is the reason to suppose that $\underset{\sim}{r}, \underset{\sim}{u} \in C^0$ and to accept negative values for the function $\underset{\sim}{r}$ - see n° 5.6 and figures 4a and 4b.

If $\underset{\sim}{r}(t) > 0$ and $\underset{\sim}{u} \in C^0$ then $\underset{\sim}{r}, \underset{\sim}{u} \in C^k$.

From (3.1) it follows that

$$\underset{\sim}{r}^2(t) = \underset{\sim}{x}_1^2(t) + \underset{\sim}{x}_2^2(t).$$

Let $t_0 \in I$ and $\underset{\sim}{r}(t_0) > 0$. We will denote by $I(t_0)$ the greatest connected subset of I such that $t_0 \in I$ and $\underset{\sim}{r}(t) > 0$ for all $t \in I(t_0)$.

The correspondence between the couple $(\underset{\sim}{x}_1, x_2)/I(t_0)$ (that is between the couple $(\underset{\sim}{x}_1, \underset{\sim}{x}_2)$ restricted to the set $I(t_0)$ and the couple $(\underset{\sim}{r}, \underset{\sim}{u})/I(t_0)$ is a bijection.

Definition 3.1. If the couple $(\underset{\sim}{r}, \underset{\sim}{u})$ corresponds to the couple $(\underset{\sim}{x}_1, \underset{\sim}{x}_2)$ at t_0 (or in $I(t_0)$).

Definition 3.2. If $(\underset{\sim}{x}_1, \underset{\sim}{x}_2)$ is a solution of the system (1.1) (or of the system (1.2)) then its polar couple $(\underset{\sim}{r}, \underset{\sim}{u})/I(t_0)$ will be called polar solution of (1.1) (or of (1.2)) at t_0 (or in $I(t_0)$).

Let $\underset{\sim}{x}_1, \underset{\sim}{x}_2 \in C^k$ where $k \geqslant 2$. For each $t_0 \in R$ such that $\underset{\sim}{r}(t_0) > 0$ the derivatives $\dot{\underset{\sim}{r}}(t_0)$, $\ddot{\underset{\sim}{r}}(t_0)$, $\dot{\underset{\sim}{u}}(t_0)$ and $\ddot{\underset{\sim}{u}}(t_0)$ exist. Especially if $(\underset{\sim}{x}_1,\underset{\sim}{x}_2)$ is a saturated solution of (1.1) or of (1.2) and $\underset{\sim}{r}(t_0) > 0$ then the polar solution $(\underset{\sim}{r},\underset{\sim}{u})_{/I(t_0)} \in C^2$. Then we have

$$(3.2)\qquad \begin{cases} \dot{\underset{\sim}{x}}_1 = \dot{\underset{\sim}{r}} \cos \underset{\sim}{u} - \underset{\sim}{r}\, \dot{\underset{\sim}{u}} \sin \underset{\sim}{u}, \\ \dot{\underset{\sim}{x}}_2 = \dot{\underset{\sim}{r}} \sin \underset{\sim}{u} + \underset{\sim}{r}\, \dot{\underset{\sim}{u}} \cos \underset{\sim}{u}. \end{cases}$$

It follows that

$$(3.3)\qquad \underset{\sim}{r}\, \dot{\underset{\sim}{u}} = -\dot{\underset{\sim}{x}}_1 \sin \underset{\sim}{u} + \dot{\underset{\sim}{x}}_2 \cos \underset{\sim}{u}.$$

Differentiating (3.2) once more, we get

$$\ddot{\underset{\sim}{x}}_1 = (\ddot{\underset{\sim}{r}} - \underset{\sim}{r}\dot{\underset{\sim}{u}}^2) \cos \underset{\sim}{u} - (2\dot{\underset{\sim}{r}}\dot{\underset{\sim}{u}} + \dot{\underset{\sim}{r}}\dot{\underset{\sim}{u}}) \sin \underset{\sim}{u},$$

$$\ddot{\underset{\sim}{x}}_2 = (2\dot{\underset{\sim}{r}}\dot{\underset{\sim}{u}} + \underset{\sim}{r}\ddot{\underset{\sim}{u}}) \cos u + (\ddot{\underset{\sim}{r}} - \underset{\sim}{r}\dot{\underset{\sim}{u}}^2) \sin \underset{\sim}{u}.$$

It follows that

$$(3.4)\qquad 2\dot{\underset{\sim}{r}}\, \dot{\underset{\sim}{u}} + \underset{\sim}{r}\, \ddot{\underset{\sim}{u}} = -\ddot{\underset{\sim}{x}}_1 \sin \underset{\sim}{u} + \ddot{x}_2 \cos \underset{\sim}{u}.$$

We will say that the function $\underset{\sim}{u}$ defined on a connected set $I \subset R$ is increasing (strictly speaking we should say it is "very strongly increasing") in I if we have $\dot{\underset{\sim}{u}}(t) > 0$ for all $t \in I$. It is decreasing in I if the function $-\underset{\sim}{u}$ is increasing in I. It is monotonic in I if it is increasing in I or decreasing in I (then we have $\dot{\underset{\sim}{u}}(t) \neq 0$ for all $t \in I$).

Definition 3.2. We will say that a couple of functions $(\underset{\sim}{x}_1,\underset{\sim}{x}_2)$ (a solution $(\underset{\sim}{x}_1,\underset{\sim}{x}_2)$ of the system (1.1) or of the system (1.2)) defined on a connected set $I \subset I(t_0)$ is winding up (is unfolding, is winding up monotonically) if for the corresponding polar couple (polar solution) $(\underset{\sim}{r},\underset{\sim}{u})$ the function $\underset{\sim}{u}$ is an increasing (is a decreasing, is a monotonic) function.

The notion of winding up of solutions of the system (1.1) of two ordinary differential equations of second order each is one of possible generalizations of the notion of oscillations of a solution of one differential equation of second order.

4. A local result

In this section we will accept the assumptions F_1 and F_2 (or H_1 and H_2 respectively).

Lemma 4.1. Let $(\underset{\sim}{r},\underset{\sim}{u})$ be a polar solution of the system (1.1) defined in I. If for a $t_0 \in I$ we have

$$\underset{\sim}{r}(t_0) > 0 \tag{4.1}$$

and

$$\dot{\underset{\sim}{u}}(t_0) = 0 \tag{4.2}$$

then

$$\ddot{\underset{\sim}{u}}(t_0) > 0. \tag{4.3}$$

Proof. If $(\underset{\sim}{x}_1,\underset{\sim}{x}_2)$ is a solution of the system (1.1) it follows from the identity (3.4) that

$$2\dot{\underset{\sim}{r}}(t_0)\dot{\underset{\sim}{u}}(t_0) + \underset{\sim}{r}(t_0)\ddot{\underset{\sim}{u}}(t_0) =$$

$$= -\underset{\sim}{f}(t_0,\underset{\sim}{x}_1(t_0),\underset{\sim}{x}_2(t_0),\dot{\underset{\sim}{x}}_1(t_0),\dot{\underset{\sim}{x}}_2(t_0))\sin \underset{\sim}{u}(t_0) +$$

$$+ \underset{\sim}{g}(t_0,\underset{\sim}{x}_1(t_0),\underset{\sim}{x}_2(t_0),\dot{\underset{\sim}{x}}_1(t_0),\dot{\underset{\sim}{x}}_2(t_0))\cos \underset{\sim}{u}(t_1)$$

By our hypotheses, the system (1.1) verifies the assumption H_2 and from (2.4), (3.3) and (4.2) it follows that

$$\underset{\sim}{r}(t_0)\cdot\ddot{\underset{\sim}{u}}(t_0) > 0.$$

Noting (4.1), we get (4.3) and this completes the proof.

Theorem A. Let the assumptions F_1 and F_2 be verified. If for a solution $(\underset{\sim}{x}_1,\underset{\sim}{x}_2)$ of the system (1.1) defined in the open (and connected) set $I \subset R$ there exists a $t_0 \in I$ such that we have (4.1) and (4.2) then this solution is unfolding in the non empty set

$$I^- := (-\infty\,; t_0) \cap I(t_0)$$

and is winding up in the non empty set

$$I^+ := (t_0; +\infty) \cap I(t_0).$$

Proof. In view of our assumptions and of the Lemma 4.1 we find that there exists an $\varepsilon > 0$ such that

$$\dot{\underset{\sim}{u}}(t) < 0 \qquad t \in (t_0 - \varepsilon;\ t_0), \tag{4.4}$$

for

$$\dot{\underset{\sim}{u}}(t) > 0 \qquad t \in (t_0;\ t_0 + \varepsilon).$$

If it is not true that $\underset{\sim}{u}$ is unfolding in I^-, that is if it is not true that $\dot{\underset{\sim}{u}}(t) < 0$ for all $t \in I^-$, then there exists a $t_1 \in I^-$ such that $\dot{\underset{\sim}{u}}(t_1) = 0$. By (4.4) the set of such t_1 is closed in the set $\left[t_1;\ t_0 - \frac{\varepsilon}{2}\right] \subset I^-$ and is nonempty. It follows that there exists a number $t_2 \in [t_1; t_0)$ such that $\dot{\underset{\sim}{u}}(t_2) = 0$ and $\dot{\underset{\sim}{u}}(t) < 0$ for all $t \in (t_2, t_0)$.

But applying a second time the Lemma 4.1 we find that there exists an $\varepsilon_2 > 0$ such that $\dot{\underset{\sim}{u}}(t) > 0$ for all $t \in (t_2; t_2+\varepsilon_2)$.

Thus, we have a contradiction and such a $t_1 \in I^-$ such that $\dot{\underset{\sim}{u}}(t_1) = 0$ does not exist. It is $\dot{\underset{\sim}{u}}(t) > 0$ for all $t \in I^-$ and our solution is unfolding in I^-.

By a similar argument we can prove that $(\underset{\sim}{x}_1, \underset{\sim}{x}_2)$ is winding up in I^+ which ends the proof.

In Lemma 2.1 we have seen that if the system (1.2) verifies the assumptions H_1 and H_2 then it is a system (1.1) which verifies the assumptions F_1 and F_2. Then it follows a lemma available to the system (1.2) analogous to the Lemma 4.1. By the same argument it can be demonstrated a theorem B available to the system (1.2) analogous to Theorem A.

Let $(\underset{\sim}{x}_1, \underset{\sim}{x}_2)$ be a solution of the system (1.1) (or (1.2)). If at a point $t_0 \in R^+$ we have (4.2) and (2.9) then (as it is known from the differential geometry) the characteristic (as a projection of the solution $(\underset{\sim}{x}_1, \underset{\sim}{x}_2)$ on the x_1x_2-plane) has a tangent. That is we have the situation as on the fig.2a. The situation as on the fig.2b (the characteristic has no tangent at t_0 such that $\dot{\underset{\sim}{u}}(t_0) = 0$) is also possible, but only in the case where

$$\dot{\underset{\sim}{x}}_1(t_0) = 0 = \dot{\underset{\sim}{x}}_2(t_0).$$

A saturated solution of (1.1) (or of (1.2)) is defined on an open connected set $I \subset R$ and cannot have any end in the interiour of the domain of existence of the functions $\underset{\sim}{f}, \underset{\sim}{g}$ (or of the functions $\underset{\sim}{p}, \underset{\sim}{q}, \underset{\sim}{k}$), that is in the interiour of R_5.

According to Theorem A for the system (1.1) (or to Theorem B for the system (1.2)) it follows that if we put

$$\underline{t} := \operatorname{Inf} I^{-} = \operatorname{Inf} I(t_o)$$

then it can be only three possible cases:

1^0 $\underline{t} = -\infty$ (then $\underset{\sim}{r}(t) > 0$ for $t \in (-\infty; t_o]$)

2^0 $\underline{t} > -\infty$ and $\lim\limits_{t \to \underline{t}^+} \underset{\sim}{r}(t) = 0$

3^0 $\underline{t} > -\infty$ and $\lim\limits_{t \to \underline{t}^+} \underset{\sim}{r}(t) = +\infty$.

(see fig.3a).

If we put $\bar{t} := \operatorname{Sup} I^{+}$ then there are three possible cases of the behaviour of solutions in $[t_o; \bar{t})$ (see also fig.3a).

Let be a $t_o \in I$ such that $r(t_o) > 0$. If there exists a $t_1 \in I(t_o)$ such that $\dot{\underset{\sim}{u}}(t_1) = 0$ then we have $9 = 3 \cdot 3$ possible cases which were considered previously. But it can be also $\dot{u}(t) \neq 0$ for all $t \in I(t_o)$. Then we will have together $18 = 9+9$ possible cases (see Fig.3b and 3c).

5. A global result

In this section we will consider the solutions of the system (1.2) only, under the assumptions H_1 and H_3. H_3 is a stronger assumption than the accepted assumption H_2 in the previous section. Under the assumption H_2 we were not able to say how much different intervals of existence $I(t_o)$ are possible for a given solution $(\underset{\sim}{x}_1, \underset{\sim}{x}_2)$? Now, the local behaviour of the solutions of system (1.2) under the assumption H_3 is the same as under the assumption H_2. But we will be able to show that for any solution it can be only one or two different intervals $I(t_o)$.

Let $C^2 \ni \underset{\sim}{x}_1, \underset{\sim}{x}_2 : I \longrightarrow R$, where $I \subset R$ is a connected set. We define a new function

(5.1) $$\underset{\sim}{w} := \underset{\sim}{x}_1\dot{\underset{\sim}{x}}_2 - \underset{\sim}{x}_2\dot{\underset{\sim}{x}}_1$$

The function $\underset{\sim}{w}$ is the scalar moment of momentum of a planar movement $(\underset{\sim}{x}_1,\underset{\sim}{x}_2)$ of unit mass point. Then we have

(5.2) $$\dot{\underset{\sim}{w}} = \underset{\sim}{x}_1\ddot{\underset{\sim}{x}}_2 - \underset{\sim}{x}_2\ddot{\underset{\sim}{x}}_1.$$

Now let $(\underset{\sim}{x}_1,\underset{\sim}{x}_2)$ be a solution of the system (1.2) and let $\underset{\sim}{w}$ be the function (5.1) corresponding to $(\underset{\sim}{x}_1,\underset{\sim}{x}_2)$. Thus from (5.2) and (1.2) we have

(5.3) $$\dot{\underset{\sim}{w}} = \bar{\underset{\sim}{k}}\underset{\sim}{w} + \underset{\sim}{m}$$

where we have put

(5.4) $$\begin{cases} \underset{\sim}{m} := -\underset{\sim}{x}_2\underset{\sim}{p}(\underset{\sim}{j},\underset{\sim}{x}_1,\underset{\sim}{x}_2,\dot{\underset{\sim}{x}}_1,\dot{\underset{\sim}{x}}_2) + \underset{\sim}{x}_1\underset{\sim}{q}(\underset{\sim}{j},\underset{\sim}{x}_1,\underset{\sim}{x}_2,\dot{\underset{\sim}{x}}_1,\dot{\underset{\sim}{x}}_2), \\ \bar{\underset{\sim}{k}} := \underset{\sim}{k}(\underset{\sim}{j},\underset{\sim}{x}_1,\underset{\sim}{x}_2,\dot{\underset{\sim}{x}}_1,\dot{\underset{\sim}{x}}_2). \end{cases}$$

We have $C^0 \ni \underset{\sim}{m}, \bar{\underset{\sim}{k}} : I \longrightarrow R$. We have assumed that the assumption H_3 is satisfied. Then we have by (5.4)

(5.5) $$\underset{\sim}{m}(t) \geqslant 0.$$

It is known that if $\underset{\sim}{w}$ is a solution of the equation (5.3) and $\underset{\sim}{v}$ is a solution of the differential equation

(5.6) $$\dot{\underset{\sim}{v}} = -\bar{\underset{\sim}{k}}\underset{\sim}{v}$$

such that for a $t_o \in I$ we have

(5.7) $$\underset{\sim}{v}(t_o) = \underset{\sim}{w}(t_o) > 0$$

then, by (5.5), we will have

(5.8) $$\underset{\sim}{w}(t) \geqslant \underset{\sim}{v}(t)$$

for all $t \in I$ such that $t \geqslant t_o$ and $\underset{\sim}{v}(t) > 0$.

But the solution of (5.6) are given by the formula

$$\underset{\sim}{v}(t) = \underset{\sim}{v}(t_o)\exp\left(-\int_{t_o}^{t}\bar{\underset{\sim}{k}}\right).$$

As we have (5.7), it will $\underset{\sim}{v}(t) > 0$ for all $t_0 \leq t \in I$. Then by (5.8) we also have

(5.9) $$\underset{\sim}{w}(t) > 0$$

for all $t_0 \leq t \in I$.

Multiplying (3.3) by $\underset{\sim}{r}$ and from (3.1) and (5.1) we get

$$\underset{\sim}{w} = \underset{\sim}{r}^2 \dot{\underset{\sim}{u}}.$$

Noting (5.9) it yields

(5.10) $$\dot{\underset{\sim}{r}}(t) > 0, \quad \dot{\underset{\sim}{u}}(t) > 0$$

for all $t_0 \leq t \in I$. Thus we have proved the following lemma.

Lemma 5.1. If for a polar solution $(\underset{\sim}{r}, \underset{\sim}{u})$ (1.2) and for a $t_0 \in I$ we have

(5.11) $$\dot{\underset{\sim}{r}}(t_0) > 0, \quad \dot{\underset{\sim}{u}}(t_0) > 0,$$

then we have (5.10) for all $t_0 \leq t \in I$.

By a similar argument we can prove the following similar lemma.

Lemma 5.2. If for a polar solution $(\underset{\sim}{r}, \underset{\sim}{u})$ of the system (1.2) and for a $t_0 \in I$ we have

(5.12) $$\dot{\underset{\sim}{r}}(t_0) > 0, \quad \dot{\underset{\sim}{u}}(t_0) < 0,$$

then we will have

(5.13) $$\dot{\underset{\sim}{r}}(t) > 0, \quad \dot{\underset{\sim}{u}}(t) < 0$$

for all $t_0 \geq t \in I$.

Directly from Lemmae 5.1 and 5.2 we have

Lemma 5.3. If there exists a $t_0 \in I$ such that we have (5.11) then

(5.14) $$\operatorname{Sup} I(t_0) = \operatorname{Sup} I =: t^+.$$

Lemma 5.4. If there exists a $t_0 \in I$ such that we have (5.12) then

$$\text{(5.15)} \qquad \operatorname{Inf} I(t_o) = \operatorname{Inf} I =: t_- .$$

Now if we have (5.11) for a $t_o \in I$, then (5.10) holds for all $t_o > t \in I$. If it is not true, then there exists a $t_o > t_1 \in I$ such that $\underset{\sim}{r}(t_1)\cdot\dot{\underset{\sim}{u}}(t_1) = 0$. Since the functions $\underset{\sim}{r}$ and $\dot{\underset{\sim}{u}}$ are continuous and we have assumed (5.11) then there exists the greatest $t_2 < t_o$ such that we have $\underset{\sim}{r}(t_2)\cdot\underset{\sim}{u}(t_2) = 0$. If $\underset{\sim}{r}(t_2) = 0$, it means that $t_2 = \operatorname{Inf} I(t_o)$. At the end if $t_2 \in I(t_o)$ then $\dot{\underset{\sim}{u}}(t_o) = 0$. The assumption H_3 implies the assumption F_2. By Lemma 4.1, there exists an $\varepsilon > 0$ such that $u(t) < 0$ for $t \in (t_2-\varepsilon; t_2)$ and by Lemma 5.4 it will be $u(t) < 0$ for all $t_2 > t \in I(t_o)$.

Thus we have proved the following lemma.

Lemma 5.5. If for a polar solution $(\underset{\sim}{r},\underset{\sim}{u})$ defined in I we have (5.11) for a $t_o \in I$ then either $I(t_o) = I$ and in this case we have one of the following possibilities:

1° $\underset{\sim}{u}$ is an increasing function in I,

2° $\underset{\sim}{u}$ is a decreasing function in I,

3° $\underset{\sim}{u}$ has exactly one minimum t_o in I (and $\dot{\underset{\sim}{u}}(t) \neq 0$ for $t \neq t_o$)

or $I(t_o) = (t_2; t^+)$ where t^+ is given by the formula (5.14), $\underset{\sim}{r}(t_2) = 0$ and $\underset{\sim}{r}(t)\cdot\dot{\underset{\sim}{u}}(t) > 0$ for all $t \in I(t_o)$.

If for a $t_o \in I$ we have (5.12) then a similar lemma (Lemma 5.6) can be formulated. Its proof (and its formulation) may be left to the reader.

From Lemmae 5.5 and 5.6 it follows

Lemma 5.7. If $(\underset{\sim}{r},\underset{\sim}{u})$ is a non trivial polar solution of the system (1.2) defined in an open connected set I, then there exists at most one $t_2 \in I$ such that $\underset{\sim}{r}(t_2) = 0$ and I is a set $I(t_1)$ (for any $t_1 \in I$) or I is the union of two disjoint sets $I(t_1)$, $I(t_2)$ and of the one point set $\{t_2\}$.

Under our assumptions and in view of Lemma 2.1, the system (1.2) admits the trivial solution $\underset{\sim}{x}_1 = \underset{\sim}{0}$, $\underset{\sim}{x}_2 = \underset{\sim}{0}$. For the non-trivial solutions we have

Lemma 5.8. If for a non trivial polar solution $(\underset{\sim}{r},\underset{\sim}{u})$ of the system (1.2) defined in I and for a $t_2 \in I$ we have

$$\underset{\sim}{r}(t_2) = 0 \tag{5.16}$$

then we will have

$$\begin{cases} \underset{\sim}{\dot{u}}(t) < 0 & t \in (-\infty;t_2) \cap I =: I_-, \\ & \text{for} \\ \underset{\sim}{\dot{u}}(t) > 0 & t \in (t_2;+\infty) \cap I =: I_+. \end{cases} \tag{5.17}$$

Proof. By Lemma 5.7 we will have $\underset{\sim}{r}(t) > 0$ for all $t \in I_-$. Let us assume that it is not true that $\underset{\sim}{\dot{u}}(t) < 0$ for all $t \in I_-$. It means that there exists at least one $t_1 \in I_-$ such that $\underset{\sim}{\dot{u}}(t_1) \geq 0$. Since $\underset{\sim}{r}(t_1) \geq 0$, then by Lemma 5.1 (in the case of $\underset{\sim}{\dot{u}}(t) > 0$) or by the Lemma 4.1 and 5.1 (in the case of $\underset{\sim}{\dot{u}}(t_1) = 0$) we will have $\underset{\sim}{r}(t_2) > 0$ which contradicts the assumption (5.16).

By a similar argument we can prove the second inequality of (5.17).

Now we will prove a local property for the characteristics in the case of existence $t_2 \in I$ such that we have (5.16).

Lemma 5.9. If for a nontrivial solution $(\underset{\sim}{x}_1, \underset{\sim}{x}_2)$ of the system (1.2) and for a $t_0 \in I$ we have (5.16), then the projection of the curve

$$x_1 = \underset{\sim}{x}_1(t), \quad x_2 = \underset{\sim}{x}_2(t) \tag{5.18}$$

on the x_1x_2-plane has a tangent at t_0.

Proof. If $(\underset{\sim}{x}_1, \underset{\sim}{x}_2)$ is a non trivial solution and we have (5.16), then we must have (4.6). Then it follows by a known result of differential geometry that there exists a unit tangent vector $[v_1, v_2]$ to the plane curve given by the parametric equation (5.18) at $t = t_0$.

From Lemmas 5.8 and 5.9 it follows that at $t_2 \in I$ such that (5.16), the plane curve given by the parametric equations (5.18) has an inflection point.

The unit tangent plane vector $[v_1, v_2]$ at t_2 to the x_1x_2-plane curve (5.18) defines for us a number $u_2 \in [0;2\pi)$ such that

$$v_1 = \underset{\sim}{\cos} u_2, \quad v_2 = \underset{\sim}{\sin} u_2. \tag{5.19}$$

This tangent vector is directed towards increasing t.

Now there is a problem. If we are traditionnally assuming (by definition) that $\underset{\sim}{r}(t) \geqslant 0$, then for such $t_0 \in I$ such that we have (5.16) it will be

$$\lim_{t \to t_0^+} \underset{\sim}{u}(t) = u_2 + 2n\pi,$$

and

$$\lim_{t \to t_0^-} \underset{\sim}{u}(t) = u_2 + 2n\pi \pm \pi$$

where u_2 is given by the formulae (5.19) and n is the same in the both last formulae (see figures 4a and 4b). We are reminding that for $t \neq t_2$ we will have $\underset{\sim}{r}(t) > 0$ and this implies the uniqueness of definition of the number $\underset{\sim}{u}(t)$.

It would be here more rational (at least for the polar solutions of our systems (1.1) and (1.2)) to accept $\underset{\sim}{r}(t) \in R$ and to take both the functions $\underset{\sim}{r}$ and $\underset{\sim}{u}$ continuous for all I. According to the definition of the polar solution, we will have $\underset{\sim}{u} \in C^2$ (and $\underset{\sim}{r} \in C^2$) not only in $I(t_2)$ but also in all I and if we have (5.16), then it would be

$$\dot{\underset{\sim}{u}}(t_2) = 0, \quad \text{and} \quad \ddot{\underset{\sim}{u}}(t_2) = 0 .$$

Taking into account all our previous results, we have

Theorem C. If the system (1.2) of two differential equations of second order each verifies the assumptions H_1 and H_3, then there exists five possible types of solutions. They are

a. The trivial solution (i.e., $\underset{\sim}{r}(t) = 0$ for all $t \in I$).

b. The solution is unfolding in I (and $\underset{\sim}{r}(t) > 0$ for all $t \in I$).

c. The solution is winding up in I (and $\underset{\sim}{r}(t) > 0$ for all $t \in I$).

d. There exists a $t_0 \in I$ such that $\dot{\underset{\sim}{u}}(t_0) = 0$ and the solution is unfolding in $(-\infty; t_0) \cap I$ and winding up in $(t_0; +\infty)$ (and $\underset{\sim}{r}(t) > 0$ for all $t \in I$). For this case it corresponds two sub-cases: (di) the characteristics have a tangent at t_0, and (dii) the characteristics have no tangent at t_0.

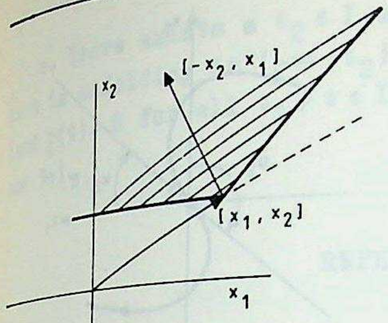
$[-x_2, x_1]$
x_2
$[x_1, x_2]$
x_1

Fig.1

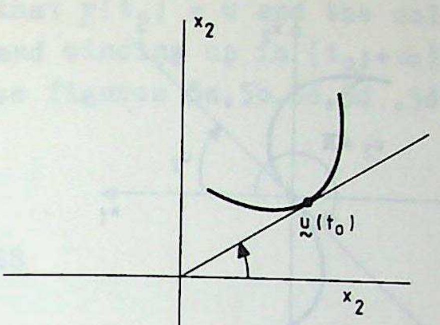
x_2
$\underset{\sim}{u}(t_0)$
x_2

Fig.2a

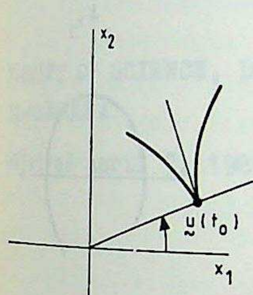
x_2
$\underset{\sim}{u}(t_0)$
x_1

Fig.2b

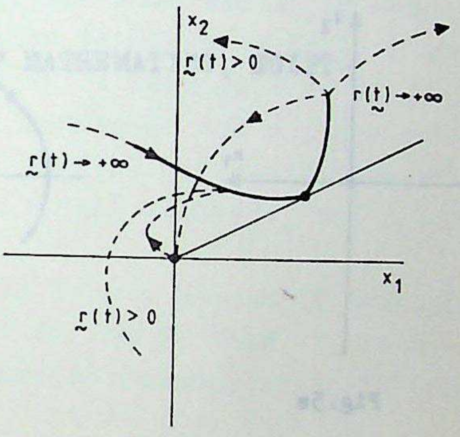
x_2
$\underset{\sim}{r}(t) > 0$
$\underset{\sim}{r}(t) \to +\infty$
$\underset{\sim}{r}(t) \to +\infty$
x_1
$\underset{\sim}{r}(t) > 0$

Fig.3a

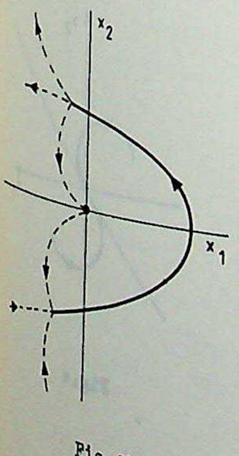

Fig.3b

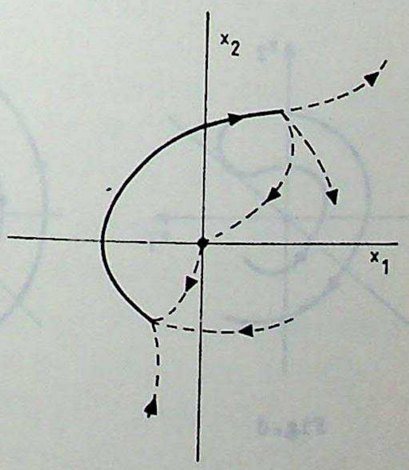

Fig.3c

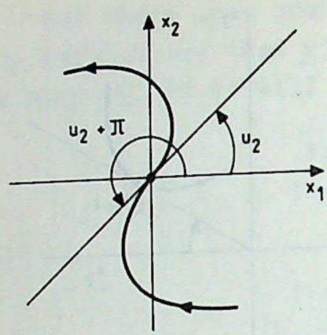
x_2
$u_2 + \pi$
u_2
x_1

Fig.4a

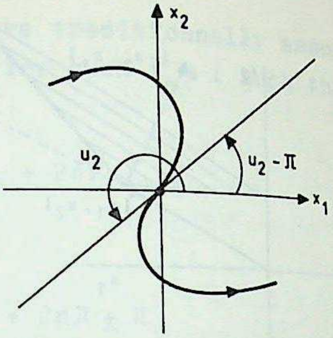
x_2
u_2
$u_2 - \pi$
x_1

Fig.4b

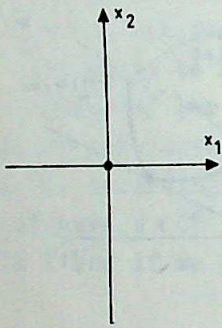

Fig.5a

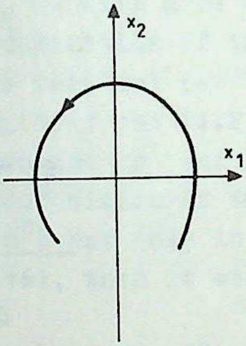

Fig.5b

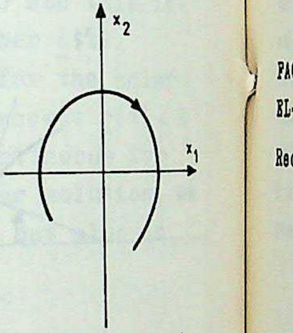

Fig.5c

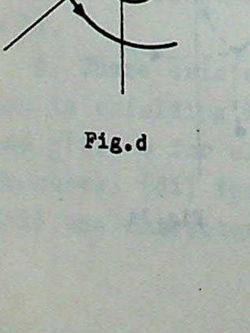

Fig.d

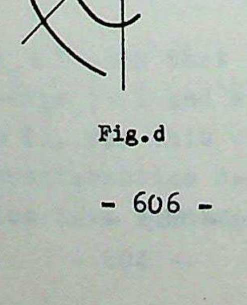

Fig.d

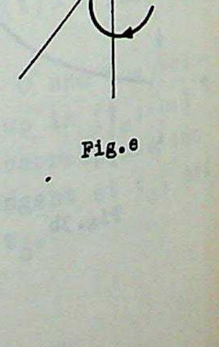

Fig.e

e. There exists a $t_2 \in I$ such that $\underset{\sim}{r}(t_2) = 0$ and the solution is unfolding in $(-\infty ; t_2) \cap I$ and winding up in $(t_2 ; +\infty)$ (and $\underset{\sim}{r}(t) > 0$ for all $t_2 \neq t \in I$) (see figures 5a,5b,5c,5d ,5d , and 5e).

REFERENCES

[1] K. Menger : Calculus, a modern approach. Boston 1955.

[2] K. Tatarkiewicz : Un cas de stabilité conditionnelle, Demonstratio Math. 7 (1974) 225-239.

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Mihai Cristea

SOME CONDITIONS FOR CONTINUITY

The main theorem of this paper generalizes some results from [2] and [3]. We shall find conditions for a map $f:E \to F$, which carries compact sets into closed sets and it is connected, to be continuous. This assertion is not true if E, F are arbitrary chosen (see [5]) and we shall require F to be locally compact and E to be locally connected and a first countable space. We shall also prove a partial generalization of a well-known theorem from the Functional Analysis, the closed graph theorem.

Throughout this paper, E,F will denote Hausdorff spaces.

Definition 1. Let $f:E \to F$. Then f is compact (connected) if f carries compact (connected) sets into compact (connected) sets.

Definition 2. Let $f:E \to F$. We say that f has closed graph if $\Gamma_f = \{(x,y) \in E\times F \mid y=f(x)\}$ is closed in $E\times F$..

Remark 1. If E is a first countable space and F locally compact, f has closed graph if and only if for every $x_n \to x$ and $y \in \{f(x_n)\}'_{n\in N}$ is results that $y = f(x)$. Indeed, let $z = (x,y) \in \Gamma_f$ and U_n a fundamental system of neighbourhoods of x and $V \in \mathcal{V}_y$ such that $\overline{V}$ is compact. Then we can find $x_n \to x$ and $b \in \{f(x_n)\}'_{n\in N}$, hence $b = f(x)$ and the assertion is proved, since f(x) belongs to an arbitrary $V \in \mathcal{V}_y$.

The following theorem is proved in [1], p.437.

Theorem A. Let E be compact and A_n a decreasing sequence of compact, connected sets. Then $\bigcap_{n=1}^{\infty} A_n$ is connected.

Theorem 1. Let E be a first countable space, F locally compact, $f:E \longrightarrow F$ such that f has closed graph, and for every $x \in E$, there exists a decreasing sequence of neighbourhoods of x, denoted by Q_n such that $\bigcap_{n=1}^{\infty} Q_n = \{x\}$, and $f(Q_n)$ is connected for every $n \in N$. Then f is continuous.

Proof. Let $x \in E$ and Q_n as in the theorem. Since $f(Q_n)$ is connected, we obtain that $\overline{f(Q_n)}$ is closed and connected. Let $X = F \cup \{\infty\}$ be the Alexandrov compactification of F. We suppose first that for every $n \in N$, $\overline{f(Q_n)}$ is not compact in F. Let $H_n = \overline{f(Q_n)} \cup \{\infty\}$. Then H_n is compact and connected in X for every $n \in N$. From Theorem A, $H = \bigcap_{n=1}^{\infty} H_n$ is connected and contains f(x) and ∞. Since $\{f(x), \infty\}$ is not connected in X, there exists $y \neq f(x)$, $y \in H \cap F$. Let V be a neighbourhood of y such that $\overline{V}$ is compact and $f(x) \notin \overline{V}$. From the definition of H, we can find $x_n \longrightarrow x$ such that $f(x_n) \in \overline{V}$. Let $z \in \{f(x_n)\}'_{n \in N}$. Since f has closed graph, $z = f(x)$, which is a contradiction, since $f(x) \notin \overline{V}$. We proved that there exists $n_0 \in N$ such that for $n \geq n_0$, $\overline{f(Q_n)}$ is compact in F. We can easily prove that if $x_n \longrightarrow x$ we obtain $f(x_n) \longrightarrow f(x)$, hence f is continuous in x, q.e.d.

Corollary 1. Let E be a metrizable linear topological space and $f:E \longrightarrow R^n$ homogeneous so that f has closed graph. Then f is continuous.

Proof. Every line is mapped in a line, hence f is connected.

Remark 2. Theorem 1 is one of the possible partial generalization of a well-known theorem from the Functional Analysis, the closed graph theorem.

Theorem 2. Let E be a first countable space, F locally compact, $f:E \longrightarrow F$ such that for every $K \subset E$ compact, f(K) is closed, and for every $x \in E$, there exists a decreasing sequence of neighboruhoods of x, denoted by Q_b such that

$\bigcap_{n=1}^{\infty} Q_n = \{x\}$, and $f(Q_n)$ is connected for every $n \in N$. Then f is continuous.

Proof. Suppose that $x_n \to x$, $y \in \{f(x_n)\}_{n\in N}$, $y \neq f(x)$ and $f(x_i) \neq f(x_j) \neq f(x)$ for every $i \neq j$. Let $A = \{x_n\}_{n\in N} \cup \{x\}$. Then A is compact, but $f(A)$ is not closed, since $y \in \overline{f(A)} \setminus f(A)$, which is a contradiction, because f carries compact sets into closed sets. Suppose now that $x_n \to x$, $f(x_n) = y$ for every $n \in N$ and $f(x) \neq y$. Let Q_n as in the theorem. Then, for every $n \in N$, $f(Q_n)$ is connected and contains $f(x)$ and y. Let $V \in \mathcal{V}_{f(x)}$ and $W_1 \in \mathcal{V}_y$ such that $\overline{V} \cap \overline{W}_1 = \emptyset$. Since $f(Q_1)$ is connected, $\mathrm{Fr}W_1 \cap f(Q_1) \neq \emptyset$, hence we can find $x_1 \in Q_1$ so that $f(x_1) \in \overline{W}_1$, $f(x_1) \neq y$ and $f(x_1) \neq f(x)$. We can find now $W_2 \in \mathcal{V}_y$ such that $f(x_1) \notin \overline{W}_2$, $W_2 \subset W_1$ and $\overline{V} \cap \overline{W}_2 = \emptyset$. Then $\mathrm{Fr}W_2 \cap f(Q_2) \neq \emptyset$, hence we can find $x_2 \in Q_2$ such that $f(x_1)$, $f(x_2)$, $f(x)$, y are different and $f(x_2) \in \overline{W}_2$. We can construct by induction a sequence $x_n \to x$ such that $f(x_i) \in \overline{W}_1$ and $f(x_i) \neq f(x_j)$ for $i \neq j$, and we can suppose that there exists $z \in \overline{W}_1$ such that $z \in \{f(x_n)\}'_{n\in N}$. From the first part of the proof, we find that $z = f(x)$, which is a contradiction, since $z \in \overline{W}_1$ and $f(x) \in \overline{V}$. We finally proved that if $x_n \to x$ and $y \in \{f(x_n)\}'_{n\in N}$, then $y = f(x)$. Using Theorem 1 and Remark 1, the theorem is proved.

Remark 3. Theorem 2 is a generalization of Theorem 2 of [3] p.723, since if $f:E \to F$ is closed, then f carries compact sets into closed sets. Also, for F locally compact, the proceeding theorem generalizes [2].

Another result in connection with [2] is the following:

Theorem 3. Let E be a first countable space, $f:E \to F$ compact such that for every $x \in E$ and $U \in \mathcal{V}_x$, $\mathrm{Card}\, f(U) \geq \aleph_0$. Then f is continuous.

Proof. a) Suppose that $x_n \to x$ and $f(x_i) \neq f(x_j)$ for $i \neq j$. Since f is compact, there exists $y \in \{f(x_n)\}'_{n\in N}$. Let $A = \{x_n\}_{n\in N} \cup \{x\}$. If $f(x) \neq y \neq f(x_n)$ then A is compact and $f(A)$ is not closed, since $y \in \overline{f(A)} \setminus f(A)$, which is a contradiction. Hence if $x_n \to x$ and $f(x_i) \neq f(x_j)$ for $i \neq j$, then $f(x_n) \to f(x)$.

b) We prove that for every $x \in E$, there exists $x_n \to x$ such that $f(x_n) \to f(x)$, $f(x_n) \neq f(x)$, and $f(x_i) \neq f(x_j)$ for $i \neq j$. Indeed, let $Q_n \in \mathcal{V}_x$ be a fundamental system of neighbourhoods of x. Since Card $f(Q_n) = \infty$ for every $n \in N$, we can find $x_n \in Q_n$ such that $f(x_i) \neq f(x_j)$ for $i \neq j$, hence $x_n \to x$ and from a), $f(x_n) \to f(x)$.

c) We now study the case when $x_n \to x$ and $f(x_n) = y \neq f(x)$ for every $n \in N$. Let $V \in \mathcal{V}_{f(x)}$ and $W \in \mathcal{V}_y$ such that $V \cap W = \emptyset$. We can suppose that $x_n \in Q_n$, where Q_n is a fundamental system of neighbourhoods of x. Let $U_1 \in \mathcal{V}_{x_1}$, such that $U_1 \subset Q_1$. From b), we can choose $z_1 \in U_1$ such that $f(z_1) \in W$ and $f(z_1) \neq y$. Let $U_2 \in \mathcal{V}_{x_2}$ such that $U_2 \subset Q_2$. From b), we can choose $z_2 \in U_2$ such that $f(z_2) \in W$ and $f(z_1) \neq f(z_2) \neq y$. By induction, we can find $z_n \to x$ such that $f(z_i) \neq f(z_j)$ for $i \neq j$ and $f(z_n) \in W$ for every $n \in N$. From a), we see that $f(z_n) \to f(x)$, which is a contradiction, since $f(x) \in V$, $f(z_n) \in W$ for every $n \in N$ and $V \cap W = \emptyset$. We proved that for $x_n \to x$, we obtain $f(x_n) \to f(x)$, hence f is continuous in x, q.e.d.

We can prove the following generalization of Theorem 1 from [3].

Theorem 4. Let E be a metrizable linear topological space, $f: E \to R$ be such that f carries compact sets into closed sets and such that for every two points x, y such that $f(x) \neq f(y)$, there exists $z \in [x,y]$ such that $f(z) \in (f(x),f(y))$ or $f(z) \in (f(y),f(x))$. Then f is continuous.

Proof. Let x,y E be such that $f(x) \neq f(y)$ and suppose that $f(x) < f(y)$. We show that $f([x,y]) \supset [f(x),f(y)]$. Since $[x,y]$ is compact, it is enough to prove that $f([x,y])$ is densely in $[f(x),f(y)]$. Suppose that $f([x,y])$ is not densely in $[f(x),f(y)]$ and let (p,q) be a connected component of $(f(x),f(y)) \setminus f([x,y])$. Then $f(x) < p < q < f(y)$. There exist $x_n, y_n \in [x,y]$ such that $f(x_n) \nearrow p$, $f(y_n) \searrow q$, hence since $f([x,y])$ is closed, there exists $x \leqslant a \leqslant y$, $x \leqslant b \leqslant y$ such that

that $f(a) = p$, $f(b) = q$, hence there exists $z \in [a,b]$ such that $f(z) \in (p,q)$, which represents a contradiction, since $f([x,y]) \cap (p,q) = \emptyset$. We proved that every segment $[x,y]$ is mapped in a segment containing $[f(x),f(y)]$, hence, for every $U \in \mathcal{V}_x$ it results that $f(U)$ is connected. We apply now Theorem 2.

I wish to thank the referee for his advices which helped me to improve the paper. The referee also communicated to me that R.J. Pawlak [4] obtained some results in connection with our results.

REFERENCES

[1] R. Engelking : General topology. Warszawa 1977.
[2] T.R. Hamlett : Compact maps, connected maps and continuity, J. London Math. Soc. 2 (1975) 22.
[3] H. Pawlak : On some condition equivalent to the continuity of closed functions, Demonstratio Math. 17 (1984) 723-732.
[4] R.J. Pawlak : On local characterisation of closed functions and functions with closed graphs, Demonstratio Math. 19 (1986) 181-188.
[5] D.J. White : Functions preserving compactness and connectedness are continuous, J. London Math.Soc. 43 (1968), 714-716.

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that $f(a) = p$, $f(b) = q$, hence there exists $y \in (a,b)$ such that $f(y) \in (p,q)$, which represents a contradiction, since $f(J) \cap (p,q) = \emptyset$. We proved that every segment $[x,y]$ is mapped on a segment containing $[f(x),f(y)]$, hence, Property 3 yields that $f(J)$ is connected. [illegible]

We wish to thank the referee for his advices which helped us to improve the paper. The referee also communicated to us that R.E. [illegible] obtained some results in connection with connectivity.

REFERENCES

[1] R. Engelking, General topology, [illegible]

[2] T.R. Hamlett, Compact maps, connected maps and continuity, J. London Math. Soc. (2) 10 (1975) 25-26.

[3] [illegible], On some conditions equivalent to the continuity of closed functions, Demonstratio Math. 17 (1984) 723-735.

[4] [illegible], On local characterization of closed functions and functions with closed graphs, Demonstratio Math. 19 (1986) 181-188.

[5] D.J. White, Functions preserving compactness and connectedness are continuous, J. London Math. Soc. 43 (1968) 714-716.

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ON THE ASYMPTOTIC BEHAVIOUR OF SOLUTION OF THE m-th ORDER DIFFERENCE EQUATION

The asymptotic behaviour of the solution of the m-th order difference equation is useful in these cases when there are difficulties to find the general solution or there is hard to obtain some conclusions about the solution on the base of the equation's shape.

Investigations of that problem and some asymptotic properties of the solutions are done in [4] for equations of the second order. In most of the other papers these problems are discussed in the cases when the solutions are bounded or tend to a constant.

The object of this paper, taking the paper [6] as a model, is to give the asymptotic behaviour of the solution of a m-th order difference equation.

Let us take the difference equation

$$\Delta^m y_n + p_n y_n^{\gamma} = f_n, \quad m \geq 2, \tag{E}$$

where γ is an arbitrary real number, $p, f\colon N_0 \to R$, $N_0 := \{n_0, n_0+1, \ldots\}$, n_0 being a given positive integer, R the set of real numbers.

As the solution of (E) we take the function $u\colon N \to R$, satisfying (E) for all $n \in \bar{N} := \{\bar{n}, \bar{n}+1, \ldots\}$, $\bar{n} \geq n_0$. If $\bar{n} = n_0$, then u is the solution of (E) in the sense of the theory of difference equations.

For the function $y\colon N_0 \to R$ we denote by Δy_n the difference $y_{n+1} - y_n$, and by $\Delta^m y_n$ the difference $\Delta(\Delta^{m-1}y_n)$, $m > 1$.

Further, by $n^{(k)}$ we shall denote the product

$$(1) \qquad n^{(k)} = \prod_{j=0}^{k-1}(n-j), \quad n \geqslant k, \quad n^{(0)} = 1$$

k being the positive integer.

Lemma 1. Let a be a non-negative integer, b an arbitrary number such that $b \leqslant a$, $u\colon N \to R$, $N = \{1,2,\dots\}$. Suppose that

$$(2) \qquad \sum_{j=1}^{\infty} j^b u_j \ \text{ is convergent,}$$

and define

$$(3) \qquad \rho_n = \sup_{T \geqslant n}\left|\sum_{j=T}^{\infty} j^b u_j\right|.$$

Then

$$(4) \qquad \left|\sum_{j=n}^{\infty}(j-n_1)^{(a)}u_j\right| \leqslant 2\rho_n n^{a-b} \quad \text{for} \quad n \geqslant n_1 + a, \ n_1 \geqslant n_0.$$

Proof. Let us denote $U_n = \sum_{j=n}^{\infty} j^b u_j$. By (2) and (3) we have

$$(5) \qquad \rho_n \geqslant \left|\sum_{j=n}^{\infty} j^b u_j\right| = |U_n|$$

and $\Delta U_n = \sum_{j=n+1}^{\infty} j^b u_j - \sum_{j=n}^{\infty} j^b u_j = -n^b u_n$ implying

$$(6) \qquad u_n = -\frac{\Delta U_n}{n^b}.$$

In virtue of (2), the series $\sum_{j=1}^{\infty} j^c u_j$ are convergent for $0 \leqslant c \leqslant b$. Hence, from the Abel theorem [2] it follows that the series

(7) $$\sum_{j=k}^{\infty} j^{(c)} u_j \text{ is convergent for } k \geq c,$$

where c is a non-negative integer.

Assume that $a > 0$. Now, we must discuss two cases:

(i) $a < b$. From (1) and (6) it follows that

(8) $$(s - n_1)^{(a)} u_s = \left[\prod_{t=n_1}^{n_1+a-1} \left(1 - \frac{t}{s}\right)\right] s^a u_s =$$

$$= -\left[\prod_{t=n_1}^{n_1+a-1} \left(1 - \frac{t}{s}\right)\right] s^{a-b} \Delta U_s \quad \text{for} \quad s \geq n_1+a,\ n_1 \geq n_0.$$

Hence

(9) $$\sum_{j=n}^{\infty} (j-n_1)^{(a)} u_j = -\sum_{j=n}^{\infty} j^{a-b} \left[\prod_{t=n_1}^{n_1+a-1} \left(1 - \frac{t}{j}\right)\right] \Delta U_j, \quad n \geq n_1+a.$$

In virtue of (7), the series on the left-hand side of (9) is convergent for $n \geq n_1+a-1$, so the series on the right-hand side is convergent too. Applying the formula for addition by parts we have

(10) $$\sum_{j=n}^{s} (j-n_1)^{(a)} u_j = -\sum_{j=n}^{s} j^{a-b} \left[\prod_{t=n_1}^{n_1+a-1} \left(1 - \frac{t}{j}\right)\right] \Delta U_j =$$

$$= -(s+1)^{a-b} U_{s+1} \prod_{t=n_1}^{n_1+a-1} \left(1 - \frac{t}{s+1}\right) + n^{a-b} U_n \prod_{t=n_1}^{n_1+a-1} \left(1 - \frac{t}{n}\right) +$$

$$+ \sum_{j=n}^{s-1} U_{j+1} \Delta \left[j^{a-b} \prod_{t=n_1}^{n_1+a-1} \left(1 - \frac{t}{j}\right)\right].$$

By (2), (3) we state that $\{\varrho_n\}_{n=1}^{\infty}$ is an non-increasing sequence tending to zero, hence the sequence $\{|U_n|\}_{n=1}^{\infty}$ tends to zero and is bounded too. So, we have

$$(11)\qquad \lim_{n\to\infty}(n+1)^{a-b}U_{n+1}\prod_{t=n_1}^{n_1+a-1}\left(1-\frac{t}{n+1}\right)=0.$$

Letting $s\to\infty$ in (10), by (9), (11), we have

$$(12)\qquad \sum_{j=n}^{\infty}(j-n_1)^{(a)}u_j = n^{a-b}U_n\prod_{t=n_1}^{n_1+a-1}\left(1-\frac{t}{n}\right)+$$

$$+\sum_{j=n}^{\infty}U_{j+1}\Delta\left[j^{a-b}\prod_{t=n_1}^{n_1+a-1}\left(1-\frac{t}{j}\right)\right],$$

the series on the right-hand side being convergent.

Let us consider the series

$$(13)\qquad \sum_{j=n}^{\infty}U_{j+1}j^{a-b}\Delta\prod_{t=n_1}^{n_1+a-1}\left(1-\frac{t}{j}\right).$$

It is obvious that the relation

$$(14)\qquad \sum_{j=n}^{\infty}\Delta\prod_{t=n_1}^{n_1+a-1}\left(1-\frac{t}{j}\right)=$$

$$=\lim_{s\to\infty}\left[\prod_{t=n_1}^{n_1+a-1}\left(1-\frac{t}{s+1}\right)-\prod_{t=n_1}^{n_1+a-1}\left(1-\frac{t}{n}\right)\right]=$$

$$=1-\prod_{t=n_1}^{n_1+a-1}\left(1-\frac{t}{n}\right)$$

holds. So, applying the Abel criterion we state the convergence of the series

$$\sum_{j=n}^{\infty}j^{a-b}\Delta\prod_{t=n_1}^{n_1+a-1}\left(1-\frac{t}{j}\right).$$

Because the sequence $\left\{ |U_n| \right\}_{n=1}^{\infty}$ is bounded, the series (13) is absolutely convergent. Applying the formula for the difference of the product and the fact that the series (12), (13) are convergent, we obtain from (12) the equality

$$\sum_{j=n}^{\infty} (j-n_1)^{(a)} u_j = n^{a-b} U_n \prod_{t=n_1}^{n_1+a-1} \left(1 - \frac{t}{n}\right) +$$

$$+ \sum_{j=n}^{\infty} U_{j+1} \left[\prod_{t=n_1}^{n_1+a-1} \left(1 - \frac{t}{j+1}\right) \right] \Delta j^{a-b} + \sum_{j=n}^{\infty} U_{j+1} j^{a-b} \Delta \prod_{t=n_1}^{n_1+a-1} \left(1 - \frac{t}{j}\right)$$

and the convergence of the series $\sum_{j=n}^{\infty} U_{j+1} \left[\prod_{t=n_1}^{n_1+a-1} \left(1 - \frac{t}{j+1}\right) \right] \Delta j^{a-b}$.

From (5), (14) and from the facts that $\left\{ \prod_{t=n_1}^{n_1+a-1} \left(1 - \frac{t}{n}\right) \right\}_{n=n_1+a-1}^{\infty}$ is a non-negative and increasing sequence for $n \geq n_1+a-1$ and $\left\{ n^{a-b} \right\}_{n=1}^{\infty}$ is a decreasing sequence for $a-b < 0$ it follows that

$$\left| \sum_{j=n}^{\infty} (j-n_1)^{(a)} u_j \right| \leq \varrho_n n^{a-b} \prod_{t=n_1}^{n_1+a-1} \left(1 - \frac{t}{n}\right) +$$

$$+ \varrho_n \sum_{j=n}^{\infty} \left[\prod_{t=n_1}^{n_1+a-1} \left(1 - \frac{t}{j+1}\right) \right] \left[-\Delta j^{a-b} \right] + \varrho_n \sum_{j=n}^{\infty} j^{a-b} \Delta \prod_{t=n_1}^{n_1+a-1} \left(1 - \frac{t}{j}\right) \leq$$

$$\leq \varrho_n n^{a-b} \prod_{t=n_1}^{n_1+a-1} \left(1 - \frac{t}{n}\right) - \varrho_n \sum_{j=n}^{\infty} \Delta j^{a-b} + \varrho_n n^{a-b} \sum_{j=n}^{\infty} \Delta \prod_{t=n_1}^{n_1+a-1} \left(1 - \frac{t}{j}\right) =$$

$$= \varrho_n n^{a-b} \prod_{t=n_1}^{n_1+a-1} \left(1 - \frac{t}{n}\right) - \varrho_n \left[\lim_{s \to \infty} (s+1)^{a-b} - n^{a-b} \right] +$$

$$+ \varrho_n n^{a-b} \left[1 - \prod_{t=n_1}^{n_1+a-1} \left(1 - \frac{t}{n}\right) \right] = 2 n^{a-b} \varrho_n .$$

(ii) a = b. Now, the formulae (6) and (8) have the form

$$u_n = -n^{-a}\Delta U_n \quad \text{and} \quad (s-n_1)^{(a)}u_s = -\left[\prod_{t=n_1}^{n_1+a-1}\left(1-\frac{t}{s}\right)\right]\Delta U_s,$$

respectively. Hence

$$\sum_{j=n}^{\infty}(j-n_1)^{(a)}u_j = -\sum_{j=n}^{\infty}\Delta U_j \prod_{t=n_1}^{n_1+a-1}\left(1-\frac{t}{j}\right).$$

Similarly to the reasonings in (i), we prove that the series in question are convergent so, applying the formula for addition by parts and, by (5), (14), we finally obtain

$$\left|\sum_{j=n}^{\infty}(j-n_1)^{(a)}u_j\right| \leqslant \rho_n \prod_{t=n_1}^{n_1+a-1}\left(1-\frac{t}{n}\right)+\sum_{j=n}^{\infty}\rho_{j+1}\,\Delta\prod_{t=n_1}^{n_1+a-1}\left(1-\frac{t}{j}\right)\leqslant$$

$$\leqslant \rho_n\left[\prod_{t=n_1}^{n_1+a-1}\left(1-\frac{t}{n}\right)+1-\prod_{t=n_1}^{n_1+a-1}\left(1-\frac{t}{n}\right)\right] = \rho_n \leqslant 2\rho_n.$$

It may well be proved that Lemma 1 holds for a = 0. This completes the proof.

Lemma 2. Let u: N → R and suppose that the series $\sum_{j=1}^{\infty} j^{m-1+\alpha}u_j$ is convergent, where α is a non-negative number. Define

$$(15)\qquad w_n = (-1)^{m-1}\sum_{j=n}^{\infty}\frac{(j+m-1-n)^{(m-1)}}{(m-1)!}\,u_j,\qquad n\in N,$$

and

$$(15')\qquad \bar{\rho}_n = \sup_{T\geqslant n}\left|\sum_{j=T}^{\infty} j^{m-1+\alpha}u_j\right|.$$

Then

$$\left|\Delta^r w_n\right| \leqslant \frac{2\bar{\rho}_n n^{-r-\alpha}}{(m-r-1)!}, \quad 0 \leqslant r \leqslant m-1, \tag{16}$$

and

$$\Delta^r w_n = o(n^{-r-\alpha}), \quad 0 \leqslant r \leqslant m-1. \tag{17}$$

Proof. By definition (15) we obtain

$$\Delta w_n = \frac{(-1)^{m-1}}{(m-1)!}\left[\sum_{j=n+1}^{\infty} (j+m-1-n-1)^{(m-1)} u_j + \right.$$

$$\left. - \sum_{j=n+1}^{\infty} (j+m-1-n)^{(m-1)} u_j - (m-1)^{(m-1)} u_n \right] =$$

$$= \frac{(-1)^m}{(m-2)!}\left[\sum_{j=n+1}^{\infty} (j+m-n-2)^{(m-2)} u_j + (m-2)!\,u_n\right] =$$

$$= (-1)^m \sum_{j=n}^{\infty} \frac{(j+m-n-2)^{(m-2)}}{(m-2)!} u_j.$$

Similarly, we get $\Delta^2 w_n$ and $\Delta^3 w_n$.

By principle of mathematical induction we show that

$$\Delta^r w_n = (-1)^{m+r-1} \sum_{j=n}^{\infty} \frac{(j+m-1-n-r)^{(m-r-1)}}{(m-r-1)!} u_j, \tag{18}$$

where $0 \leqslant r \leqslant m-1$. Hence

$$\left|\Delta^r w_n\right| \leqslant \frac{1}{(m-r-1)!}\left|\sum_{j=n}^{\infty} (j+m-1-n-r)^{(m-r-1)} u_j\right|.$$

Now, we use the Lemma 1 taking $n_1 = -(m - 1 - r - n)$, $a = m - 1 - r$ and $b = m - 1 + \alpha$ which obviously satisfy the conditions of Lemma 1. So, $a \leqslant b \Rightarrow a - b = -(r+\alpha) \leqslant 0$ and $n = n_1 + a$. Hence we have (16) with $\bar{\rho}_n$ given by (15′). Further, by (15′), (16), we have

$$\frac{\left|\Delta^r w_n\right|}{n^{-r-\alpha}} \leqslant \frac{2}{(m-r-1)!} \sup_{T \leqslant n} \left| \sum_{j=T}^{\infty} j^{m-1+\alpha} u_j \right| ,$$

the sequence $\sum_{j=1}^{\infty} j^{m-1+\alpha} u_j$ being convergent by assumption; hence $\lim_{T \to \infty} \sum_{j=T}^{\infty} j^{m-1+\alpha} u_j = 0$ which implies the thesis (17).

Now, we come to the main problem of this paper: to the investigation of the equation (E). The following theorem holds

Theorem. Let us assume that k and m are integers such that $0 \leqslant k \leqslant m-1$; α is a non-negative number such that

$$\alpha + k > 0; \tag{19}$$

γ is a real number; the series

$$\sum_{j=n_0}^{\infty} j^{m-1+\alpha+k\gamma} p_j \quad \text{is absolutely convergent,} \tag{20}$$

$$\sum_{j=n_0}^{\infty} j^{m-1+\alpha} f_j \quad \text{is convergent,} \tag{21}$$

where $p, f \colon N_0 \to R$. Let us denote

$$q_n = \sum_{j=0}^{k} a_j \frac{n^j}{j!} , \tag{22}$$

where $a_0, a_1, \dots, a_k$ are given real constants, with $a_k > 0$. Then the equation (E) has a solution x satisfying the relation

(23) $$\Delta^r x_n = \Delta^r q_n + o(n^{-r-\alpha}), \quad 0 \leqslant r \leqslant m-1.$$

Proof. Without loss of generality we can assume that $n_0 \geqslant m-1$. Denote by $m(n^{-\alpha})$ the Banach space of sequences $h = \{h_n\}_{n=n_0}^{\infty}$ such that $h_n = O(n^{-\alpha})$, with the norm

(24) $$\|h\| = \sup_{n \geqslant n_0} \left\{ n^{\alpha} |h_n| \right\}.$$

For $M > 0$ we define a set

(25) $$m_M(n^{-\alpha}) = \left\{ h \in m(n^{-\alpha}) \ : \ \|h\| \leqslant M \right\}.$$

By (22), $\lim\limits_{n \to \infty} \frac{q_n}{n^k} = \frac{a_k}{k!}$. So, there exists such n_1 that for each $n \geqslant n_1$ we have

$$\frac{3}{4} \frac{a_k}{k!} \leqslant \frac{q_n}{n^k} \leqslant \frac{5}{4} \frac{a_k}{k!}.$$

Take $h \in m_M(n^{-\alpha})$, hence $n^{\alpha}|h_n| \leqslant M$. This implies $n^{-k}|h_n| \leqslant M n^{-k-\alpha}$. In virtue of (19), we obtain $\lim\limits_{n \to \infty} \frac{|h_n|}{n^k} = 0$. So, there exists n_2 such that for each $n \geqslant n_2$ we get

$$-\frac{1}{4} \frac{a_k}{k!} \leqslant \frac{h_n}{n^k} \leqslant \frac{1}{4} \frac{a_k}{k!}.$$

Therefore for $n \geqslant n_3 = \max(n_1, n_2)$ we have

(26) $$\frac{1}{2} \frac{a_k}{k!} n^k \leqslant q_n + h_n \leqslant \frac{3}{2} \frac{a_k}{k!} n^k.$$

Introduce the following notations

$$(27)\qquad C_1 = \begin{cases} \left(\frac{1}{2}\,\frac{a_k}{k!}\right)^{\gamma} & \text{for } \gamma < 0 \\ \left(\frac{3}{2}\,\frac{a_k}{k!}\right)^{\gamma} & \text{for } \gamma \geqslant 0, \end{cases}$$

$$(28)\qquad C_2 = \begin{cases} \left(\frac{1}{2}\,\frac{a_k}{k!}\right)^{\gamma-1} & \text{for } \gamma < 1 \\ \left(\frac{3}{2}\,\frac{a_k}{k!}\right)^{\gamma-1} & \text{for } \gamma \geqslant 1. \end{cases}$$

The sequence $\left\{n^{-k-\alpha}\right\}_{n=1}^{\infty}$ being decreasing, (20), (21) imply that there exists a finite $n_4 \in N$ such that

$$(29)\qquad n_4 = \sup\left\{t \in N : \left|\sum_{j=t}^{\infty} j^{m-1+\alpha} f_j\right| + \right.$$

$$\left. + \, C_1 \sum_{j=t}^{\infty} j^{m-1+\alpha+k\gamma} |p_j| > \frac{M(m-1)!}{2}\right\} =: \sup A$$

and $n_5 \in N$ such that

$$(30)\qquad n_5 = \sup\left\{t \in N : C_2 |\gamma| t^{-k-\alpha} \frac{1}{(m-1)!} \sum_{j=t}^{\infty} j^{m-1+k\gamma+\alpha} |p_j| \geqslant \frac{1}{2}\right\} =:$$

$$=: \sup B.$$

If A is an empty set (for example: when $f \equiv 0$ and $p \equiv 0$), then we put $n_4 = 1$. Similarly, we put $n_5 = 1$, if B is empty. Let $n_6 = \max(n_3, n_4, n_5)$. In the set $m_M(n^{-\alpha})$ we define a mapping $\hat{h} = Th$ in the below given manner

$$(31)\quad \hat{h}_n = \begin{cases} 0, & n \leq n_6, \\ (-1)^{m-1} \displaystyle\sum_{j=n}^{\infty} \frac{(j+m-1-n)^{(m-1)}}{(m-1)!}\left[-f_j + p_j(q_j+h_j)^{\gamma}\right], & n > n_6. \end{cases}$$

Now, we shall show that the series

$$(32)\quad \sum_{j=n}^{\infty} \frac{(j+m-1-r-n)^{(m-r-1)}}{(m-r-1)!}\, f_j,$$

$$(33)\quad \sum_{j=n}^{\infty} \frac{(j+m-1-r-n)^{(m-r-1)}}{(m-r-1)!}\, p_j(q_j + h_j)^{\gamma}, \quad 0 \leq r \leq m-1,$$

are convergent. In virtue of the convergence of the series (21) and because the sequence $\left\{j^{-\alpha-r}\right\}_{j=n}^{\infty}$ is non-increasing ($\alpha + r \geq 0$ for an arbitrary $0 \leq r \leq m-1$) and bounded (in virtue of the Abel criterion) the series $\sum_{j=n}^{\infty} j^{m-r-1} f_j = \sum_{j=n}^{\infty} j^{-\alpha-r} j^{m-1+\alpha} f_j$ is convergent. Because the sequence $\left\{\frac{(j+m-r-n-1)^{(m-r-1)}}{j^{m-r-1}}\right\}_{j=n}^{\infty}$ is increasing and bounded, applying once more the Abel criterion, we state the convergence of the series

$$(34)\quad \sum_{j=n}^{\infty} (j+m-r-1-n)^{(m-r-1)} f_j, \quad 0 \leq r \leq m-1.$$

By (26), (27) with $n \geq n_3$, we get

$$\sum_{j=n}^{\infty} j^{m-r-1} |p_j|\, |q_j + h_j|^{\gamma} \leq C_1 \sum_{j=n}^{\infty} j^{m-1+k\gamma+\alpha} |p_j|\, j^{-r-\alpha},$$

the series on the right-hand side being convergent (in virtue of (20), the Abel criterion and because the sequence $\left\{j^{-r-\alpha}\right\}_{j=n}^{\infty}$ is non-increasing and bounded); so the series on the left-hand side is convergent too. Hence, the series $\sum_{j=n}^{\infty} j^{m-r-1} p_j(q_j + h_j)^{\gamma}$, $0 \leq r \leq m-1$, is convergent. Because the sequence $\left\{\dfrac{(j+m-r-n-1)^{(m-r-1)}}{j^{m-r-1}}\right\}_{j=n}^{\infty}$ is increasing and bounded, in virtue of the Abel criterion, the series

$$(35)\qquad \sum_{j=n}^{\infty} (j+m-r-n-1)^{(m-r-1)} p_j(q_j + h_j)^{\gamma}, \quad 0 \leqslant r \leqslant m-1,$$

is convergent. The convergence of the series (34), (35) implies the convergence of their sum i.e.

$$(36)\qquad (-1)^{m-r-1} \sum_{j=n}^{\infty} \frac{(j+m-r-n-1)^{(m-r-1)}}{(m-r-1)!} \left[-f_j + p_j(q_j + h_j)^{\gamma}\right].$$

Similarly, one can prove that the series

$$(37)\qquad \sum_{j=n}^{\infty} j^{m-1+\alpha} \left[-f_j + p_j(q_j + h_j)^{\gamma}\right]$$

is convergent too.

Now, we prove that the mapping $\hat{h} = Th$ is a contraction of $m_M(n^{-\alpha})$ into itself.

By (31) and the convergence of (37), applying Lemma 2 with $u_j = -f_j + p_j(q_j + h_j)^{\gamma}$, we have $|\hat{h}_n| \quad \dfrac{2\bar{\rho}_n n^{-\alpha}}{(m-1)!}$.

But (26), (27) yield (as $n \geqslant \max(n_3, n_4)$)

$$\bar{\rho}_n = \sup_{T \geqslant n} \left|\sum_{j=T}^{\infty} j^{m-1+\alpha} \left[-f_j + p_j(q_j + h_j)^{\gamma}\right]\right| \leqslant$$

$$\leqslant \sup_{T \geqslant n} \left\{\left|\sum_{j=T}^{\infty} j^{m-1+\alpha} f_j\right| + C_1 \sum_{j=T}^{\infty} j^{m-1+\alpha+k\gamma} |p_j|\right\} \leqslant$$

$$\leqslant \sup_{T \geqslant n} \left\{\frac{M(m-1)!}{2}\right\} = \frac{M(m-1)!}{2}.$$

Hence $\|\hat{h}\| = \sup\limits_{n \geq n_6} \left\{ n^{\alpha} |\hat{h}_n| \right\} \dfrac{2\bar{\varrho}_n}{(m-1)!} \leq M$. Therefore T maps $m_M(n^{-\alpha})$ into itself. Now, we prove that the mapping T is a contraction of $m_M(n^{-\alpha})$. Let

$$\hat{h}_{1,n} = \begin{cases} 0, & n \leq n_6 \\ (-1)^{m-1} \displaystyle\sum_{j=n}^{\infty} \frac{(j+m-1-n)^{(m-1)}}{(m-1)!} \left[-f_j + p_j (q_j + h_{1,j})^{\gamma} \right], & n > n_6, \end{cases}$$

$$\hat{h}_{2,n} = \begin{cases} 0, & n \leq n_6 \\ (-1)^{m-1} \displaystyle\sum_{j=n}^{\infty} \frac{(j+m-1-n)^{(m-1)}}{(m-1)!} \left[-f_j + p_j (q_j + h_{2,j})^{\gamma} \right], & n > n_6, \end{cases}$$

both of above series being convergent, by (36). Now, applying the mean value theorem to the function $G(x) = x^{\gamma}$, we get

$$(q_j + h_{1,j})^{\gamma} - (q_j + h_{2,j})^{\gamma} = \gamma \xi_j^{\gamma-1} (h_{1,j} - h_{2,j}).$$

From (26) we have

$$\frac{1}{2} \frac{a_k}{k!} j^k \leq \min(q_j + h_{1,j},\ q_j + h_{2,j}) \leq \xi_j \leq$$
$$\leq \max(q_j + h_{1,j},\ q_j + h_{2,j}) \leq \frac{3}{2} \frac{a_k}{k!} j^k$$

and, by (28), we can write $|\xi_j|^{\gamma-1} = \xi_j^{\gamma-1} \leq C_2 j^{(\gamma-1)k}$. Therefore

$$|\hat{h}_{1,n} - \hat{h}_{2,n}| \leq \frac{C_2 |\gamma|}{(m-1)!} \sum_{j=n}^{\infty} (j+m-1-n)^{(m-1)}\ j^{(\gamma-1)k} |p_j| |h_{1,j} - h_{2,j}|.$$

Because $j+m-n-1 \leq j$, we have $(j+m-n-1)^{(m-1)} \leq j^{(m-1)} \leq j^{m-1}$ and consequently

$$n^{\alpha}\left|\hat{h}_{1,n} - \hat{h}_{2,n}\right| \leqslant \frac{C_2\,|\gamma|}{(m-1)!}\,\|h_1 - h_2\|\, n^{-k-\alpha}\sum_{j=n}^{\infty} j^{m-1+k\gamma+\alpha}|p_j|.$$

Hence, by $n \geqslant n_5$ and (30), we have the relation

$$\sup_{n\geqslant n_5}\left\{n^{\alpha}\left|\hat{h}_{1,n} - \hat{h}_{2,n}\right|\right\} \leqslant \frac{1}{2}\,\|h_1 - h_2\|$$

which implies $\|\hat{h}_1 - \hat{h}_2\| \leqslant \frac{1}{2}\|h_1 - h_2\|$. It means that T is a contraction. Therefore there exists (in virtue of the contraction theorem) exactly one such a sequence $h_0 \in m_M(n^{-\alpha})$ that $h_0 = T\,h_0$.

Now, we show that $x = q + h_0$ is the solution of the equation (E). Let $\Delta^m x_n = \Delta^m(q_n + h_{0,n}) = \Delta^m q_n + \Delta^m h_{0,n} = \Delta^m h_{0,n}$ ($\Delta^m q_n = 0$ as the m-th difference of the polynomial q of order $k \leqslant m-1$). By (31), (18) for $\hat{h}_0 = h_0$ and $r = m-1$, we have

$$\Delta^{m-1}\,\hat{h}_{0,n} = \sum_{j=n}^{\infty}\left[-f_j + p_j(q_j + h_{0,j})^{\gamma}\right].$$

Hence $\Delta^m\,\hat{h}_{0,n} = f_n - p_n(q_n + h_{0,n})^{\gamma}$ and finally $\Delta^m x_n = f_n - p_n(q_n + h_{0,n})^{\gamma} = f_n - p_n x_n^{\gamma}$ or x is the solution of (E).

By Lemma 2, (17), we have $\Delta^r h_{0,n} = o(n^{-r-\alpha})$ for $n > n_6$. Therefore

$$\Delta^r x_n = \Delta^r(q_n + h_{0,n}) = \Delta^r q_n + o(n^{-r-\alpha}), \quad 0 \leqslant r \leqslant m-1.$$

This completes the proof of the theorem.

Example 1. Take $n_0 = 1$, $m = 2$, $\gamma = 3$, $p_n = 3\left[2^{n+2}(1+n+2^{-n})^3\right]^{-1}$, $f_n = 2^{-n}$, then (E) has the form

$$(E_1) \qquad \Delta^2 y_n + 3\left[2^{n+2}(1+n+2^{-n})^3\right]^{-1} y_n^3 = 2^{-n},$$

further let $k = 1$ and $\alpha = 1$. It is easy to prove the convergence of the series $\sum_{j=1}^{\infty} j^2 2^{-j}$ (cf. (21)) and $\sum_{j=1}^{\infty} 3\left[2^{j+2}(1+j+2^{-j})^3\right]^{-1} j^5$ (cf. (20)), so the assumptions of the theorem are satisfied. Let $q_n = 1 + n$. One can see that $x_n = 1 + n + 2^{-n}$ is the solution of the equation (E_1). Moreover, x_n satisfies the condition (23), since

$$\frac{x_n - q_n}{n^{-1}} = \frac{n}{2^n} \to 0, \quad \text{as} \quad n \to \infty, \text{ for } r = 0,$$

$$\frac{\Delta x_n - \Delta q_n}{n^{-2}} = -\frac{n^2}{2^{n+1}} \to 0, \quad \text{as} \quad n \to \infty, \text{ for } r = 1.$$

Asymptotic properties of solutions of difference equations were studied e.g. in [4]. The authors studied (E) with $m = 2$ and $f \equiv 0$ and presented conditions by which (E) possesses oscillatory, nonoscillatory solutions and these solutions tend monotonically to infinity or to zero. In [6] the authors consider systems of summary-difference equations and give conditions for the solutions tending to constant. The most similar to our results were these obtained in [3], [5]; it is true that the equations considered in [3], [5] are more general, but the assumptions on p, f, and γ are different: so the results presented in [5] are general, provided that we take in our investigations $\gamma \leq 1$, $\alpha = 0$, $k = m-1$, $f > 0$.

REFERENCES

[1] R.P. Agarwal : Properties of solutions of higher order nonlinear difference equations II, An. Sti Univ. Al. I. Cuza Iasi 29 (1983), 85-96.

[2] G.M. Fichtenholz : Rachunek różniczkowy i całkowy. T. II, Warszawa 1966.

[3] V.G. Golovina: St'ep'ennaja asimptotika res'enij nielin'ejnych raznostnych urawn'enij, Issl'edowanija po integrodifferencialnych urawn'enijam. (1980), 310-313.

[4] J.W. Hooker, W.T. Patula: A second-order nonlinear difference equation: oscillation and asymptotic behaviour, J.Math.Anal.Appl. 91 (1983), 9-29.

[5] J. Popenda: On the asymptotic behaviour of the solutions of an n-th order difference equation, Ann. Polon. Math. 44.1 (1984), 95-111.

[6] W.F. Trench: Asymptotic integration of $y^{n}+q(t)y^{\gamma}=$ $= f(t)$ under mild integral smallness conditions, Funkcialaj Ekvacioj, 26 (1983), 197-209.

[7] J.A. V'ed', M.K. Kaparov: O str'eml'enii k kon'ecnym pr'ed'elam r'es'enij sist'em summarno-raznostnych uravn'enij, Issl'edowanija po integrodifferencialnych urawn'enijam. (1980), 299-309.

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A NOTE ON THE STABILIZATION OF LINEAR SYSTEMS IN HILBERT SPACES

0. Introduction

Let U, X, Y be Hilbert spaces, and A, B, C linear operators. Assume that $B \in L(U,X)$ and $C \in L(X,Y)$, where the symbol "L(,)" denotes the boundedness of the two operators. A is supposed to be the infinitesimal generator of a C_0-semigroup, with (dense) domain D(A) in X.

Let us consider the observed process:

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{1}$$

$$y(t) = Hx(t), \quad t \geq 0. \tag{2}$$

The aim of this paper is to study the stabilization properties of the system (1) by means of a direct feedback $u(t) = Dx(t)$ (with $D \in L(X,U)$), or through the output (2) with $u(t) = Ky(t) = KHx(t)$ (where $K \in L(Y,U)$).

Stabilization problems have gained the interest of many authors. Yet we shall refer to W.M. Wonham [8] for finite dimensional systems, and to A.J. Pritchard - J. Zabczyk [6] for the infinite dimensional case (see also L. Pandolfi [5], M. Megan - R. Reghis [4] for evolution processes in infinite dimensional spaces).

Section 1 deals with stabilization by direct feedback. A generalization of results in [7] will be given via a similar decomposition of the state space. The second section will be reserved to some generalization of statements on stabilization by output (see W.M. Wonham [8]).

1. Stabilization by direct feedback

Let $S_D(t)$ denote the semigroup generated by the operator A+BD.

Definition 1. The system (1) is called stabilizable if and only if:

$$\exists D \in L(X,U), \quad \forall x_0 \in X; \quad S_D(t)x_0 \to 0 \quad (t \to +\infty).$$

Definition 2. The system (1) is called exponentially stabilizable if and only if:

$$\exists N, \ \exists \varepsilon > 0, \quad \exists D \in L(X,U); \quad \| S_D(t) \| \leqslant Ne^{-\varepsilon t}.$$

Let us put $L = \left\{ x \in X : B^* S^*(t)x = 0, \ t \geqslant 0 \right\}$, $L^\perp$ the orthogonal complement of the subspace L, and P the orthogonal projection operator on $L^\perp$ (so that I-P is the orthogonal projector associated with L). L is then clearly an A^*-invariant, closed subspace of X, i.e. $A^*(L \cap D(A^*)) \subset L$, where A^* is the adjoint of A.

Applying P and I-P to both sides of (1) yields:

$$\dot{x}_1(t) = PAx_1(t) + PAx_2(t) + PBu(t) \tag{3}$$

$$\dot{x}_2(t) = (I-P)Ax_2(t), \quad t \geqslant 0 \tag{4}$$

with $x_1(t) = Px(t)$ and $x_2(t) = (I-P)x(t)$, $t \geqslant 0$.

Theorem 1. When the operator PA(I-P) is bounded, the system (1) is exponentially stabilizable if and only if:

(i) The system $\dot{x}_1 = PAx_1 + PBu$ is exponentially stabilizable in $L^\perp$,

(ii) The system $\dot{x}_2 = A^* x_2$ is exponentially stable in L.

Proof. Only if: Assume that system (1) is exponentially stabilizable, then (i) is obvious: one can take as a feedback operator the restriction of D to the subspace $L^\perp$. Now to show (ii), let $x_2 \in L$, so that, [6]:

$$S_D^*(t)x_2 = S^*(t)x_2 + \int_0^t S_D^*(t)D^*B^*S^*(t-s)x_2\,ds =$$

$$= S^*(t)x_2, \quad \text{since} \quad B^*S^*(t)x_2 = 0 \quad \text{for} \quad t \geqslant 0.$$

On the other hand, from exponential stabilizability assumption we get

$$\|S^*(t)x_2\| \leqslant N_\varepsilon e^{-\varepsilon t}\|x_2\|,$$

which states the exponential stability of the system $\dot{x}_2 = A^*x_2$.

If: Let D_1 be the feedback operator for the system $\dot{x}_1 = PAx_1 + PBu$, and denote by $S_1(t)$ the semigroup generated by $PA + PBD_1$ in the subspace L.

As feedback operator for the system (1), let us take $D = \{D_1, 0\}$, i.e. the operator defined by the relation $Dx = D_1Px$; then for $x_0 \in D(A)$, we get

$$x(t) = S_D(t)x_0 = x_1(t) + x_2(t),$$

where

(5) $$x_1(t) = S_1(t)x_1(0) + \int_0^t S_1(t-s)PAx_2(s)\,ds,$$

(6) $$x_2(t) = (I-P)S_D(t)(I-P)x_2(0) = (I-P)S(t)(I-P)x_2(0).$$

We shall first consider $x_2(t)$. We have

$$\|x_2(t)\| \leqslant \|(I-P)S(t)(I-P)\|\,\|x_2(0)\| \leqslant$$

$$\leqslant \|(I-P)S^*(t)(I-P)\|\,\|x_2(0)\|.$$

On the other hand, $(I-P)S^*(t)(I-P)$ is a semigroup generated by the restriction of A^* to the subspace L. So, by assumption (ii) we get

$$(7) \qquad \|x_2(t)\| \leq N_2 e^{-\varepsilon_2 t}\|x_2(0)\|$$

where ε_2 and N_2 are positive constants.

Now, assumption (i) gives

$$(8) \qquad \|S_1(t)\| \leq N_1 e^{-\varepsilon_1 t},$$

and from (7) and (8) and the continuity of PA on L, we are lead to

$$\|x_1(t)\| \leq \|S_1(t)x_1(0)\| + N_1N_2\|PA(I-P)\| \int_0^t e^{-\varepsilon_1 t} e^{(\varepsilon_2-\varepsilon_1)s}\|x_2(0)\|ds.$$

After some computations, we can obtain

$$(9) \qquad \|x_1(t)\| \leq N\, e^{-\varepsilon t}\|x_0\|,$$

with positive constants N and ε.

Therefore, (7) and (9) give

$$\|x(t)\| = \|S_D(t)x_0\| \leq N\, e^{-\varepsilon t}\|x_0\|, \quad \text{for all } x_0 \text{ in } D(A).$$

Finally, from density of D(A) in X, it follows that

$$\|S_D(t)\| \leq N\, e^{-\varepsilon t},$$

which means the exponential stabilizability of system (1).

Remark. The boundedness assumption of PA(I-P) is satisfied in particular in the following cases:

(i) $\operatorname{Im}(I-P) = L \subset D(A)$,

(ii) $\dim L < \infty$,

(iii) $PA(I-P) = 0$, i.e., L is A-invariant (for example when A is self-adjoint).

We can also give a more particular result for the stabilization of system (1), when the operator A is self-adjoint.

Theorem 2. If A is self-adjoint, the system (1) is stabilizable if and only if:

(i) The system (1) is stabilizable on $L^{\perp}$,

(ii) The system (1) is asymptotically stable on L.

Proof. Only if: Assume that the system (1) is stabilizable. Then condition (i) is trivial, and there exists $D \in L(X,U)$ such that we have:

$$\lim_{t \to \infty} (S_D(t)x,y) = 0, \quad \text{for all } x \text{ in } X \text{ and all } y \text{ in } L.$$

Notice that $(S_D(t)x,y) = (x,S_D^*(t)y)$, where $S_D^*(t)$ is the semigroup generated by the operator $(A+BD)^* = A^* + D^*B^*$, and satisfies [1]:

$$S_D^*(t)y = S^*(t)y + \int_0^t S_D^*(t)D^*B^*S^*(t-s)yds.$$

But since $y \in L$, then $B^*S^*(t-s)y = 0$ for all $s \in [0,t]$, and so: $S_D^*(t)y = S^*(t)y$. Therefore we get

$$\lim_{t \to \infty} (S(t)x,y) = \lim_{t \to \infty} (x,S^*(t)y) = \lim_{t \to \infty} (x,S_D^*(t)y) =$$

$$= \lim_{t \to \infty} (S_D(t)x,y) = 0,$$

which means that the system $\dot{x} = (A+BD)x$ is weakly-asymptotically-stable. From [2], it follows that this system is asymptotically stable, indeed, since A is self-adjoint, S(t) is self-adjoint and we have for all x in X:

$$\|S(t)x\|^2 = (S(t)x,S(t)x) = (S(2t)x,x) \longrightarrow 0 \ (t \longrightarrow \infty).$$

If: From assumption (i), there exists a linear operator D_1, $D_1 \in L(L^\perp,U)$ such that the system (3) is stabilizable. Remark that, since A is self-adjoint, $PAx_2 = 0$ for all x_2 in L, so that system (3) becomes

$$\dot{x}_1 = PAx_1 + PBu. \tag{3'}$$

Let us take as a feedback control for the system (1), the function $u(t) = D_1Px(t) = D_1x_1(t)$; the generalised solution of (1) with this control is: $x(t) = x_1(t) + x_2(t)$, where

$$x_1(t) = S_D(t)x_1(0) = PS_{D_1}(t)x_1(0),$$

$$x_2(t) = S_2(t)x_2(0),$$

$S_2(t)$ standing for the semigroup generated by the operator $(I-P)A$. Since L is A-invariant (because $A = A^*$), the operators $S_2(t)$ and $S(t)$ coincide on L. So, from (ii) we have

$$\lim_{t \to \infty} \|x_2(t)\| = 0,$$

and from (i)

$$\lim_{t \to \infty} \|x_1(t)\| = 0.$$

Therefore, we deduce that $\lim_{t \to \infty} \|x(t)\| = 0$, which means that the system (1) is stabilizable.

2. Stabilization by indirect feedback

In this section we shall attempt to extend some of Wonham's finite-dimensional results [8] to the case where A is a bounded operator acting in a Hilbert space (likewise B and H).

Let X_1 be the subspace $\overline{\text{Lin}}\,\{\text{KerH}, \text{AKerH}, A^2\text{KerH},\ldots\}$, and $X_2 = X_1^{\perp}$ its orthogonal complement. These two closed subspaces of X are respectively A and A^*-invariant.

Set Q and I-Q for the orthogonal projectors on the subspaces X_1 and X_2 respectively, and apply these operators to both sides of the system (1)-(2) to obtain

$$\dot{x}_1 = QAx_1 + QAx_2 + QBu, \tag{10}$$

$$\dot{x}_2 = (I-Q)Ax_2 + (I-Q)Bu, \tag{11}$$

and the output

$$y(t) = Hx_1(t) + Hx_2(t), \tag{12}$$

with $x_1 = Qx$, $x_2 = (I-Q)x$.

Theorem 3. The system (1) (with A bounded) is exponentially stabilizable by the output if the following conditions are satisfied:

(i) The system (1) is exponentially stabilizable,

(ii) The system $\dot{x} = Ax$ is exponentially stable on X_1,

(iii) Im H (i.e., the range of H) is a closed subspace.

Proof. Assumption (i) assures the existence of a bounded linear operator $D_2 \in L(X_2, U)$ which makes the system (11) exponentially stabilizable. Since $X_2 \subset \overline{\mathrm{Im}\, H^*}$, then with (iii) and from a result of R.G. Douglas [3], there exists a linear operator $F_2 \in L(Y, X_2)$ such that $I - Q = F_2 H$. So the system (11) is exponantially stabilizable by means of the feedback $u(t) = K_2 H x_2(t) = D_2 F_2 H x_2(t)$.

On the other hand, Y is a Hilbert space, and Im H is closed. Consequently

$$Y = H(X) \oplus \mathrm{Ker} H^* = H(X_1) \oplus H(X_2) \oplus \mathrm{Ker} H^*.$$

Now let us define a bounded linear operator K by

$$Ky = \begin{cases} 0, & y \in H(X_1); \\ K_2 y, & y \in H(X_2); \\ 0, & y \in \mathrm{Ker}\, H^*, \end{cases}$$

and consider the initial system (1) with the control $u(t) = KHx(t) = Ky(t)$. For x_0 in X, the solution of (1) is: $x(t) = x_1(t) + x_2(t)$, where

(13) $$x_1(t) = e^{QAt} x_1(0) + \int_0^t e^{QA(t-s)} (QA + QBK_2H) x_2(s)\, ds,$$

(14) $$x_2(t) = e^{(I-Q)(A+BK_2H)t} x_2(0).$$

By construction of K_2, we have

(15) $$\|x_2(t)\| \leq N_2 e^{-\alpha_2 t} \|x_2(0)\|,$$

with positive constants N_2 and α_2. For its part, assumption (ii) gives

$$(16) \qquad \|e^{QAt}x_1(0)\| \leqslant N_1 e^{-\alpha_1 t}\|x_1(0)\|.$$

Then, we can get from (13) and (16) after some computations

$$(17) \qquad \|x_1(t)\| \leqslant Me^{-\alpha t}\|x(0)\|,$$

for some positive constants M and α. Therefore, from (15) and (17), we deduce the inequality $\|x(t)\| \leqslant Ne^{-\alpha t}x_0$, for all $x_0 = x(0)$ in X; that means that the system (1) is exponentially stabilizable by output feedback.

We can derive as a corollary the following known result [8] in finite dimensional spaces.

Corollary. If the system (1) is stabilizable, and X_1 contained in the stable part of the homogeneous system $\dot{x} : Ax$ (i.e. $X_1 \subset X^-(A)$), then (1) is stabilizable by output feedback.

Remark. The conditions of Theorem 3 are sufficient but not necessary for the stabilization by output. (Examples can be easily constructed).

From more investigations in this way, it appears that a necessary condition for stabilization by output is that the unobservable part of the system i.e. the subspace $\mathcal{L} = \bigcap_i \operatorname{Ker} HA^i$, must be contained in the stable part of the homogeneous system $\dot{x} = Ax$.

Let R be the orthogonal projection operator on $\mathcal{L}$, I-R that of $\mathcal{L}^\perp$ and $x_1 = Rx$, $x_2 = (I-R)x$ for all x in X. Applying R and I-R to each side of (1) yields

$$(17) \qquad \dot{x}_1 = RAx_1 + RAx_2 + RBu$$

$$(18) \qquad \dot{x}_2 = (I-R)Ax_2 + (I-R)Bu$$

with the output: $y(t) = Hx_1(t) + Hx_2(t) = Hx_2(t)$.

We can give presently necessary and sufficient conditions for stabilization by output feedback in the case where the operator A is unbounded.

Theorem 4. If the operator $RA(I-R)$ is bounded, then the system (1) is exponentially stabilizable by output feedback if and only if:

(a) The system (1) is exponentially stabilizable by output feedback on $\mathcal{L}^{\perp}$,

(b) The system $\dot{x} = Ax$ is exponentially stable on $\mathcal{L}$.

Proof. Only if: Since (1) is exponentially stabilizable on X, it must be so on $\mathcal{L}^{\perp}$ (and $\mathcal{L}$ too). Then (a) is obvious, and there exists a bounded linear operator $K \in L(Y,U)$ such that the system $\dot{x} = (A+BKH)x$ is exponentially stable in $\mathcal{L}$. Thus, we have

$$\| S_{KH}(t)x \| \leqslant Ne^{-\alpha t}\| x \|,$$

for some positive constants N and α, and for all x in $\mathcal{L}$. Furthermore, for all x in $\mathcal{L}$, $BKH\, x = 0$. So, on the subspace $\mathcal{L}$, the semigroup generated by A coincide with that generated by the restriction on $\mathcal{L}$ of the operator A+BKH. Then,

$$\| S_{KH}(t)x \| = \| S(t)x \| \leqslant Ne^{-\alpha t}\| x \|, \quad \text{for all } x \text{ in } \mathcal{L}.$$

This proves that the system $\dot{x} = Ax$ is exponentially stable on $\mathcal{L}$.

If: Let us consider the system (18). From assumption (a), there exists an operator $K \in L(Y,U)$ such that the generalised solution of (8) with the control $u(t) = KHx_2(t)$ satisfies

$$\| x_2(t) \| \leqslant M_2 e^{-\varepsilon_2 t} \| x_2(0) \|, \tag{19}$$

for some positive constants M_2, ε_2.

As a feedback control for the system (1), we shall take the control $u(t) = KHx_2(t)$. If $S_1(t)$ denotes the semigroup generated by the operator RA, then we get for x_0 in $D(A)$: $x(t) = x_1(t) + x_2(t)$, where $x_2(t)$ is the solution of (18) and $x_1(t)$ that of (17) defined by

$$(20)\qquad x_1(t) = S_1(t)x_1(0) + \int_0^t S_1(t-s)(RA+RBKH)x_2(s)ds.$$

Since $RAx_1 = Ax_1$ for all x_1 in $\mathcal{L}$, then $S_1(t)x_1 = S(t)x_1$. But,

$$\|S_1(t)x_1(0)\| \leqslant M_1 e^{-\varepsilon_1 t}\|x_1(0)\|,$$

and since the operator RA is continuous on $\mathcal{L}$, we obtain from (20):

$$\|x_1(t)\| \leqslant M_1 e^{-\varepsilon_1 t}\|x_1(0)\| + M_1M_2(\|RA(I-R)\| + \|RBKH\|) \times$$

$$\times \int_0^t e^{-\varepsilon_2(t-s)}\, e^{-\varepsilon_1 s}\|x_1(0)\|ds.$$

After some computations, we are led to

$$(21)\qquad \|x_1(t)\| \leqslant Me^{-\alpha t}\, x_0$$

with positive constants M and α, and for all $x_0 = x(0)$ in D(A). Therefore, from (19) and (21), we deduce

$$\|x(t)\| = \|S_{KH}(t)x_0\| \leqslant Ne^{-\alpha t}\, x_0,$$

for all x_0 in D(A). Now as $\overline{D(A)} = X$, then: $\|S_{KH}(t)\| \leqslant Ne^{-\alpha t}$. So the system (1) is exponentially stabilizable with the control: $u(t) = Ky(t) = KHx_2(t)$.

REFERENCES

[1] A.V. Balakrishnan : Applied functional analysis. Springer-Verlag, 2nd Ed., New-York, 1981.

[2] C.D. Benchimol : Feedback stabilization in Hilbert space, Appl. Math. Optim., 4 (1978) 225-248.

[3] R.G. Douglas : On majorization, factorization and range inclusion of operators on Hilbert space, Proc. Matem. Soc., 17, N° 2, 1966.

[4] M. Megan, R. Reghis : On controllability and stabilizability in Hilbert space, Anal. Univ. din Timisoara. Seria Stünte Matematica, 15, 1, 1977.

[5] L. Pandolfi : Stabilization of control processes in Hilbert space, Proc. Royal Society of Edinburgh, 81A (1978) 247-258.

[6] A.J. Pritchard, J. Zabczyk : Stability and stabilizability for infinite dimensional systems, SIAM Review, 23, N° 1, 1981.

[7] R. Rabah : О точной управляемости и стабилизации в бесконечномерных пространствах. Вестник Харьковского Университета, серия ЛММ, выпуск 43, 1978.

[8] W.M. Wonham : Linear multivariable control: a geometric approach. Springer-Verlag, N.-Y., 1979.

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[3] R.G. Douglas : On majorization, factorization and range inclusion of operators on Hilbert space, Proc. Amer. Math. Soc., 17, N° 2, 1966.

[4] M. Megan, R. Reghis : On controllability and stabilizability in Hilbert spaces, Anal. Univ. din Timisoara. Seria Stiinte Matematice, 15, 1, 1977.

[5] L. Pandolfi : Stabilization of control processes in Hilbert spaces, Proc. Royal Society of Edinburgh, 81A (1978) 247-258.

[6] A.J. Pritchard, J. Zabczyk : Stability and stabilizability for infinite dimensional systems, SIAM Review, 23, N° 1, 1981.

[7] S. Rebah : О точной управляемости и стабилизации в бесконечномерных пространствах. Вестник Харьковского Университета, серия АМ, выпуск 43, 1978.

[8] W.M. Wonham : Linear multivariable controls: a geometric approach. Springer-Verlag, N.-Y., 1979.

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Leszek Sołtysik

IRRESOLUTE MAPPINGS

1. Foundations

Let X be a topological space. The closure and the interior of a set $A \subset X$ will be denoted by $\bar{A}$ and, $\operatorname{Int} A$ respectively.

Definition 1.1. [3]. A set A is called semi-open if there is an open set $U \subset X$ such that $U \subset A \subset \bar{U}$. A set B is called semi-closed if its complementary $X \setminus B$ is semi-open.

Any semi-open set U such that $x \in U$ $(A \subset U)$ will be called the semi-neighbourhood of a point x (a set A). The family of all semi-neighbourhoods of a point x (a set A) will be denoted by $S.N.(x)$ (resp. $S.N.(A)$).

Definition 1.2. [1]. The intersection of all semi-closed sets containing a set A is called semi-closure of a set A and is denoted by $\underline{A}$.

Proposition 1.1. A point x belongs to $\underline{A}$ if and only if for every its semi-neighbourhood V we have $V \cap A = \emptyset$.

Definition 1.3. [5]. A topological space X is said to be s-regular if for every closed set $A \subset X$ and every $x \in X \setminus A$ there are two disjoint sets: $U \in S.N.(x)$ and $V \in S.N.(A)$.

Definition 1.4. [4]. If for each pair of disjoint closed subsets A, B of a space X there is a pair of disjoint sets $U \in S.N.(A)$ and $V \in S.N.(B)$ then the space X is called s-normal.

Proposition 1.2. [5], [4]. A space X is s-regular (s-normal) iff for every $x \in X$ (every closed set $B \subset X$)

and every its semi-neighbourhood V there is a set $U \in S.N.(x)$ ($U \in S.N.(B)$) such that $\underline{U} \subset V$.

Definition 1.5 ([6], pp.74-75). A topological space X is said to be Baire space iff every its open subset is of the second category.

Let X, Y be topological spaces and let $\varphi(Y)$, $\mathcal{C}(Y)$ be classes of all not-empty and not-empty closed subsets of Y respectively. For a multivalued map $F : X \to \varphi(Y)$ we will denote:

$$F(A) = \{F(x) \subset Y : x \in A\}$$

$$F^{-}(B) = \{x \in X : F(x) \cap B \neq \emptyset\}$$

$$F^{+}(B) = \{x \in X : F(x) \subset B\}$$

for any sets $A \subset X$, $B \subset Y$.

Definition 1.6. [2]. A multivalued map $F : X \to \varphi(Y)$ is said to be l-irresolute (u-irresolute) at a point $x_0 \in X$ if for any semi-open set $W \subset Y$ such that $F(x_0) \cap W \neq \emptyset$, ($F(x_0) \subset W$) there exists a semi-neighbourhood U of the point x_0 such that $F(x) \cap W \neq \emptyset$, (resp. $F(x) \subset W$) for any $x \in U$.

A multivalued map $F : X \to \varphi(Y)$ is called l-irresolute (u-irresolute) if it is l-irresolute (u-irresolute) at every point of the space X.

A multivalued map which is l-irresolute and u-irresolute is called irresolute.

Proposition 1.3. [7]. A multivalued map $F : X \to \varphi(Y)$ is u-irresolute (l-irresolute) at a point $x \in X$ if and only if for every open neighbourhood U of x and every $V \in S.N.(F(x))$ (every semi-open set $V \subset Y$ such that $V \cap F(x) \neq \emptyset$) there is a notempty open set $U_1 \subset U$ such that $F(y) \subset V$ (resp. $F(y) \cap V \neq \emptyset$) for every $y \in U_1$.

Proposition 1.4. [7]. A multivalued map $F : X \to \varphi(Y)$ is u-irresolute (l-irresolute) if and only if the set $F^{+}(U)$ (resp. $F^{-}(U)$) is semi-open in X for every semi-open set $U \subset Y$.

Definition 1.7. [1]. An ordinary map $f : X \to Y$ is said to be irresolute if for every semi-open set $V \subset Y$ the set $f^{-1}(V)$ is semi-open in X.

An ordinary map $f: X \to Y$ may be interpreted as a multivalued map, which assigns at every point $x \in X$ the set $\{f(x)\}$. Moreover, we have $f^{-1}(A) = f^{-}(A) = f^{+}(A)$ for every $A \subset Y$. And so we have:

Proposition 1.5. [7]. A map $f: X \to Y$ is irresolute if and only if it is l-irresolute or u-irresolute as a multivalued map.

2. Irresolute mappings and dense subsets of their domains

Let X, Y be topological spaces. Let $f: X \to Y$ be irresolute and let $D \subset X$ be dense. It is evident that the restriction $f|_D$ is irresolute. In this part of the paper we will show that under some assumptions a map which is irresolute on dense subset of its domain is irresolute on whole domain. The example of Dirichlet's function shows that in general this fact does not hold.

Definition 2.1. We will say that a family $\mathcal{A} \subset \varphi(Y)$ is s-u-dense in a family $\mathcal{B} \subset \varphi(Y)$ if for every $B \in \mathcal{B}$ and every $G \in S.N.(B)$ there is $A \in \mathcal{A}$ such that $A \subset G$. However if for every $B \in \mathcal{B}$ and $y \in B$, and every $V \in S.N.(y)$ there is $A \in \mathcal{A}$ such that $A \cap V \neq \emptyset$ then we will say that $\mathcal{A}$ is s-l-dense in $\mathcal{B}$.

Proposition 2.1. If $F: X \to \varphi(Y)$ is u-irresolute (l-irresolute) then for every dense set $D \subset X$ and every open $U \subset X$ the family $F(D \cap U)$ is s-u-dense (s-l-dense) in the family $F(U)$.

Proof. We will prove this proposition supposing that F is u-irresolute, for l-irresolute mappings the proof is similar to this.

Let D be a dense subset of a space X. Let $P \in F(U)$ where U is an open subset of X, and let $G \subset Y$ be a semi-open set containing P. Evidently $F^{+}(G) \cap U \neq \emptyset$ and because $F^{+}(G)$ is semi-open we have $\operatorname{Int} F^{+}(G) \cap U \neq \emptyset$, so $F^{+}(G) \cap U \cap D \neq \emptyset$. There is a point $x \in U \cap D$ such that $F(x) \subset G$, obviously $F(x) \in F(U \cap D)$ and the proof is complete.

Making use of similar methods one can easily proved the following:

Proposition 2.2. If $f : X \rightarrow Y$ is irresolute then for every dense $D \subset X$ and every open $U \subset X$ we have $f(U) \subset \underline{f(U \cap D)}$.

If a family $\mathcal{A}$ is s-u-dense and s-l-dense in a family $\mathcal{B}$ then we will say that a family $\mathcal{A}$ is s-dense in a family $\mathcal{B}$.

Theorem 2.1. Let $F : X \rightarrow \mathcal{C}(Y)$ be a multivalued map having closed values in s-normal space Y. If there is a dense subset D of X such that for every open $U \subset X$ the family $F(U \cap D)$ is s-dense in the family $F(U)$ and $F|_D$ is u-irresolute then F is u-irresolute.

Proof. First we establish: a point $x_0 \in X$, its neigbourhood U and a semi-neighbourhood V of the set $F(x_0)$. Let V_1 be semi-open set such that

$$(1) \qquad F(x_0) \subset V_1 \subset \underline{V_1} \subset V.$$

Because $F(U \cap D)$ is s-u-dense in $F(U)$ there is a point $x_1 \in U \cap D$ such that $F(x_1) \subset V_1$. $F|_D$ is u-irresolute, so there is not-empty open set $G \subset U$ such that for every $x \in G \cap D$ we have

$$(2) \qquad F(x) \subset V_1.$$

Theorem will be proved if we shall show that $F(x) \subset V$ for every $x \in G$. Assume that, contrary, there is a point $z \in G$ such that $F(z) \cap (Y \setminus V) \neq \emptyset$. Let $q \in F(z) \cap (Y \setminus V)$. By (1) there is $W \in S.N.(q)$ such that

$$(3) \qquad W \cap \underline{V_1} = \emptyset.$$

On the other hand, because the family $F(G \cap D)$ is s-l-dense in the family $F(G)$, there is a point $y \in G \cap D$ such that

$$(4) \qquad F(y) \cap W \neq \emptyset.$$

But from (2) we have $F(y) \subset V_1$ and this is contradiction to (3) so the proof is completed.

Theorem 2.2. Let Y be s-regular space. If for a closed-valued map $F : X \to \mathcal{C}(Y)$ there is a dense subset D of the space X such that $F|_D$ is 1-irresolute and for every open set $U \subset X$ the family $F(U \cap D)$ is s-dense in the family $F(U)$, then F is 1-irresolute.

The proof is similar to that in the previous theorem.

From Theorems 2.1 and 2.2 we have for singlevalued maps following:

Theorem 2.3. Let Y be s-regular. If for a map $f : X \to Y$ there is a dense subset D of the space X such that $f|_D$ is irresolute and for every open $U \subset X$ we have $f(U) \cap \underline{f(U \cap D)}$ then f is irresolute.

3. Irresolute mappings and product spaces

For any map $F : X \times Y \to \varphi(Z)$ ($f : X \times Y \to Z$), points $x \in X$ and $y \in Y$ we define maps $F_x : Y \to \varphi(Z)$ and $F^y : X \to \varphi(Z)$ ($f_x : Y \to Z$ and $f^y : X \to Z$) by:

$$F_x = F|_{\{x\} \times Y} \quad (f_x = f|_{\{x\} \times Y})$$

and

$$F^y = F|_{X \times \{y\}} \quad (f^y = f|_{X \times \{y\}}).$$

The main aim of this part is to show two following properties of mappings on product spaces.

Theorem 3.1. Let X be a Baire space, Y a locally second countable space and let Z be s-regular. If a multivalued map $F : X \times Y \to \mathcal{C}(Z)$ is such that:

1) F_x is 1-irresolute for every $x \in X \setminus M$ where M is of the first category,

2) F^y is irresolute for every $y \in Y$,

then F is 1-irresolute.

Theorem 3.2. Let X be a Baire space, Y a locally second countable space and let Z be s-normal. If a multivalued map $F : X \times Y \to \mathcal{C}(Z)$ is such that:

1) F_x is u-irresolute for every $x \in X \setminus M$ where M is of the first category,

2) F^y is irresolute for every $y \in Y$,

then F is u-irresolute.

Proof of the Theorem 3.1: Assume that F is not l-irresolute. By Proposition 1.4 there is a point $(x_0,y_0) \in X \times Y$ and its neighbourhood $U_0 \times V_0 \subset X \times Y$ and a semi-open set $G \subset Z$ satisfying $G \cap F(x_0,y_0) \neq \emptyset$, such that for every not-empty open sets $U' \subset U_0$, $V' \subset V_0$ there is a point $(x_1,y_1) \in U' \times V'$ such that:

$$F(x_1,y_1) \subset Z \setminus G.$$

Because Y is locally second countable there is a neighbourhood D of y_0 which has a countable base. Obviously the set $V = D \cap V_0$ has the same property. Let $\{V_n\}_{n=1}^{\infty}$ be a countable base of the set V. Let $z \in F(x_0,y_0) \cap G$, by s-regularity of the space Z there is a semi-open set $G_0 \subset Z$ such that:

$$z \in G_0 \subset \underline{G_0} \subset G.$$

F^y is l-irresolute at x_0, so we have:

$$W_1 = \mathrm{Int}(F^{y_0})^-(G_0) \cap U_0 \neq \emptyset.$$

Because X is a Baire space $W = W_1 \cap M \neq \emptyset$ is of the second category. For every natural n take:

$$A_n = \{x \in W : V_n \subset (F_x)^-(G_0)\}.$$

Evidently $\bigcup_{n=1}^{\infty} A_n \subset W$. Let $x \in W$, then $F^y(x) \cap G_0 \neq \emptyset$ so $F_x(y_0) \cap G \neq \emptyset$ and $y_0 \in (F_x)^-(G_0) \cap V$. Because F_x is l-irresolute we have:

$$\mathrm{Int}\left[(F_x)^-(G_0) \cap V\right] \neq \emptyset.$$

There is $n \in N$ such that $V_n \subset (F_x)^-(G_0)$, therefore $W = \bigcup_{n=1}^{\infty} A_n$. We will show that for every $n \in N$ the set A_n is nowhere dense what is contradiction to the fact that W is of the second category. Let $S \subset U$ be any not-empty open set, and let $n \in N$. Let us consider a set $S \times V_n$. There is $(x_1,y_1) \in S \times V_n$ such that

$$F(x_1,y_1) \subset Z \setminus G.$$

Let $G_1 = Z \setminus \underline{G_0}$, G_1 is semi-open and $F(x_1,y_1) \subset G_1$. Because F^{y_1} is u-irresolute $(F^{y_1})^+(G_1)$ is semi-open and $x_1 \in (F^{y_1})^+(G_1) \cap S$. So there is a not-empty open set $S_1 \subset S$ such that for every $x \in S_1$ we have $F(x,y_1) \subset G_1$. Therefore $F(x,y_1) \cap G_0 = \emptyset$ and because $y_1 \in (F_x)^-(G_0)$ we obtain $V_n \not\subset (F_x)^-(G_0)$ what mean that $x \notin A_n$, and consequently $S_1 \cap A_n = \emptyset$. This is the end of the proof.

The proof of Theorem 3.2 is similar to that in Theorem 3.1. As an immediate consequence of Theorems 3.1 and 3.2 we obtain:

Corollary 3.1. Let X be a Baire space, Y be a locally second countable space and let Z be s-regular and s-normal. If a multivalued map $F: X \times Y \to \mathcal{C}(Z)$ is such that for every $x \in X \setminus M$, where M is of the first category, and for every $y \in Y$ the maps F_x and F^y are irresolute, then F is irresolute.

By Proposition 1.5 we have:

Theorem 3.3. Let X,Y,Z satisfy the assumptions of Theorem 3.1. If $f : X \times Y \to Z$ is such that for every $x \in X \setminus M$ where M is of the first category, and for every $y \in Y$ the maps f_x and f^y are irresolute then f is irresolute.

REFERENCES

[1] S.G. Crossley, S.K. Hildebrand: Semi-topological properties, Fund. Math. 74 (1972) 233-254.

[2] J. Ewert, T. Lipski: Quasi-continuous multivalued mappings, Math. Slovaca 33 (1983) 69-74.

[3] N. Levine: Semi-open sets and semi-continuity in topological spaces. Amer. Math. Monthly 70 (1963) 69-74.

[4] S.N. Maheshwari, R. Prasad: On s-normal spaces, Bull. Math. Soc. Sci. Math. R.S. Roumanie 22(70) (1978) 27-29.

[5] S.N. Maheshwari, R. Prasad: On s-regular spaces, Glasnik Mat. 10 (30) (1975) 347-350.

[6] J. Okstoby: Miera i kategorija. Moskwa 1974.

[7] V. Popa: On characterizations of irresolute multifunctions, Czech. Math. J. (to appear).

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REPRESENTATION OF FUNCTIONS OF THE CLASS $\varphi(L)$ BY THE FABER-SCHAUDER SERIES ON $\langle 0,1\rangle^2$

1. Preliminaries

Let $\{r_m\}_{m=0}^{\infty}$ be the sequence of non-negative numbers $r_0 = 0$, $r_1 = 1$ and $r_m = r_{2^i+k} = \dfrac{2k-1}{2^{i+1}}$ with $1 \leqslant k \leqslant 2^i$, $i = 0,1,2,\ldots$. Consider the Faber-Schauder system $\{\varphi_m\}_{m=0}^{\infty}$ of non-negative continuous functions defined on the interval $\langle 0,1\rangle$ by the identities

$$\varphi_0(t) = 1,\ \varphi_1(t) = t \text{ and, if } m = 2^i + k,\ (1 \leqslant k \leqslant 2^i),$$

$$\varphi_m(t) = \begin{cases} 0 & \text{for } t \in \langle 0, \frac{k-1}{2^i}\rangle \cup \langle \frac{k}{2^i}, 1\rangle, \\ 1 & \text{for } t = r_m, \\ \text{linear on } \langle \frac{k-1}{2^i}, r_m\rangle, \langle r_m, \frac{k}{2^i}\rangle, & \text{respectively.} \end{cases}$$

It is the basis of the space of all real-valued functions continuous on $\langle 0,1\rangle$ (see [1]).

Let us introduce the functions φ_{mn} with values $\varphi_{mn}(x_1,x_2) = \varphi_m(x_1)\varphi_n(x_2)$ for $x_1,x_2 \in \langle 0,1\rangle$ and arrange them as follows $\varphi_{00},\varphi_{10},\varphi_{01},\varphi_{11},\ldots,\varphi_{i-1,i-1},\varphi_{i0},\varphi_{i1},\ldots,\varphi_{i,i-1},\varphi_{0i},\varphi_{1i},\ldots,\varphi_{ii},\ldots$ If the index ij precedes the index mn in this ordering, we will write $(i,j) < (m,n)$.

Let E be the square $\langle 0,1\rangle^2$, and μ the Lebesgue measure on E. Suppose that $\varphi: R \to R^+$ is an even non-negative function on $(-\infty,\infty)$, non-decreasing on $\langle 0,\infty)$ such that $\lim\limits_{t\to\infty} \varphi(t) = \infty$.

Denote by $\varphi(L)$ the set of all μ-measurable functions $f:E \to R$ such that

$$\int_E \varphi(f(x))d\mu < \infty.$$

Write $C(E)$ for the subset of $\varphi(L)$, consisting of all continuous functions on E. If $f,g \in \varphi(L)$, the non-negative number

$$d_\varphi(f,g) = \int_E \varphi(f(x) - g(x))d\mu$$

is called the φ-distance of f and g.

Given any function f of the set $\varphi(L)$, let us put

$$(1.1)\qquad \begin{cases} a_{00} = a_{00}[f] = f(r_0,r_0), \\ a_{mn} = a_{mn}[f] = f(r_m,r_n) - \sum\limits_{(i,j)<(m,n)} a_{ij}\varphi_{ij}(r_m,r_n) \end{cases}$$

and

$$(1.2)\qquad S_{mn}[x;f] = \sum_{(i,j)\leqslant(m,n)} a_{ij}\varphi_{ij}(x), \quad x = (x_1,x_2).$$

It can easily be observed that $S_{mn}[(r_i,r_j);f] = f(r_i,r_j)$, if $(i,j) \leqslant (m,n)$. In the case of $f \in C(E)$

$$(1.3)\qquad \lim_{m,n\to\infty} S_{mn}[x;f] = f(x) \quad \text{uniformly in } x \in E \quad \text{(see [2]).}$$

Definition 1.1. We say that a function φ satisfies the ω-condition, if there are constants $t_0 > 0$ and $K > 0$ such that

$$\varphi(t + 1) \leqslant K\varphi(t) \quad \text{for} \quad t \geqslant t_0.$$

Definition 1.2. We say that a function $f \in \varphi(L)$ is represented, in the sense of φ-convergence, by a series $\sum\limits_{i,j=0} d_{ij}\varphi_{ij}$ in the Faber-Schauder system $\{\varphi_{mn}\}_{m,n=0}$ (or shortly, by the Faber-Schauder series) if

$$\lim_{m,n\to\infty} \int_E \varphi\Big(f(x) - \sum_{(i,j)\leqslant(m,n)} d_{ij}\varphi_{ij}(x)\Big)d\mu = 0.$$

In this paper we give necessary and sufficient conditions for all functions $f \in \varphi(L)$ [resp. $f \in C(E)$] to be representable (in the sense of φ-convergence) by series in the Faber-Schauder system. Our results can be treated as two-dimensional analogues of those obtained by P.L. Ul'yanov in [5], [6].

2. Auxiliary lemmas

Lemma 2.1 (see Lemma 3.1 in [5]). If a function φ satisfies the ω-condition, then there are constants $A > 2$ and $B > 0$ such that

$$\varphi(t+1) \leqslant A\,\varphi(t) + B \quad \text{for} \quad t \geqslant 0,$$

$$\varphi(t+n) \leqslant A^n\{\varphi(t) + B\} \quad \text{for} \quad t \geqslant 0 \quad \text{and} \quad n = 1,2,\ldots$$

Lemma 2.2 (see [4]). If $\varphi(t+0) = \varphi(t)$ for all $t \geqslant 0$ and a function φ satisfies the ω-condition, then for any function $f \in \varphi(L)$ and for any real number $\varepsilon > 0$ there is a $\delta = \delta(f,\varepsilon) > 0$ such that

$$\int_E \varphi(|f(x)| + \delta)\,d\mu < \varepsilon + \int_E \varphi(f(x))\,d\mu.$$

Lemma 2.3. Suppose that the inequalities

(a) $d_\varphi(\tau,g) < \eta$,

(b) $\int_E \varphi(g(x))d\mu \leqslant 2\int_E \varphi(\tau(x))d\mu$

hold for the functions $\tau, g \in \varphi(L)$ and for some $\eta > 0$. If h is a μ-measurable function on E such that

$$h(x)\cdot g(x) \geqslant 0 \quad \text{and} \quad |h(x)| \leqslant |g(x)| \quad \text{for all } x \in E,$$

then

$$d_\varphi(\tau,h) \leqslant \eta + 3\int_E \varphi(\tau(x))\,d\mu.$$

Proof. Put

$$E_1 = \{x \in E:\ \tau(x) \geqslant 0 \text{ and } h(x) \geqslant 0\},$$
$$E_2 = \{x \in E:\ \tau(x) \geqslant 0 \text{ and } h(x) < 0\},$$
$$E_3 = \{x \in E:\ \tau(x) < 0 \text{ and } h(x) > 0\},$$
$$E_4 = \{x \in E:\ \tau(x) < 0 \text{ and } h(x) \leqslant 0\}.$$

The sets E_i, $1 \leqslant i \leqslant 4$, being pairwise disjoit and $E = \bigcup_{i=1}^{4} E_i$. We have

$$\int_{E_1} \varphi(\tau(x) - h(x))d\mu \leqslant \int_{E_1'} \varphi(\tau(x) - h(x))d\mu +$$

$$+ \int_{E_1''} \varphi(\tau(x) - h(x))d\mu \leqslant \int_{E_1} \varphi(\tau(x))d\mu + \int_{E_1} \varphi(g(x))d\mu,$$

where $E_1' = E_1 \cap \{x: 0 \leqslant h(x) \leqslant \tau(x)\}$, $E_1'' = E_1 \cap \{x: 0 \leqslant \tau(x) < h(x)\}$ and, similarly,

$$\int_{E_4} \varphi(\tau(x) - h(x))d\mu \leqslant \int_{E_4} \varphi(\tau(x))d\mu + \int_{E_4} \varphi(g(x))d\mu,$$

$$\int_{E_i} \varphi(\tau(x) - h(x))d\mu \leqslant \int_{E_i} \varphi(\tau(x) - g(x))d\mu, \quad i=2,3.$$

Hence

$$d_\varphi(\tau,h) = \sum_{i=1}^{4} \int_{E_i} \varphi(\tau(x) - h(x))d\mu \leqslant \eta + 3\int_{E} \varphi(\tau(x))d\mu,$$

which proves Lemma 2.3.

Lemma 2.4. Let N be an arbitrary natural number, μ, ν integers such that $\mu > \nu > N$ and l_{ij}, $1 \leqslant i,j \leqslant 2^{\nu}$, real numbers. Write

$$I_i = \left\langle \frac{i-1}{2^\nu} + \frac{1}{2^\mu} , \frac{i}{2^\nu} - \frac{1}{2^\mu} \right\rangle, \quad J_i' = \left\langle \frac{i-1}{2^\nu} , \frac{i-1}{2^\nu} + \frac{1}{2^\mu} \right\rangle ,$$

$$J_i'' = \left\langle \frac{i}{2^\nu} - \frac{1}{2^\mu} , \frac{i}{2^\nu} \right\rangle .$$

Let $\psi \in C(E)$ be the function of $x = (x_1, x_2)$ defined by

$$(2.1)\quad \psi(x) = \begin{cases} l_{ij} & \text{for } x \in I_i \times I_j, \\ (2^\mu x_1 - 2^{\mu-\nu}(i-1))l_{ij} & \text{for } x \in J_i' \times I_j, \\ (-2^\mu x_1 + 2^{\mu-\nu} i)l_{ij} & \text{for } x \in J_i'' \times I_j, \\ (2^\mu x_2 - 2^{\mu-\nu}(j-1))l_{ij} & \text{for } x \in I_i \times J_j', \\ (-2^\mu x_2 + 2^{\mu-\nu} j)l_{ij} & \text{for } x \in I_i \times J_j'', \\ (2^{2\mu}x_1x_2 - 2^{2\mu-\nu}(j-1)x_1 - 2^{2\mu-\nu}(i-1)x_2 + & \\ \qquad +2^{2\mu-2\nu}(i-1)(j-1))l_{ij} & \text{for } x \in J_i' \times J_j', \\ (-2^{2\mu}x_1x_2 + 2^{2\mu-\nu}jx_1 + 2^{2\mu-\nu}(i-1)x_2 - 2^{2\mu-2\nu}(i-1)j)l_{ij} & \\ & \text{for } x \in J_i' \times J_j'', \\ (2^{2\mu}x_1x_2 - 2^{2\mu-\nu}jx_1 - 2^{2\mu-\nu}ix_2 + 2^{2\mu-2\nu}ij)l_{ij} & \\ & \text{for } x \in J_i'' \times J_j'', \\ (-2^{2\mu}x_1x_2 + 2^{2\mu-\nu}(j-1)x_1 + 2^{2\mu-\nu}ix_2 - 2^{2\mu-2\nu}i(j-1))l_{ij} & \\ \text{for } x \in J_i''\times J_j', \text{ and } 1 \leqslant i,j \leqslant 2^\nu \text{ in all cases above} & \\ 0 \text{ for } x \in (\langle 0,1\rangle \times \frac{j}{2^\nu} \cup \frac{i}{2^\nu} \times \langle 0,1\rangle) \text{ and } 0 \leqslant i,j \leqslant 2^\nu . & \end{cases}$$

Then ψ is a polynomial in the Faber-Schauder system

$$(2.2)\quad \psi(x) = \sum_{i,j=2^\nu+1}^{2^\mu} c_{ij}\varphi_{ij}(x)$$

for which the sums $S_{mn}[x;\psi]$ $(m,n \geqslant 0)$ defined by (1.2) satisfy the inequalities

(2.3) $S_{mn}[x;\psi]\cdot\psi(x) \geqslant 0$ and $|S_{mn}[x;\psi]| \leqslant |\psi(x)|$ for all $x \in E$.

Proof. By (1.3) we have

$$\psi(x) = \lim_{m,n\to\infty} \sum_{(i,j)\leqslant(m,n)} a_{ij}[\psi]\,\varphi_{ij}(x), \quad (x \in E).$$

Applying (1.1) we obtain:

$$a_{oo}[\psi] = \psi(0,0),$$
$$a_{1o}[\psi] = \psi(1,0) - \psi(0,0),$$
$$a_{o1}[\psi] = \psi(0,1) - \psi(0,0),$$
$$a_{11}[\psi] = \psi(1,1) - \psi(1,0) - \psi(0,1) + \psi(0,0),$$

and, if $m = 2^i + k$, $n = 2^j + l$, where $1 \leqslant k \leqslant 2^i$, $1 \leqslant l \leqslant 2^j$, $i,j = 0,1,\ldots$

$$a_{mo}[\psi] = \psi(r_m,0) - \tfrac{1}{2}\psi\left(\tfrac{k-1}{2^i},0\right) - \tfrac{1}{2}\psi\left(\tfrac{k}{2^i},0\right),$$
$$a_{on}[\psi] = \psi(0,r_n) - \tfrac{1}{2}\psi\left(0,\tfrac{l-1}{2^j}\right) - \tfrac{1}{2}\psi\left(0,\tfrac{l}{2^j}\right),$$

$$a_{m1}[\psi] = \psi(r_m,1) - \tfrac{1}{2}\psi\left(\tfrac{k-1}{2^i},1\right) - \tfrac{1}{2}\psi\left(\tfrac{k}{2^i},1\right) - \psi(r_m,0) + \tfrac{1}{2}\psi\left(\tfrac{k-1}{2^i},0\right) + \tfrac{1}{2}\psi\left(\tfrac{k}{2^i},0\right),$$

$$a_{1n}[\psi] = \psi(1,r_n) - \tfrac{1}{2}\psi\left(1,\tfrac{l-1}{2^j}\right) - \tfrac{1}{2}\psi\left(1,\tfrac{l}{2^j}\right) - \psi(0,r_n) + \tfrac{1}{2}\psi\left(0,\tfrac{l-1}{2^j}\right) + \tfrac{1}{2}\psi\left(0,\tfrac{l}{2^j}\right),$$

$$a_{mn}[\psi] = \psi(r_m,r_n) - \tfrac{1}{2}\psi\left(r_m,\tfrac{l-1}{2^j}\right) - \tfrac{1}{2}\psi\left(r_m,\tfrac{l}{2^j}\right) - \tfrac{1}{2}\psi\left(\tfrac{k-1}{2^i},r_n\right) - \tfrac{1}{2}\psi\left(\tfrac{k}{2^i},r_n\right) + \tfrac{1}{4}\psi\left(\tfrac{k-1}{2^i},\tfrac{l-1}{2^j}\right) + \tfrac{1}{4}\psi\left(\tfrac{k-1}{2^i},\tfrac{l}{2^j}\right) + \tfrac{1}{4}\psi\left(\tfrac{k}{2^i},\tfrac{l-1}{2^j}\right) + \tfrac{1}{4}\psi\left(\tfrac{k}{2^i},\tfrac{l}{2^j}\right).$$

Thus $a_{ij}[\psi] = 0$ if $(i,j) \leqslant (2^\nu, 2^\nu+1)$ or $2^\nu+2 \leqslant i \leqslant 2^\mu$ and $0 \leqslant j \leqslant 2^\nu$ or $0 \leqslant i \leqslant 2^\nu$ and $2^\nu+2 \leqslant j \leqslant 2^\mu$ or $(i,j) > (2^\mu,2^\mu)$. Hence ψ can be written in the form (2.2).

Further, $S_{mn}[x;\psi] = 0$, if $(m,n) < (2^{\nu}+1, 2^{\nu}+1)$, and $S_{mn}[x;\psi] = \psi(x)$, if $(m,n) \geqslant (2^{\mu}, 2^{\mu})$ for every $x \in E$. Let

$$E^{ij}_{kl} = \left\langle \frac{i-1}{2^{\nu}} + \frac{k-1}{2^{\mu}}, \frac{i-1}{2^{\nu}} + \frac{k}{2^{\mu}} \right\rangle \times \left\langle \frac{j-1}{2^{\nu}} + \frac{l-1}{2^{\mu}}, \frac{j-1}{2^{\nu}} + \frac{l}{2^{\mu}} \right\rangle$$

for $1 \leqslant k,l \leqslant 2^{\mu-\nu}$ and $1 \leqslant i,j \leqslant 2^{\nu}$; then

$$\bigcup_{k,l=1}^{2^{\mu-\nu}} \bigcup_{i,j=1}^{2^{\nu}} E^{ij}_{kl} = E.$$

Let $x \in E^{ij}_{11}$ $(i \leqslant j)$ be arbitrary chosen. Then

$$S_{mn}[x;\psi] = \begin{cases} 0 & \text{if } (m,n) < (2^{\nu}+i, 2^{\nu}+j), \\ l_{ij}\varphi_{2^{\nu}+i,2^{\nu}+j}(x) & \text{if } (2^{\nu}+i, 2^{\nu}+j) \leqslant (m,n) < \\ & < (2^{\nu+1}+(i-1)\cdot 2+1, 2^{\nu}+j), \\ l_{ij}\left(\varphi_{2^{\nu}+i,2^{\nu}+j}(x) + \frac{1}{2}\varphi_{2^{\nu+1}+(i-1)\cdot 2+1,2^{\nu}+j}(x)\right) & \\ \quad \text{if } (2^{\nu+1}+(i-1)\cdot 2+1, 2^{\nu}+j) \leqslant (m,n) < & \\ \quad < (2^{\nu}+i, 2^{\nu+1}+(j-1)\cdot 2+1), & \\ l_{ij}\left(\varphi_{2^{\nu}+i,2^{\nu}+j}(x) + \frac{1}{2}\varphi_{2^{\nu+1}+(i-1)\cdot 2+1,2^{\nu}+j}(x) + \right. & \\ \quad \left. + \frac{1}{2}\varphi_{2^{\nu}+i,2^{\nu+1}+(j-1)\cdot 2+1}(x)\right) & \\ \quad \text{if } (2^{\nu}+i, 2^{\nu+1}+(j-1)\cdot 2+1) \leqslant (m,n) < & \\ \quad <(2^{\nu+1}+(i-1)\cdot 2+1, 2^{\nu+1}+(j-1)\cdot 2+1), & \\ l_{ij}\left(\varphi_{2^{\nu}+i,2^{\nu}+j}(x) + \frac{1}{2}\varphi_{2^{\nu+1}+(i-1)\cdot 2+1,2^{\nu}+j}(x) + \right. & \\ \quad + \frac{1}{2}\varphi_{2^{\nu}+i,2^{\nu+1}+(j-1)\cdot 2+1}(x) + & \end{cases}$$

$$+\frac{1}{4}\varphi_{2^{\nu+1}+(i-1)\cdot 2+1,\,2^{\nu+1}+(j-1)\cdot 2+1}(x)\Big) \text{ if}$$

$$(2^{\nu+1}+(i-1)\cdot 2+1,\,2^{\nu+1}+(j-1)\cdot 2+1) \leqslant (m,n) <$$

$$< (2^{\nu+2}+(i-1)\cdot 2^{2}+1,\,2^{\nu}+j),$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$l_{ij}\Big(\varphi_{2^{\nu}+i,\,2^{\nu}+j}(x) + \ldots$$

$$+\frac{1}{4}\varphi_{2^{\mu-1}+(i-1)\cdot 2^{\mu-\nu-1}+1,\,2^{\mu-1}+(j-1)\cdot 2^{\mu-\nu-1}+1}(x)\Big) =$$

$$= \psi(x) \text{ if } (2^{\mu-1}+(i-1)\cdot 2^{\mu-\nu-1}+1,\,2^{\mu-1}+(j-1)\cdot 2^{\mu-\nu-1}+1) \leqslant$$

$$\leqslant (m,n) < (2^{\mu},2^{\mu}).$$

This implies the validity of (2.3) for all $x \in E_{11}^{ij}$ $(i \leqslant j)$. The analogical computation gives (2.3) for all sets E_{kl}^{ij} $(1 \leqslant k,l \leqslant 2^{\mu-\nu}$ and $1 \leqslant i,j \leqslant 2^{\nu})$. This proves Lemma 2.4.

Lemma 2.5. If $\varphi(0) = 0$, $\varphi(t+0) = \varphi(t)$ for $t \geqslant 0$ and if a function φ satisfies the ω-condition, then for any function $f \in \varphi(L)$, any natural number N and every real number $\eta > 0$ there are integers $\mu > \nu > N$, real numbers l_{ij} $(1 \leqslant i,j \leqslant 2^{\nu})$ and a function $\psi \in C(E)$ of the form (2.1) such that

$$d_{\varphi}(f,\psi) < \eta, \quad \int_{E} \varphi(\psi(x))d\mu \leqslant 2\int_{E} \varphi(f(x))d\mu.$$

Proof. If $\int_{E} \varphi(f(x))d\mu = 0$, then the assertion of the lemma is obvious, since it is enough to take $\psi(x) \equiv 0$. Consequently we suppose that

$$\alpha \equiv \int_{E} \varphi(f(x))d\mu > 0.$$

Let $\varepsilon = \min\left(\frac{\alpha}{3}, \frac{\eta}{3}\right)$. As in the proof of Lemma 2.2 in [3] we can find a number $\delta > 0$ and a compact set $P \subset E$ such that f is a continuous function on P, $\mu(E \setminus P) < \delta$ and

$$\int_F \varphi(f(x))d\mu < \frac{\varepsilon}{3} \quad \text{if} \quad F\subset E \text{ and } \mu F<\delta. \tag{2.3}$$

We can choose $\delta_1 > 0$ so that

$$\delta_1 < \min\left(\frac{\varepsilon}{3A^mB}, \frac{\varepsilon}{\varphi(m)}\right) \tag{2.4}$$

and

$$\int_{F_1} \varphi(f(x))d\mu < \frac{\varepsilon}{3A^m} \quad \text{if} \quad F_1\subset E \quad \text{and} \quad \mu F_1<\delta_1, \tag{2.5}$$

where $\sup_{x\in P}|f(x)| = M < 1+[M] = m$. After this we find a compact set V (see the proof of Lemma 2.2 in [3]) such that $P\subset V\subset E$ and $\mu(V\setminus P)<\delta_1$ and we choose a function $g\in C(E)$ as follows

$$\begin{cases} g(x) = f(x) & \text{for} \quad x\in P, \\ g(x) = 0 & \text{for} \quad x\in E\setminus V, \\ g \text{ attains its supremum and infimum on the set } P. \end{cases} \tag{2.6}$$

By (2.6) and (2.3), we have

$$\int_P \varphi(f(x)-g(x))d\mu = 0, \qquad \int_{E\setminus V} \varphi(f(x)-g(x))d\mu < \frac{\varepsilon}{3}$$

and by Lemma 2.1, (2.4) and (2.5)

$$\int_{V\setminus P} \varphi(f(x)-g(x))d\mu \leqslant \int_{V\setminus P} \varphi(|f(x)|+m)d\mu \leqslant$$

$$\leqslant A^m\left(\frac{\varepsilon}{3A^m} + B\,\frac{\varepsilon}{3A^mB}\right) = \frac{2}{3}\varepsilon.$$

Thus $d_\varphi(f,g)<\varepsilon$. By (2.6) and (2.4) we obtain

$$\int_E \varphi(g(x))d\mu \leqslant \int_P \varphi(f(x))d\mu + \varphi(m)\mu(V\setminus P) < \int_E \varphi(f(x))d\mu + \varepsilon. \tag{2.7}$$

By Lemma 2.2, we can find $\gamma\in(0,1)$ such that

$$(2.8)\qquad \int_E \varphi(|f(x)-g(x)|+\gamma)d\mu<\varepsilon+\int_E \varphi(f(x)-g(x))d\mu$$

$$(2.9)\qquad \int_E \varphi(|g(x)|+\gamma)d\mu<\varepsilon+\int_E \varphi(g(x))d\mu.$$

Since $g\in C(E)$, for a sufficiently large integer $\nu>N$, the function

$$g_1(x)=\begin{cases} g\left(\frac{i-1}{2^\nu},\frac{j-1}{2^\nu}\right) & \text{for } x\in\left\langle\frac{i-1}{2^\nu},\frac{i}{2^\nu}\right)\times\left\langle\frac{j-1}{2^\nu},\frac{j}{2^\nu}\right) \\ & (1\leqslant i,j\leqslant 2^\nu), \\ 0 & \text{for } x\in(\langle 0,1\rangle\times 1\cup 1\times\langle 0,1\rangle), \end{cases}$$

satisfies the inequality

$$(2.10)\qquad |g_1(x)-g(x)|<\gamma \quad \text{for all} \quad x\in E.$$

We now choose a natural number $\mu>\nu$ so large that

$$(2.11)\quad 2^{\nu+2-\mu}\varphi(m)<\varepsilon,\ \int_{F_2}\varphi(|f(x)|+m)d\mu<\varepsilon \ \text{ if } F_2\subset E,\ \mu F_2\leqslant 2^{\nu+2-\mu}.$$

We define sets I_i, J_i' and J_i'' $(1\leqslant i\leqslant 2^\nu)$ as in Lemma 2.4 and write

$$l_{ij}=g_1(x) \quad \text{for} \quad x\in I_i\times I_j \quad (1\leqslant i,j\leqslant 2^\nu).$$

Now, we construct a function $\psi\in C(E)$ of the form (2.1) and we put

$$E_1=\bigcup_{i,j=1}^{2^\nu} I_i\times I_j,\quad E_2=E\setminus E_1.$$

By (2.10), (2.8) and (2.11) we have

$$d_\varphi(f,\psi)\leqslant\int_{E_1}\varphi(f(x)-g_1(x))d\mu+\int_{E_2}\varphi(|f(x)|+m)d\mu\leqslant$$

$$\leqslant\int_{E_1}\varphi(|f(x)-g(x)|+\gamma)d\mu+\varepsilon<3\varepsilon\leqslant\eta.$$

By (2.10), (2.9), (2.7) and (2.11) we obtain

$$\int_E \varphi(\psi(x))d\mu = \int_{E_1} \varphi(g_1(x))d\mu + \int_{E_2} \varphi(\psi(x))d\mu \leqslant$$

$$\leqslant \int_E \varphi(|g_1(x) - g(x)| + |g(x)|)d\mu +$$

$$+ \varphi(m)\mu E_2 < 3\varepsilon + \int_E \varphi(f(x))d\mu \leqslant \alpha + \int_E \varphi(f(x))d\mu =$$

$$= 2\int_E \varphi(f(x))d\mu.$$

This proves Lemma 2.5.

3. Main results

Theorem 3.1. If $\varphi(t+0) = \varphi(t)$ for $t \geqslant 0$, then any function $f \in \varphi(L)$ can be represented by a series in the Faber-Schauder system in the sense of φ-convergence if and only if $\varphi(0) = 0$ and φ satisfies the ω-condition.

Proof. Sufficiency. We shall show that for any function $f \in \varphi(L)$ and any natural number N there is a series

(3.1) $$\sum_{(N,N)\leqslant(i,j)}^{\infty} d_{ij}\varphi_{ij}$$

such that

$$\lim_{m,n\to\infty} \int_E \varphi\Big(f(x) - \sum_{(N,N)\leqslant(i,j)\leqslant(m,n)} d_{ij}\varphi_{ij}(x)\Big)d\mu = 0.$$

Put $\eta_k = \frac{1}{2^k}$ $(k = 1,2,\ldots)$. Let N be an arbitrary natural number. Following Lemma 2.5 we construct a function $\psi_1 \in C(E)$ of the form (2.1) with $\mu_1 > \nu_1 > N$ so that

$$\int_E \varphi(f(x) - \psi_1(x))d\mu < \eta_1, \quad \int_E \varphi(\psi_1(x))d\mu \leqslant 2\int_E \varphi(f(x))d\mu.$$

Further, according to Lemma 2.5, we construct by induction the natural numbers $\mu_1 < \nu_2 < \mu_2 < \dots < \nu_k < \mu_k < \dots$ $(k \geqslant 2)$ and functions $\psi_k \in C(E)$ such that

$$\int_E \varphi\Big(f(x) - \sum_{p=1}^{k} \psi_p(x)\Big)d\mu < \eta_k \quad (k \geqslant 2) \tag{4.2}$$

and

$$\int_E \varphi(\psi_k(x))d\mu \leqslant 2 \int_E \varphi\Big(f(x) - \sum_{p=1}^{k-1} \psi_p(x)\Big)d\mu \quad (k \geqslant 2). \tag{3.3}$$

By Lemma 2.4 we have

$$\psi_k(x) = \sum_{i,j=2^{\nu_k}+1}^{2^{\mu_k}} c_{ij}\varphi_{ij}(x) \quad (k \geqslant 1)$$

and for $m,n = 0,1,2,\dots$ we have the inequalities

$$S_{mn}[x;\psi_k]\cdot\psi_k(x) \geqslant 0, \quad \big|S_{mn}[x;\psi_k]\big| \leqslant \big|\psi_k(x)\big| \quad \text{for all } x \in E. \tag{3.4}$$

Now we take the series $\sum_{k=1}^{\infty} \psi_k$, which can be written in the form (3.1).

Let $m,n > 2^{\nu_2}$ be arbitrary natural numbers. Then there is $k \geqslant 2$ so that $2^{\nu_k}+1 \leqslant \max(m,n) \leqslant 2^{\nu_{k+1}}$. In this case

$$\sum_{(N,N)\leqslant(i,j)\leqslant(m,n)} d_{ij}\varphi_{ij}(x) = \sum_{p=1}^{k-1} \psi_p(x) + S_{mn}[x;\psi_k].$$

By (3.2), (3.4), (3.3) and Lemma 2.3 we have

$$\int_E \varphi\Big(f(x) - \sum_{(N,N)\leqslant(i,j)\leqslant(m,n)} d_{ij}\varphi_{ij}(x)\Big)d\mu =$$

$$= \int_E \varphi\Big(f(x) - \sum_{p=1}^{k-1} \psi_p(x) - S_{mn}[x;\psi_k]\Big)\, d\mu \leqslant$$

$$\leqslant \eta_k + 3\int_E \varphi f(x) - \sum_{p=1}^{k-1} \psi_p(x))d\mu \leqslant$$

$$\leqslant \eta_k + 3\eta_{k-1} \longrightarrow 0,$$

because $k \to \infty$ as $m,n \to \infty$.

Necessity follows from Theorem 4.2 in [3].

Theorem 3.2. An arbitrary function $f \in C(E)$ can be represented in the sense of φ-convergence by a series in the Faber-Schauder system if and only if $\varphi(0+) = 0$.

Proof. Sufficiency. From (1.3) it follows at once that

$$\lim_{m,n\to\infty} \int_E \varphi(f(x) - S_{mn}[x;f])d\mu = 0.$$

Necessity. If $\varphi(0+) = \alpha > 0$, then the function $f(x) = e^{x_1+x_2}$, satisfies the inequality

$$\int_E \varphi(f(x) - p(x))d\mu \geqslant \alpha$$

for any polynomial p in the Faber-Schauder system.

REFERENCES

[1] Z. Ciesielski: On Haar functions and on the Schauder basis of the space $C\langle 0,1\rangle$, Bull. Acad. Polon. Sci. Sér. Sci. Math., Astronom., Phys. 7 (1959) 227-232.

[2] H.W. Ellis, D.G. Kuchner: On Schauder bases for spaces of continuous functions, Canadian Math. Bull. 3, No 2 (1960) 173-184.

[3] D. Stachowiak-Gniżka: The generalized Wierstrass theorem in the class $\varphi(L)$, Funct. Approximatio Comment. Math. 4 (1976) 7-17.

[4] D. Stachowiak-Gniłka: Remarks on topological properties of the class φ(L), Funct. Approximatio Comment. Math. 5 (1977) 32-38.

[5] П.Л. Ульянов: Представление функций рядами и классы φ(L), Usp. Mat. Nauk 27, 2 (164) (1972) 3-52.

[6] П.Л. Ульянов: Представление функций класса φ(L) рядами, Trudy Mat. Inst. im. V.A.Steklova 112 (1971) 372-384, 372-384.

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ω-SPECIAL FIELDS IN THE DEFORMATION ALGEBRA

In this paper we introduce the concept of ω-special field in the deformation algebra of two linear connections (Section 1). In Section 2 we study those fields in the deformation algebra associated to the natural connections on Lie groups and in Section 3 we obtain a characterization of the hypersurfaces of the Euclidean space. In Section 4 we give a characterisation of Riemann two-dimensional spaces with constant curvature and in Section 5 we study the lift of the ω-special fields to the bundle of infinitely near points associated to a differentiable manifold.

1. Introduction

Let M be a manifold of dimension n and let $\mathcal{F}(M)$ be the algebra of all differentiable functions on M. We denote by $\mathcal{T}^r_s(M)$ the $\mathcal{F}(M)$-module of the tensor fields of type (r,s). In particular, for $\mathcal{T}^1_0(M)$, resp. $\mathcal{T}^0_1(M)$ we shall use also the notation $\mathfrak{X}(M)$, resp. $\Lambda^1(M)$. By $\delta \in \mathcal{T}^1_1(M)$ we shall denote the Kronecker tensor. If $T \in \mathcal{T}^r_s(M)$, then we denote by $C^p_q T$ the tensor field C^p_q-contracted of the tensor field T.

Let $D \in \mathcal{T}^1_2(M)$. If one defines the product of two vector fields $X,Y \in \mathfrak{X}(M)$ by the formula:

$$X \circ Y = D(X,Y)$$

then the $\mathcal{F}(M)$-module $\mathfrak{X}(M)$ becomes an algebra over $\mathcal{F}(M)$ [6]. This algebra is called the associated algebra to D and it will be denoted by $\mathcal{U}(M,D)$.

2. ω-Special Fields

Let be $\omega \in \Lambda^1(M)$.

Definition 2.1. An element $X \in \mathfrak{X}(M)$ is called ω-special field in the algebra $\mathcal{U}(M,D)$ iff

$$D(X,Y) = \omega(X)Y + \omega(Y)X \quad \text{for} \quad Y \in \mathfrak{X}(M). \tag{2.1}$$

Observation 2.2. If $\omega = 0$, then (2.1) shows that X is a special field [4].

Proposition 2.3. All the elements of $\mathcal{U}(M,D)$ are ω-special fields, where

$$D = \omega \otimes \delta + \delta \otimes \omega. \tag{2.2}$$

Proof. One must use relation (2.1).

Corollary 2.4. Let ∇ be a linear symmetric connection on a manifold M and let be $\bar{\nabla} = \nabla + D$. All the elements of $\mathcal{U}(M,D)$ are ω-special fields iff the connections ∇ and $\bar{\nabla}$ are projectively related in the classical sense,

$$\bar{\nabla}_X Y = \nabla_X Y + \omega(X)Y + \omega(Y)X, \quad \text{for} \quad X,Y \in \mathfrak{X}(M).$$

Remark 2. Let ∇ and $\bar{\nabla}$ be two symmetric connections on M. All the elements of $\mathcal{U}(M, \bar{\nabla} - \nabla)$ are ω-special fields iff every submanifold of M has the same second fundamental form with respect to ∇ as it does with respect to $\bar{\nabla}$.

Proposition 2.5. Let $X \in \mathcal{U}(M,D)$ be an ω-special field. Then X is a proper field in the algebra $\mathcal{U}(M,D)$ its proper value being the function $2\omega(X)$.

Proof. X is on ω-special field so relation (2.1) takes place for every $Y \in \mathfrak{X}(M)$. In particular, for $Y = X$, from (2.1) we obtain

$$D(X,X) = 2\omega(X)X$$

which shows that X is a proper field in the algebra $\mathcal{U}(M,D)$ [4], its proper value being the function $2\omega(X)$.

Proposition 2.6. The set $S_\omega(M)$ of the ω-special fields in the algebra $\mathcal{U}(M,D)$ is an $\mathcal{F}(M)$-submodule of the module $\mathfrak{X}(M)$.

Proof. It is easy to see that for every $X,Y \in S_\omega(M)$ and for every $f \in \mathcal{F}(M)$ we have $X+Y \in S_\omega(M)$ and $f.X \in S_\omega(M)$.

Definition 2.7. Let X be an element of the set $S_\omega(M)$. If the field X does not vanish over M then X is called ω-special direction field. The trajectories of an ω-special direction field are called ω-special curves.

Observation 2.8. Let D^i_{jk}, ω_i and X^i be the components of D, respectively ω and X in a system of locally coordinates. Then we rewrite relation (2.1) like that:

$$(D^i_{jk} - \delta^i_j\omega_k - \delta^i_k\omega_j)X^j = 0. \tag{2.1'}$$

and by this we obtain the differential equations of the ω-special curves

$$(D^i_{jk} - \delta^i_j\omega_k - \delta^i_k\omega_j)\,\frac{dx^j}{dt} = 0. \tag{2.1''}$$

3. Special fields on Lie groups

Let G be a Lie group and L(G) its Lie algebra and let $\{E_1,E_2,\dots,E_n\}$ be a basis in L(G). We consider the linear connections $\bar{\nabla}$, $\overset{+}{\nabla}$, $\overset{\circ}{\nabla}$ defined by

$$\bar{\nabla}_{E_i} E_j = 0, \quad \overset{+}{\nabla}_{E_i} E_j = [E_i,E_j], \quad \overset{\circ}{\nabla}_{E_i} E_j = \frac{1}{2}[E_i,E_j] \tag{3.1}$$

for every $i,j \in \{1,2,\dots,n\}$.

It is obviously clear that the linear connections $\bar{\nabla}$, $\overset{+}{\nabla}$, $\overset{\circ}{\nabla}$ are left-invariant.

We consider the following deformation tensor fields

$$\bar{D} = \bar{\nabla} - \overset{\circ}{\nabla}, \quad \overset{+}{D} = \overset{+}{\nabla} - \overset{\circ}{\nabla}, \quad D = \overset{+}{\nabla} - \bar{\nabla}. \tag{3.2}$$

By (3.1) and (3.2), it follows:

$$D = 2\overset{+}{D} = -2\bar{D}, \quad D(E_i,E_j) = [E_i,E_j]. \tag{3.3}$$

Proposition 3.1. Let ω be a 1-form on the Lie group G and let $X \in \mathfrak{X}(G)$. The following assertions are equivalent:

(i) X is a ω-special field in the algebra $\mathcal{U}(G,D)$;

(ii) X is a $\frac{1}{2}\,\omega$-special field in the algebra $\mathcal{U}(G,\overset{+}{D})$;

(iii) X is a $-\frac{1}{2}\,\omega$-special field in the algebra $\mathcal{U}(G,\overset{-}{D})$.

Proof. Straightforward verification by using (3.3) and (2.1).

Proposition 3.2. Let us assume that the Lie group G is convex and let be $\omega = C_1^1 D$. The following assertions are equivalent:

(i) All the elements of the algebra $\mathcal{U}(G,D)$ are ω-special fields;

(ii) The algebra L(G) is abelian;

(iii) The Lie group G is abelian;

(iv) The left-invariant vector fields are coincident with the right-invariant vector fields;

(v) The adjoint representation of the Lie group is trivial:

$$Ad\,a = Id_{L(G)}, \quad (\forall)\ a \in G.$$

(vi) The adjoint representation of the Lie algebra L(G) is trivial

$$ad\,X = 0, \quad \text{for} \quad X \in L(G).$$

Proof. (i) $\Rightarrow$ (ii). All the elements of the algebra $\mathcal{U}(G,D)$ are ω-special fields, so it turns out that

$$D(X,Y) = \omega(X)Y + \omega(Y)X, \quad \text{for} \quad X,Y \in \mathfrak{X}(G). \tag{3.4}$$

In particular, from (3.4) we obtain

$$D(E_i,E_j) = \omega(E_i)E_j + \omega(E_j)E_i, \quad \text{for} \quad i,j \in \{1,2,\dots,n\}. \tag{3.5}$$

By (3.5) and (3.3) there results $D(E_i,E_j) = 0$ for $i,j \in \{1,\dots,n\}$ and so $[E_i,E_j] = 0$, i.e. the Lie algebra L(G) is abelian.

(ii) $\Rightarrow$ (i). By $[E_i,E_j] = 0$ there results $D = 0$, and this implies $\omega = C_1^1 D = 0$ and therefore any field $X \in \mathcal{U}(G,D)$ verifies relation (2.1).

(ii) $\Leftrightarrow$ (iii) $\Leftrightarrow$ (iv) $\Leftrightarrow$ (v) $\Leftrightarrow$ (vi) to see [1].

4. ω-special fields on hypersurfaces

Let M be a hypersurface in the Euclidean space E^{n+1}. Let g, b and h be the first, the second and respectively the third fundamental form of M. We assume that b is nondegenerated. Let $\nabla(I)$, $\nabla(II)$ and $\nabla(III)$ be the linear conncections associated to g, b and respectively h, and let be $D = \nabla(I) - \nabla(II)$, $D' = \nabla(II) - \nabla(III)$, $D'' = \nabla(I) - \nabla(III)$.

The following formulas are well-known [4]:

$$b(D(X,Y),Z) = b(D'(X,Y),Z) = 2b(D''(X,Y),Z) = \tag{4.1}$$

$$= -\frac{1}{2}(\nabla(I)_X b)(Y,Z).$$

Theorem 4.1. Let M be a hypersurface in the Euclidean space E^{n+1} $(n \geq 2)$. Let us assume that the second fundamental form of M is nondegenerated. Let $\omega = C_1^1 D$, where D is given by (4.1). The following assertions are equivalent:

(i) All the elements of the algebra $\mathcal{U}(M,D)$ are ω-special fields;

(ii) All the elements of the algebra $\mathcal{U}(M,D')$ are ω-special fields;

(iii) All the elements of the algebra $\mathcal{U}(M,D'')$ are $\frac{1}{2}\omega$-special fields;

(iv) $\nabla(I)_X b = 0$, $(\forall)\ X \in \mathfrak{X}(M)$;

(v) $\nabla(I)$-geodesics are $\nabla(II)$-geodesics;

(vi) $\nabla(II)$-geodesics are $\nabla(I)$-geodesics;

(vii) M is a spheric hypersurface.

Proof. (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii). We have by using (4.1)

$$D = D' = 2D'', \quad \omega = C_1^1 D = C_1^1 D' = 2C_1^1 D''.$$

(i) $\Rightarrow$ (iv). For every $X,Y \in \mathfrak{X}(M)$ we have

$$D(X,Y) = \omega(X)Y + \omega(Y)X \tag{4.2}$$

because all the elements of the algebra $\mathcal{U}(M,D)$ are ω-special fields.

Let b_{ij}, D^i_{jk} respectively $\omega_i = D^i_{ik}$ be the components of b, D respectively ω in a system of locally coordinates. Then one can rewrite the relation (4.1):

$$(4.1') \qquad D^i_{jk} = D'^i_{jk} = 2D''^i_{jk} = -\frac{1}{2}\, b^{is}\, \nabla(I)_s b_{jk}$$

and relation (4.2) becomes

$$(4.2') \qquad D^i_{jk} = \delta^i_j \omega_k + \delta^i_k \omega_j .$$

From (4.1') and (4.2') it results

$$(4.3) \qquad -\frac{1}{2}\, \delta^{is} \nabla(I)_s b_{jk} = \delta^i_j \omega_k + \delta^i_k \omega_j$$

We multiply (4.3) by b_{ir} and we add

$$(4.4) \qquad -\frac{1}{2}\, \nabla(I)_r b_{jk} = b_{jr}\omega_k + b_{kr}\omega_j .$$

Using Codazzi's equations

$$\nabla(I)_r b_{jk} = \nabla(I)_j b_{rk}$$

and by (4.4) we have

$$(4.5) \qquad b_{kr}\omega_j - b_{kj}\omega_r = 0 .$$

We multiply relation (4.5) by b^{kr} and we add; also we obtain

$$(n-1)\omega_j = 0 .$$

We have $\omega_j = 0$ because $n \geq 2$. Using this and relation (4.1') and (4.2') we obtain finally (iv).

(iv) $\Rightarrow$ (i). From (iv) and (4.1) we have

$$(4.6) \qquad b(D(X,Y),Z) = 0, \quad \text{for} \quad X,Y,Z \in \mathfrak{X}(M).$$

and because b is nondegenerated, from (4.6) it comes out that $D = 0$.

So relation (2.1) is verified by every field $X \in \mathfrak{X}(M)$.

(iv) $\Leftrightarrow$ (v) $\Leftrightarrow$ (vi) $\Leftrightarrow$ (vii). One should use [5] (Theorem D).

5. A property of Riemann two-dimensional spaces

Let $(\mathcal{M},g)$ be a Riemann space of dimension two with the property that Ricci´s tensor S is nondegenerated and let ∇ and $\bar{\nabla}$ be the linear connections associated to g, respectively S.

For every $X,Y \in \mathfrak{X}(\mathcal{M})$ we have relation

(5.1) $$\nabla_X Y - \nabla_Y X = [X,Y] = \bar{\nabla}_X Y - \bar{\nabla}_Y X$$

and

(5.2) $$\nabla_X g = \bar{\nabla}_X S = 0.$$

We have n = 2 and that implies the existence of a function $\rho \in \mathcal{F}(\mathcal{M})$ so that

(5.3) $$S = \rho g.$$

Because the rank of S is equal to two, we have $\rho(p) \neq 0$, for $p \in M$.

Let $D = \bar{\nabla} - \nabla$. Using (5.1), (5.2) and (5.3) we obtain:

(5.4) $$X(\ln|\rho|)\, g(Y,Z) + Y(\ln|\rho|) g(X,Z) - $$
$$- Z(\ln|\rho|) g(X,Y) = 2g(D(X,Y),Z)$$

for every $X,Y,Z \in \mathfrak{X}(M)$.

Theorem 5.1. Let $(\mathcal{M},g)$ be a convex Riemannian manifold of dimension two, with the Ricci tensor nondegenerated. Let $D = \bar{\nabla} - \nabla$, where ∇ and $\bar{\nabla}$ the linear symmetric connections associated to g and respectively to S are and let $\omega \in \Lambda^1(M)$. If all the elements of the algebra $\mathcal{U}(M,D)$ are ω-special fields, then $(\mathcal{M},g)$ is a space with constant curvature (which is not zero).

Proof. Because all the elements of the algebra $\mathcal{U}(\mathcal{M},D)$ are ω-special fields, we have

(5.5) $$D(X,Y) = \omega(X)Y + \omega(Y)X, \quad \text{for } X,Y \in \mathfrak{X}(M).$$

For X = Y it comes out from (5.5)

$$(5.5')\qquad D(X,X) = 2\omega(X)X, \quad \text{for} \quad X \in \mathfrak{X}(M).$$

If in (5.4) we take Y = X, we obtain

$$(5.6)\qquad 2X(\ln(\rho))g(X,Z) - Z(\ln(\rho))g(X,X) = $$
$$= 2g(D(X,X),Z), \quad \text{for} \quad X,Z \in \mathfrak{X}(M).$$

From (5.6) and (5.5′) it comes out that

$$(5.7)\qquad 2X(\ln(\rho))g(X,Z) - Z(\ln(\rho))g(X,X) = $$
$$= 4\omega(X)g(X,Z), \quad \text{for} \quad X,Z \in \mathfrak{X}(M).$$

In particular, taking in (5.7) Z = X we have

$$(5.8)\qquad X_p(\ln(\rho)) = 4\omega_p(X_p), \quad \text{for} \quad p \in M \quad \text{and} \quad X_p \in T_pM-\{0\}$$

where $T_p\mathcal{M}$ is the tangent space in the point $p \in \mathcal{M}$. Relation (5.7) and (5.8) imply

$$(5.9)\qquad \omega_p(Z_p)g_p(X_p,X_p) = \omega_p(X_p)g_p(X_p,Z_p)$$

for $p \in M$ and $X_p, Z_p \in T_pM-\{0\}$.

For every $p \in M$ and for every $Z_p \in T_pM-\{0\}$ there exists $X_p \in T_pM-\{0\}$ so that $g_p(X_p,Z_p) = 0$. Using (5.9) it results $\omega_p(Z_p) = 0$, for $p \in M$ and $Z_p \in T_mM-\{0\}$. So $\omega_p = 0$ and relying on (5.8) we obtain $\rho = \text{const}$.

Theorem 5.2. Let $(\mathcal{M},g)$ be a Riemannian convex manofild of dimension two with the Ricci tensor nondegenerated. We consider the deformation tensor field $D = \bar{\nabla} - \nabla \in \mathcal{T}^1_2(\mathcal{M})$, where ∇ and $\bar{\nabla}$ the linear symmetric connexions associated to g and respectively S and let the 1-form $\omega = C^1_1 D$.

The following assertions are equivalent:

(i) All the elements of the algebra $\mathcal{U}(\mathcal{M},D)$ are ω-special fields.

(ii) $(\mathcal{M},g)$ is a space with constant nonzero curvature.

Proof. (i) ⇒ (ii). This implication results by using Theorem 5.1.

(ii) $\Rightarrow$ (i) From $\rho = \text{const}$ it comes out $X(\ln(\rho)) = 0$ for $X \in \mathfrak{X}(\mathcal{M})$ and considering (5.4) we obtain $D = 0$ and so $\omega = C_1^1 D = 0$. It follows that every element of the algebra $\mathcal{U}(\mathcal{M}, D)$ verifies relation (2.1).

6. Prolongation of ω-special vector fields to bundles of infinity near points

In this section we study the lift of the ω-special fields to bundle of infinitely near points M^A associated to the variety M and to the local algebra A in the sense of A. Weil [7] and A. Morimoto [3].

Definition 6.1. An associative and commutative algebra over the field of the real numbers $\mathbb{R}$, with unit (denoted by 1), of finite dimension over $\mathbb{R}$ and admitting a unique maximal ideal m, so that A/m is one-dimensional over $\mathbb{R}$ and $m^{h+1} = (0)$ for some nonnegative integer h, is called a local algebra.

Identifying the field $\mathbb{R}$ with the subspace of A consisting of all scalar multiples of 1, A is the direct sum of the vector subspaces $\mathbb{R}$ and m. If $a \in A$ the scalar $a_0 \in R$ defined by $a \equiv a_0 (\text{mod } m)$ is called the finite part of a.

Let $\mathbb{R}[p] = \mathbb{R}[[X_1, \ldots, X_p]]$ be the algebra of all formal power series in the indeterminates $X_1, \ldots, X_p$, let m_p be the maximal ideal of $\mathbb{R}[p]$ consisting of all formal power series without constant terms and let a be an ideal of $\mathbb{R}[p]$ so that $\dim \mathbb{R}[p]/a < \infty$. Then $A = \mathbb{R}[p]/a$ is a local algebra with the maximal ideal $m = m_p/a$ and every local algebra is isomorphic to such a local algebra.

Definition 6.2. An algebra homomorphism $\mathfrak{X}' : \mathfrak{F}(\mathcal{M}) \to A$ so that $\mathfrak{X}(f) \equiv f(\mathfrak{X}) (\text{mod } m)$ for $f \in \mathfrak{F}(\mathcal{M})$ is called an A point of $\mathcal{M}$ near to $\mathfrak{X}$ or infinitely near point to $\mathfrak{X}$ on $\mathcal{M}$ of A-kind.

We denote by $M^A_{\mathfrak{X}}$ the set of all A-points of M near to $\mathfrak{X}$ and by M^A the union $\bigcup_{\mathfrak{X} \in \mathcal{M}} M^A_{\mathfrak{X}}$ and we define $\pi_A : M^A \longrightarrow M$ by $\pi_A(M^A_{\mathfrak{X}}) = \mathfrak{X}$ for every $\mathfrak{X} \in M$.

In particular, if $a = (m_1)^2$ and $C = \mathbb{R}[1]/a$ then the notion of C-point is nothing but the notion of tangent vector on M. We shall denote by $\tau = \pi(X_1)$ where X_1 is the indeterminate in $\mathbb{R}[1]$ and $\pi : \mathbb{R}[1] \to \mathbb{C}$ is the natural projection. A tangent vector $X \in T\,M_{\mathfrak{X}}$ is identified with $\mathfrak{X}' \in \mathcal{M}^{\mathbb{C}}_{\mathfrak{X}}$ defined by $\mathfrak{X}'(f) = f(\mathfrak{X}) + (Xf)\cdot\tau$ for every $f \in \mathcal{F}(\mathcal{M})$.

Let U be a coordinate neighbourhood of $\mathfrak{X}_0$ in M with the coordinate system $\{\mathfrak{X}_1,\dots,\mathfrak{X}_n\}$ and $\{1 = B^0, B^1,\dots,B^N\}$ a basis of the local algebra A so that $B^1,\dots,B^N$ span the maximal ideal m of A. We define $\mathfrak{X}_{i,\lambda} : \pi_A^{-1}(\mathcal{U}) \to \mathbb{R}$ by

$$\sum_{\lambda=0}^{N} \mathfrak{X}_{i,\lambda}(\mathfrak{X}')\cdot B^{\lambda} = \mathfrak{X}'(\mathfrak{X}_i) \quad \text{for any } \mathfrak{X}' \in \pi_A^{-1}(\mathcal{U}). \tag{6.1}$$

M^A becomes a differentiable manofild of dimension $n(N+1)$ by the coordinate neighbourhoods $\pi_A^{-1}(\mathcal{U})$ with coordinate system $\{\mathfrak{X}_{i,\lambda} \mid i = 1,\dots,n;\ \lambda = 0,1,\dots,N\}$ induced by the coordinate system $\{\mathfrak{X}_1,\dots,\mathfrak{X}_n\}$ on $\mathcal{U}$.

Definition 6.3. The differentiable manifold M^A with the projection $\pi_A : M^A \to M$ will be called the bundle of A-points of (or bundle of infinitely near points of M of A-kind).

Let $F : M \to M'$ be a differentiable map of a manifold M into a manofild M'. Then the map $F^A : M^A \to M'^A$ defined by

$$(F^A(\mathfrak{X}'))(g) = \mathfrak{X}'(g \circ F) \tag{6.2}$$

for every $x' \in M^A$ and $g \in \mathcal{F}(M')$ is differentiable.

$\mathbb{R}^A$ can be identified with A by $\mathbb{R}^A \ni \mathfrak{X}' \to \mathfrak{X}'(t) \in A$, where t is the natural coordinate on $\mathbb{R}$.

For any $f \in \mathcal{F}(\mathcal{M})$ one can define the map $f^A : \mathcal{M}^A \to \mathbb{R}^A = A$ which is called the lift of the map f.

Lemma 6.4. For any $X \in \mathcal{F}^1_0(\mathcal{M})$ and $f \in \mathcal{F}(\mathcal{M})$ there exists a unique vector field $X^A \in \mathcal{T}^1_0(\mathcal{M}^A)$ so that

$$(Xf)^A = X^A f^A \quad \text{and} \quad (f\cdot X)^A = f^A X^A. \tag{6.3}$$

Definition 6.5. The vector field X^A is called the lift of the vector field $X \in \mathcal{T}_0^1(\mathcal{M})$ to the bundle M^A.

Lemma 6.6. For any $\sigma \in \mathcal{T}_1^0(\mathcal{M})$, $X \in \mathcal{T}_0^1(\mathcal{M})$ and $f \in \mathcal{F}(\mathcal{M})$ there exists a unique 1-form $\sigma^A \in \mathcal{T}_1^0(\mathcal{M})$ so that

$$(f\cdot\sigma)^A = f^A\cdot\sigma^A \quad \text{and} \quad (\sigma(X))^A = \sigma^A(X^A). \tag{6.4}$$

Let ∇ be an affine connection defined on M.

Theorem 6.7. There exists a unique affine connection ∇^A on M^A so that

$$\nabla^A_{X^A} Y^A = (\nabla_X Y)^A \tag{6.5}$$

for every $X, Y \in \mathcal{F}_0^1(M)$ [3].

Theorem 6.8. If X is ω-special field on the manifold M, then the field X^A is ω^A-special on the manifold M^A.

Proof. We have $D(X,Y) = \omega(X)Y + \omega(Y)X$ for any $Y \in \mathcal{T}_0^1(\mathcal{M})$.

Let $\bar\nabla$ and ∇ be affine connections on the manifold M so that $D = \bar\nabla - \nabla$ and we make the following notation $D^A = \bar\nabla^A - \nabla^A \in \mathcal{T}_2^1(\mathcal{M}^A)$. Then

$$DA_{X^A}Y^A = \bar\nabla^A_{X^A}Y^A - \nabla^A_{X^A}Y^A = (\bar\nabla_X Y)^A - (\nabla_X Y)^A =$$

$$= (D(X,Y))^A = (\omega(X)Y)^A + (\omega(Y)X)^A = \omega^A(X^A)Y^A + \omega^A(Y^A)X^A$$

and this shows that X^A is ω^A-special field on the manifold M^A.

REFERENCES

[1] G. Greub, S. Halperin, R. Vanstone: Connections, Curvature and Cohomology, Vol. II, Academic Press, New York, (1973).

[2] R. Hermann: Differential geometry and the calculus of variations, Academic Press, (1968).

[3] A. Morimoto: Prolongation of connections to bundles of infinitely near points, J. Differential Geometry, 11 (1976) 479-498.

[4] L. Nicolescu: Sur la géométrie de l'algèbre associé à un champ tensoriel du type (1,2), Tensor, N.S., 38 (1982), 235-241.

[5] U. Simon: On the inner geometry of the second fundamental form, Michigan J. Math. 9 (1972), 129-132.

[6] I. Vaisman: Sur quelques formules du calcul de Ricci global, Comm. Math. Helv. 41 (1966-67), 73-87.

[7] A. Weil: Théorie des points proches sur les variétiés différentiables, Colloque Geom. Diff. Strasbourg (1963), 111-117.

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EQUIVALENCE SYSTEMS DERIVED FROM PARTIAL ORDERS

Let R be a binary relation on a set X, and let $S(R) = \{(x,y) \in X\times X \mid xRy\}$. We define equivalence relations R_1 and R_2 on S(R) by $(x_1,x_2)R_i(y_1,y_2)$ if and only if $x_i = y_i$ (for $i=1,2$). We show that if R is a partial order then (X,R) is uniquely determined up to isomorphism or dual isomorphism by the derived equivalence system $(S(R),\{R_1,R_2\})$, and that it has the same automorphism group. Furthermore, every group is isomorphic to the automorphism group of an equivalence system derived from a partial order. We give criteria to decide whether an equivalence system $(Y,\{e_1,e_2\})$ is isomorphic to the derived system of a partial order, or of an equivalence relation.

0. Introduction

An *equivalence system* (X,E) is a pair consisting of a set X and a set E of equivalence relations on X. Let R be a binary relation on a set X, and let $S(R) = \{(x,y) \in X^2 \mid xRy\}$. We define relations R_1, R_2 on S(R) by $(x_1,x_2)R_i(y_1,y_2)$ if and only if $x_i = y_i$ for $i = 1,2$. Clearly R_1 and R_2 are equivalence relations on S(R). Note that the meet of R_1 and R_2 in the lattice of equivalence relations on S(R) is the equality relation on S(R). We say that $S(R),\{R_1,R_2\})$ is the *equivalence system derived* from (X,R). Let X,X′ be sets and T,T′ sets of binary relations on X,X′ respectively. We say that the relational systems (X,T) and (X′,T′) are *isomorphic* if and only if there exist bijections $\phi: X \to X'$ and $\psi: T \to T'$ such that for all $x,y \in X$ and $t \in T$ we have xty if and only if $(x\phi)(t\psi)(y\phi)$.

Let X be a set and T a set of binary relations on X, and let ϕ be a permutation of X. We call ϕ an automorphism of (X,T) if for all $x,y \in X$ and $t \in T$ we have xty if and only if $(x\phi)t(y\phi)$. It is easy to see that the set $\mathrm{Aut}(X,T)$ of all automorphisms of (X,T) forms a group.

Note that if $\bar{R}$ is the inverse relation of R then $(S(R),\{R_1,R_2\})$ is isomorphic to $(S(\bar{R}),\{\bar{R}_1,\bar{R}_2\})$; where $\phi: S(R) \to S(\bar{R})$ and $\psi: \{R_1,R_2\} \to \{\bar{R}_1,\bar{R}_2\}$, are given by $(x,y)\phi = (y,x)$ for all $(x,y) \in S(R)$, $R_1\psi = \bar{R}_2$ and $R_2\psi = \bar{R}_1$.

1. Partial orders and automorphisms

Let $(Z,\{e_1,e_2\})$ be an equivalence system. We say that $z \in Z$ has property $(*)$ if the following holds:

$(*)$ whenever $z',z'' \in Z$ are such that ze_1z' and ze_2z'' then there exists $\bar{z} \in Z$ such that $\bar{z}e_1z''$ and $\bar{z}e_2z'$.

Note that property $(*)$ is preserved by isomorphism. We now consider property $(*)$ in equivalence systems derived from partial orders.

Lemma 1.1. Let (X,P) be a partially ordered set, let $(S(P),\{P_1,P_2\})$ be the derived equivalence system, and let $z \in S(P)$. Then z has property $(*)$ if and only if there exists $x \in X$ such that $z = (x,x)$.

Proof. First suppose that there exists $x \in X$ such that $z = (x,x)$. Let $z' = (x',y')$ and $z'' = (x'',y'')$. As zP_1z' and zP_2z'' it follows that $x' = x$ and $y'' = x$. Therefore we have $x''Px$ and xPy', hence by transitivity $x''Py'$, and therefore $\bar{z} := (x'',y') \in S(P)$. Now we clearly have $\bar{z}P_1z''$ and $\bar{z}P_2z'$, thus z has property $(*)$.

Conversely, suppose that z has property $(*)$. Let $x,y \in X$ such that $z = (x,y)$. Define $z' = (x,x)$ and $z'' = (y,y)$. Thus we have zP_1z' and zP_2z''. Hence there exists $\bar{z} = (\bar{x},\bar{y}) \in S(P)$ such that $\bar{z}P_1z''$ and $\bar{z}P_2z'$. It follows that $\bar{x} = y$ and $\bar{y} = x$, therefore yPx. As xPy also holds, we get $x = y$, which proves the lemma. ∎

If (X,P) is a partially ordered set then the inverse relation $\bar{P}$ defined by $x\bar{P}y$ if and only if yPx is also a partial order, and $(X,\bar{P})$ is called the dual of (X,P).

Theorem 1.2. Let (X,P) and (X′,P′) be partially ordered sets, and suppose that $(S(P),\{P_1,P_2\})$ and $(S(P'),\{P_1',P_2'\})$ are isomorphic. Then (X,P) is isomorphic to (X′,P′) or to its dual.

Proof. Without loss of generality, we can assume that the isomorphism is given by $\varphi: S(P) \to S(P')$, and that $P_i\psi = P_i'$ for i = 1,2. If $x \in X$, then we have $z := (x,x) \in S(P)$. By Lemma 1.1, the element z has property (*). By the isomorphism, it follows that $z\varphi$ also has property (*), and hence there exists a unique $\hat{x} \in X'$ such that $z\varphi = (\hat{x},\hat{x})$. We thus can define a mapping $\gamma: X \to X'$ by $x\gamma = \hat{x}$ where $\hat{x}$ is the unique element of X′ such that $(x,x)\varphi = (\hat{x},\hat{x})$. In the same way it follows that γ is bijective.

Now let $x,y \in X$ such that xPy and $x \neq y$. If we set $z = (x,y)$ then z is the unique element of S(P) such that $zP_1(x,x)$ and $zP_2(y,y)$. Then $z\varphi$ is the unique element of S(P′) such that $(z\varphi)P_1'((x,x)\varphi)$ and $(z\varphi)P_2'((y,y)\varphi)$. As $(x,x)\varphi = (x\gamma,x\gamma)$ and $(y,y)\varphi = (y\gamma,y\gamma)$, we thus have $(x\gamma,y\gamma) = z\varphi \in S(P')$, hence $(x\gamma)P'(y\gamma)$, and γ is order-preserving. Similarly we also get that the inverse of γ is order-preserving, and hence γ is an isomorphism. ■

In essentially the same way it follows that the following holds.

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Theorem 1.3. Let (X,P) be a partially ordered set. Then $\mathrm{Aut}(S(P),\{P_1,P_2\})$ is isomorphic to the group Aut(X,P) of order-automorphisms of (X,P).

In [1] the author has shown that for every group G there exists an equivalence system (X,E), that is, a set X and a set E of equivalence relations on X, such that G is isomorphic to Aut(X,E), and that we can take X and E such that $|X| = 2|G| + 1$ and $|E| = |G| + 1$. It is natural to ask the question if the size of E can be made smaller. Using Theorem 1.3 we can show that this can be done (then, however, the size of X will usually become larger).

Theorem 1.4. Let G be a group. Then there exists a set Y and two equivalence relations e_1 and e_2 on Y whose intersection is the identity and whose join is the trivial equivalence relation on Y such that G is isomorphic to $\mathrm{Aut}(Y,\{e_1,e_2\})$. If G is finite then Y can be taken to be finite, and if G is infinite then Y can be taken to be of the same cardinality as G.

Proof. First of all note that a partially ordered set (X,P) is connected if and only if the join of P_1 and P_2 in the lattice of equivalence relations on $S(P)$ is equal to the trivial equivalence relation on $S(P)$. Next it is well known that for every group G there exists a connected partially ordered set (X,P) such that $G \cong \mathrm{Aut}(X,P)$, where X is finite if G is finite and where $|X| = |G|$ if G is infinite (see, for example, [2] or [3]). The rest now follows from Theorem 1.3. ∎

2. Recognizing partial orders and equivalence relations

We now give necessary and sufficient conditions for an equivalence system $(Z,\{e_1,e_2\})$ to be isomorphic to the derived system of a partial order or of an equivalence relation.

Theorem 2.1. Let $(Z,\{e_1,e_2\})$ be an equivalence system. The following are equivalent.

(1) There exists a partially ordered set (X,P) such that $(Z,\{e_1,e_2\})$ is isomorphic to $(S(P),\{P_1,P_2\})$.

(2) The meet of e_1 and e_2 is the equality relation and the following properties hold.

(a) For every $z \in Z$ there exist $z',z'' \in Z$ with property $(*)$ such that ze_1z' and ze_2z''.

(b) If $x,y \in Z$ are such that there exists $z \in Z$ with property $(*)$ such that ze_2x and ze_1y, then there exists $z' \in Z$ with $z'e_1x$ and $z'e_2y$.

(c) If $x,y \in Z$ are such that there exist $z,z' \in Z$ with property $(*)$ such that xe_1z, xe_2z', ye_1z', and ye_2z, then $x = y$.

Proof. First assume (1). To prove (2), it is enough to show that $(S(P),\{P_1,P_2\})$ satisfies the conditions (a), (b), (c). By Lemma 1.1, all elements $(x,x) \in S(P)$ have property $(*)$, and thus we have (a). Let $(x_1,x_2),(y_1,y_2) \in S(P)$, and suppose there exists $(z_1,z_2) \in S(P)$ with property $(*)$ such that $(z_1,z_2)P_2(x_1,x_2)$ and $(z_1,z_2)P_1(y_1,y_2)$. Then we have $z_2 = x_2$ and $z_1 = y_1$, and Lemma 1.1 gives $z_1 = z_2$, and thus $x_2 = y_1$. Now we have x_1Px_2 and y_1Py_2, by transitivity therefore x_1Py_2 and $(x_1,y_2) \in S(P)$. As $(x_1,y_2)P_1(x_1,x_2)$ and $(x_1,y_2)P_2(y_1,y_2)$, we have (b). Let $(x_1,x_2),(y_1,y_2) \in S(P)$ such that there exist $(z_1,z_2),(z'_1,z'_2) \in S(P)$ with property $(*)$, such that $(x_1,x_2)P_1(z_1,z_2)$, $(x_1,x_2)P_2(z'_1,z'_2)$, $(y_1,y_2)P_1(z'_1,z'_2)$ and $(y_1,y_2)P_2(z_1,z_2)$. Then $x_1 = z_1$, $x_2 = z'_2$, $y_1 = z'_1$ and $y_2 = z_2$. By Lemma 1.1, also $z_1 = z_2$ and $z'_1 = z'_2$. Thus $x_1 = y_2$ and $x_2 = y_1$. But then x_1Px_2 and x_2Px_1, hence by antisymmetry, we have $x_1 = x_2 = y_1 = y_2$, and thus $(x_1,x_2) = (y_1,y_2)$.

Conversely, let $(Z,\{e_1,e_2\})$ be an equivalence system which satisfies (2). We define $X = \{z \in Z \mid z \text{ has property } (*)\}$. We define a relation P on X by xPy if and only if there exists $z \in Z$ with ze_1x and ze_2y. We first show that P is a partial order on X.

If $x \in X$, then clearly xe_1x and xe_2x, and thus xPx, hence P is reflexive. Let $x,y,z \in X$ such that xPy and yPz. Then there exist $v,w \in Z$ such that ve_1x, ve_2y, we_1y, we_2z. By (b), there exists $y' \in Z$ with $y'e_1v$ and $y'e_2w$. By transitivity of e_1 and e_2, we get $y'e_1x$ and $y'e_2z$, and therefore xPz. Hence P is transitive. Let $x,y \in X$ such that xPy and yPx. Then there exist $z,\bar{z} \in Z$ such that xe_1z, ye_2z, $ye_1\bar{z}$, $xe_2\bar{z}$. By (c), we have $z = \bar{z}$. By transitivity of e_1, e_2, we get xe_1y and xe_2y. As the meet of e_1 and e_2 is the equality relation, we have $x = y$, and thus P is antisymmetric.

It remains to prove that if we take this partially ordered set (X,P) then $(S(P),\{P_1,P_2\})$ is isomorphic to $(Z,\{e_1,e_2\})$. We define $P_i\psi = e_i$ for $i = 1,2$. Note that if $(x,y) \in S(P)$, then there exists a unique $z \in Z$ such that ze_1x and ze_2y (otherwise we would get a contradiction to the fact that the intersection of e_1 and e_2 is the equality). Denote this element by $(x,y)\varphi$. Clearly the mapping $\varphi: S(P) \rightarrow Z$ is injective.

From (a) it follows that φ is also surjective. Let $(x_1,x_2),(y_1,y_2) \in S(P)$. If $(x_1,x_2)P_i(y_1,y_2)$, then $x_i = y_i$, and $((x_1,x_2)\varphi)e_ix_i$ and $((y_1,y_2)\varphi)e_iy_i$. Thus $((x_1,x_2)\varphi)e_i((y_1,y_2)\varphi)$ for $i = 1,2$. For the converse, first note that whenever $x,y \in Z$ have property (*) and xe_1y or xe_2y then from (c) it follows that $x = y$. Now let $z_1,z_2 \in Z$. Let $z_i', z_i'' \in X$ be such that $z_ie_1z_i'$ and $z_ie_2z_i''$ for $i = 1,2$. Note that $z_i = (z_i', z_i'')\varphi$. Now if $z_1e_1z_2$ then we have $z_1'e_1z_2'$, and hence $z_1' = z_2'$, and thus $(z_1',z_1'')P_1(z_2',z_2'')$, and the same holds for e_2 and P_2. Therefore it follows that $(S(P),\{P_1,P_2\})$ is isomorphic to $(Z,\{e_1,e_2\})$.■

Theorem 2.2. Let $(Z,\{e_1,e_2\})$ be an equivalence system. Then the following are equivalent.

(1) There exists a set X and an equivalence relation f on X such that $(Z,\{e_1,e_2\})$ is isomorphic to $(S(f),\{f_1,f_2\})$.

(2) The meet of e_1 and e_2 is the equality relation, every element $z \in Z$ has property (*), and if $z \in Z$, then the classes of e_1 and e_2 containing z have the same cardinality.

(3) The meet of e_1 and e_2 is the equality relation, every element $z \in Z$ has property (*), and if x is an equivalence class of e_1, then there exists an element $z \in x$ such that the equivalence class of e_2 containing z has the same cardinality as x.

Proof. First of all we show that (1) implies (2). We prove that every $z \in S(f)$ has property (*). Let $z,z',z'' \in S(f)$ such that zf_1z' and zf_2z'', and let $a,a',a'',b,b',b'' \in X$ such that $z = (a,b)$, $z' = (a',b')$, $z'' = (a'',b'')$. We then have afb, $a'fb'$, $a''fb''$, and furthermore $a = a'$ and $b' = b''$. Let $\bar{z} = (a'',b')$. By transitivity of f, we get $a''fb'$, and hence $\bar{z} \in S(f)$, and we have $\bar{z}f_1z''$ and $\bar{z}f_2z'$. Therefore z has property (*). If $z \in S(f)$, $z = (a,b)$ with $a,b \in X$, then clearly the classes of f_1 and f_2 containing z both have the same cardinality as the class of f containing a, and hence (2) holds. The fact that (2) implies (3) is trivial, and therefore it remains to prove that (3) implies (1).

Assume (3). Let g be the join of e_1 and e_2 in the lattice of equivalence relations on Z. Let $C(g)$ and $C(e_1)$

be the set of equivalence classes of g and e_1 respectively, and for $c \in C(g)$, let $A(c) = \{c' \in C(e_1) | c' \subseteq c\}$. By the axiom of choice, and as all the sets $A(c)$ are disjoint, there exists a subset X_1 of $C(e_1)$ such that $|X_1 \cap A(c)| = 1$ for all $c \in C(g)$. For each $x \in X_1$, there exists an element $t_x \in x$ and a bijection ϕ_x between x, and the class of e_2, containing t_x such that $t_x \phi_x = t_x$. As t_x has the property (*), for each $y \in x$ there exists an element $d(y) \in Z$ such that $d(y) e_2 y$ and $d(y) e_1 (y\phi_x)$. Note that given t_x and ϕ_x, the element $d(y)$ is unique as the intersection of e_1 and e_2 is the equality. Let $X = \{d(y) | y \in x \text{ for some } x \in X_1\}$. We define a relation f on X by $d(y) f d(y')$ if and only if there exists $x \in X_1$ such that $\{y, y'\} \subseteq x$. Clearly f is reflexive and symmetric. Suppose $d(y) f d(y')$ and $d(y') f d(y'')$. Then there exist $x, x' \in X_1$ such that $\{y, y'\} \subseteq x$ and $\{y', y''\} \subseteq x'$. As x, x' are equivalence classes of e_1, it follows that $x = x'$, and hence $d(y) f d(y'')$. Thus f is an equivalence relation.

Next we note that as every element of Z has property (*), whenever z, z' are elements of Z then z and z' are in the same class of g if and only if there exists $z'' \in Z$ such that $z e_1 z''$ and $z'' e_2 z'$ (and also $\hat{z} \in Z$ such that $z e_2 \hat{z}$ and $\hat{z} e_1 z'$). Now if $z \in Z$, then there exists a unique class c of g which contains z. Then there exists a unique $x \in X_1$ contained in c. It then follows that there exist unique $y_1, y_2 \in X_1$ such that $y_2 e_2 z$ and $(y_1 \phi_x) e_1 z$. Then there are unique $d(y_1), d(y_2) \in X$ such that $d(y_2) e_2 z$ and $d(y_1) e_1 z$. We claim that the mapping which maps z to $(d(y_1), d(y_2))$ is a bijection from Z onto $S(f)$ such that if z' is mapped to $(d(y_1'), d(y_2''))$ then $z e_i z'$ if and only if $(d(y_1), d(y_2)) f_i d(y_1'), d(y_2'))$ for $i = 1,2$. The mapping is clearly well-defined and bijective. Also we have $(d(y_1), d(y_2)) f_1 (d(y_1'), d(y_2'))$ if and only if $d(y_1) = d(y_1')$, which is equivalent to $y_1 = y_1'$, and then also to $y_1 \phi_x = y_1' \phi_x$ and $z e_1 z'$. Similarly the corresponding result for e_2 and f_2 follows, and we have (1). ■

REFERENCES

[1] G. Behrendt: Equivalence structures and their automorphisms, European J. Comb. (to appear).

[2] G. Birkhoff: Sobre los grupos de automorfismos, Revista Unión Mat. Argentina 11 (1946), 155-157.

[3] R. Frucht: On the construction of partially ordered systems with a given group of automorphisms, Amer. J. Math. 72 (1950), 195-199.

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CAUCHY STRUCTURES AND THE KERÉKJÁRTÓ-STOILOW COMPACTIFICATION

1. Introduction

Let $\mathfrak{X}$ be a connected, locally connected and locally compact topological space. The Kerékjártó-Stoilow compactification of $\mathfrak{X}$ is a couple $(\tilde{\mathfrak{X}},\rho)$ consisting of a topological space $\tilde{\mathfrak{X}}$ and of a map: $\rho:\mathfrak{X}\to\tilde{\mathfrak{X}}$, with the following properties:

(i) $(\tilde{\mathfrak{X}},\rho)$ is a compactification of $\mathfrak{X}$ (i.e. $\tilde{\mathfrak{X}}$ is compact, and ρ induces a homeomorphism onto an open and dense subset of $\tilde{\mathfrak{X}}$);

(ii) The ideal boundary, $\tilde{\mathfrak{X}}\setminus\rho(\mathfrak{X})$, is a totally disconnected and nowhere disconnecting subset of $\tilde{\mathfrak{X}}$. (A closed subset, B, of a topological space, $\mathfrak{X}$, is nowhere disconnecting iff any point $x\in B$ has a fundamental system of open neighbourhoods such that $U\setminus B$ is connected for any U in the system, [4]).

The Kerékjártó-Stoilow boundary is the finest totally disconnected and nowhere disconnecting boundary, that is, it satisfies the following universality property: if $(\mathcal{Y},f)$ is another compactification of $\mathfrak{X}$, such that the ideal boundary, $\mathcal{Y}\setminus f(\mathfrak{X})$, is totally disconnected, then there exists a unique continuous map $f':\tilde{\mathfrak{X}}\to\mathcal{Y}$ which makes the diagram

$$\begin{tikzcd}
\mathfrak{X} \arrow[rr, "\rho"] \arrow[dr, "f"'] & & \tilde{\mathfrak{X}} \arrow[dl, "f'"] \\
& \mathcal{Y} &
\end{tikzcd}$$

commutative.

The ideal boundary $\tilde{\mathfrak{X}} \setminus \rho(\mathfrak{X})$ is a fundamental invariant which plays a main role in the topological classification of open orientable surfaces. It has been studied in [7], [3], [9] (see also [1], [2]). The previous axiomatic definition of it belongs to M. Jurchescu, [5]. The aim of this paper is to give a constructive proof of its existence, using as a main tool the Cauchy structures.

2. Cauchy structures

The Cauchy structures were first considered by M. Jurchescu in order to construct an axiomatic theory of completion ([6], [8]). Another goal to be pursued with their aid, is to bring together, in the same theoretical framework, both completion and compactification.

Let $\mathfrak{X}$ be a topological space, and let $\mathcal{T}$ denote its topology.

Definition 1. A set, $\mathcal{C}$, of filters on $\mathfrak{X}$ is said to be a Cauchy structure on $\mathfrak{X}$ if it satisfies the following five axioms:

(C1) If γ is a filter on $\mathfrak{X}$, and there exists a filter $\gamma' \in \mathcal{C}$ such that $\gamma \supseteq \gamma'$, then $\gamma \in \mathcal{C}$.

(C2) For any $\gamma \in \mathcal{C}$ there exists a minimal filter $\gamma_0 \in \mathcal{C}$, such that $\gamma_0 \subseteq \gamma$. (We call γ_0 minimal, if from $\gamma' \in \mathcal{C}$ and $\gamma' \subseteq \gamma_0$, it follows $\gamma' = \gamma_0$).

(C3) Any minimal element of $\mathcal{C}$ has an open base.

(C4) For any point $x \in \mathfrak{X}$, the filter of neighbourhoods of x, denoted $\mathcal{T}_{\mathfrak{X},x}$ (or simply $\mathcal{T}_x$ when no ambiguity is possible) is a minimal element of $\mathcal{C}$.

(C5) For any minimal element $\gamma \in \mathcal{C}$ and for any $M \in \gamma$, there is a $V \in \gamma$ with the following property: if $\gamma' \in \mathcal{C}$ is a minimal element, such that $M' \cap V \neq \emptyset$ for any $M' \in \gamma'$, then it follows that $M \in \gamma'$.

From (C5), called the regularity axiom, it follows that $\overline{V} \subseteq M$, and, consequently, that $\mathfrak{X}$ is quasiregular. Conversely, any quasiregular space can be canonically endowed with a Cauchy structure, by choosing $\mathcal{C}$ to be the set of all convergent filters of $\mathfrak{X}$.

If $\mathfrak{X}$ is compact, the set of all convergent filters of $\mathfrak{X}$ defines the unique Cauchy structure on $\mathfrak{X}$.

Another consequence of (C5) is that given any pair of minimal elements $\gamma_1, \gamma_2 \in \mathcal{C}$, $\gamma_1 \neq \gamma_2$, one can find a pair of sets $U_1 \in \gamma_1$, $U_2 \in \gamma_2$, such that $U_1 \cap U_2 = \emptyset$.

Definition 2. A *Cauchy space* is a triple $(\mathfrak{X},\mathcal{T},\mathcal{C})$, where $\mathfrak{X}$ is a topological space, $\mathcal{T}$ its topology, and $\mathcal{C}$ a Cauchy structure on $\mathfrak{X}$. The elements of $\mathcal{C}$ will be called the *Cauchy filters* of $\mathfrak{X}$.

Let $(\mathfrak{X},\mathcal{T}_{\mathfrak{X}},\mathcal{C}_{\mathfrak{X}})$ and $(\mathcal{Y},\mathcal{T}_{y},\mathcal{C}_{y})$ be two Cauchy spaces.

Definition 3. A map $f:\mathfrak{X} \to \mathcal{Y}$ is a *Cauchy morphism* iff

(i) $f^{-1}(\mathcal{T}_y) \subseteq \mathcal{T}_{\mathfrak{X}}$ (continuity),

(ii) $f(\mathcal{C}_{\mathfrak{X}}) \subseteq \mathcal{C}_y$ (f carries the Cauchy filters of $\mathfrak{X}$ to the Cauchy filters of $\mathcal{Y}$).

A map $f:\mathfrak{X} \to \mathcal{Y}$ is a *Cauchy isomorphism* iff

(i) $f^{-1}(\mathcal{T}_y) = \mathcal{T}_{\mathfrak{X}}$,

(ii) $f(\mathcal{C}_{\mathfrak{X}}) = \mathcal{C}_y$.

3. The fundamental theorem of the axiomatic theory of completion

Definition 4. A Cauchy space $(\mathfrak{X},\mathcal{T},\mathcal{C})$ is called *quasicomplete* if and Cauchy filter of $\mathfrak{X}$ is convergent (that is, if the only minimal elements of $\mathcal{C}$ are the neighbourhood filters), and *complete* if $\mathfrak{X}$ is Hausdorff's and quasicomplete space.

The following proposition provides the relationship between completeness of topological spaces and of Cauchy spaces. The proof is in [6], [8].

Proposition. Let $(\mathfrak{X},\mathcal{T},\mathcal{C})$ be a Cauchy space such that any minimal Cauchy filter has a countable base. Then, we have equivalence between the following statements:

(a) $(\mathfrak{X},\mathcal{T},\mathcal{C})$ is quasicomplete,

(b) Any Cauchy sequence of $\mathfrak{X}$ is convergent.

Definition 5. Given a Cauchy space $\mathfrak{X}$, we will call a completion of $\mathfrak{X}$ a couple $(\mathfrak{X}',\rho)$, where $\mathfrak{X}'$ is a Cauchy space and $\rho : \mathfrak{X} \to \mathfrak{X}'$ is a Cauchy morphism (called the structural morphism), satisfying the following two conditions:

(i) $\mathfrak{X}'$ is complete,

(ii) For any other couple $(\mathcal{Y}, f)$, where $\mathcal{Y}$ is a complete Cauchy space and $f : \mathfrak{X} \to \mathcal{Y}$ is a Cauchy morphism, there exists a unique continuous map $f' : \mathfrak{X}' \to \mathcal{Y}$ which makes the diagram

$$\begin{tikzcd}
\mathfrak{X} \arrow[r, "\rho"] \arrow[dr, "f"'] & \mathfrak{X}' \arrow[d, "f'"] \\
& \mathcal{Y}
\end{tikzcd}$$

commutative.

Note that from the completeness of $\mathfrak{X}'$ it follows that f' is a Cauchy morphism.

The following theorem is fundamental in the axiomatic theory of completion.

Theorem 1. 1°. Any Cauchy space has a completion.

2°. A couple $(\mathfrak{X}',\rho)$, where $\mathfrak{X}'$ is a Cauchy space and $\rho : \mathfrak{X} \to \mathfrak{X}'$ a Cauchy morphism, is a completion of the Cauchy space $\mathfrak{X}$ iff $\mathfrak{X}'$ is complete and the structural morphism ρ is a dense Cauchy isomorphism ("dense" in the topological sense, that is $\rho(\mathfrak{X}) = \mathfrak{X}'$).

We will present the important facts which appear in the proof of this result without entering into the details of the proof, which can be found in ([6], [8]).

A canonical completion $(\hat{\mathfrak{X}},\rho)$ of the Cauchy space $(\mathfrak{X},\mathcal{T}_{\mathfrak{X}},\mathcal{C})$ can be obtained by defining $\hat{\mathfrak{X}}$ to be the set of all minimal elements of $\mathcal{C}$, and $\rho : \mathfrak{X} \to \hat{\mathfrak{X}}$ to be the map which carries every $x \in \mathfrak{X}$ to its filter of neighbourhoods $\mathcal{T}_{\mathfrak{X},x}$. The set $\hat{\mathfrak{X}}$ can be endowed with the topology $\mathcal{T}_{\hat{\mathfrak{X}}}$ generated by the base formed of sets of the form

$$c(U) := \{\gamma \in \mathcal{C} \mid U \in \gamma\}$$

defined for every open set $U \in \mathcal{T}_{\mathfrak{X}}$. Note that the family of the $c(U)$'s with U open sets belonging to a minimal element $\gamma \in \mathcal{C}$ form a base for the filter of neighbourhoods of γ in $\hat{\mathfrak{X}}$, Furthermore, the map $c: \mathcal{T}_{\mathfrak{X}} \to \mathcal{T}_{\hat{\mathfrak{X}}}$ is injective, and $\rho^{-1}(c(U)) = U$, for every $U \in \mathcal{T}_{\mathfrak{X}}$.

Using essentially axiom (C5), it can be shown that $\hat{\mathfrak{X}}$ is a quasiregular space; consequently, $\hat{\mathcal{C}}$, the family of all convergent filters on $\hat{\mathfrak{X}}$, defines a Cauchy structure on $\hat{\mathfrak{X}}$. Since $\hat{\mathfrak{X}}$ is also Hausdorff´s space $(\hat{\mathfrak{X}}, \mathcal{T}_{\hat{\mathfrak{X}}}, \hat{\mathcal{C}})$ becomes a complete Cauchy space, and the structural morphism $\rho: \mathfrak{X} \to \hat{\mathfrak{X}}$ is a dense Cauchy isomorphism.

If $(\mathcal{Y}, \mathcal{T}_{\mathcal{Y}}, \mathcal{C}_{\mathcal{Y}})$ is another complete Cauchy space, and $f: \mathfrak{X} \to \mathcal{Y}$ a Cauchy morphism, then $f': \hat{\mathfrak{X}} \to \mathcal{Y}$ will be constructed in the following way: for an element $\gamma \in \hat{\mathfrak{X}}$, consider its filter of neibhbourhoods $\mathcal{T}_{\hat{\mathfrak{X}},\gamma}$. Its preimage $\rho^{-1}(\mathcal{T}_{\hat{\mathfrak{X}},\gamma})$, will be a Cauchy filter of $\mathfrak{X}$ (because ρ is a Cauchy isomorphism), and $f(\rho^{-1}(\mathcal{T}_{\hat{\mathfrak{X}},\gamma}))$ will be a Cauchy filter of $\mathcal{Y}$ (because f is a Cauchy morphism). Since $\mathcal{Y}$ is complete, $f(\rho^{-1}(\mathcal{T}_{\hat{\mathfrak{X}},\gamma}))$ is convergent to some point $y \in \mathcal{Y}$, which is unique, because $\mathcal{Y}$ is Hausdorff´s space and we define $f'(\gamma) = y$.

4. Construction of the Kerékjártó-Stoilow compactification

We are now able to formulate and to prove the main result which makes the subject of this paper.

Theorem 2. Let $\mathfrak{X}$ be a connected, locally connected, and locally compact topological space and define $\mathcal{C}$ the family of those connected filters on $\mathfrak{X}$ for which card ad $\gamma \in \{0,1\}$. Then:

1°. $\mathcal{C}$ defines a Cauchy structure on $\mathfrak{X}$,

2°. The canonical completion $\hat{\mathfrak{X}}$, of $\mathfrak{X}$ is precisely the Kerékjártó-Stoilow compactification of $\mathfrak{X}$. The Kerékjártó-Stoilow boundary is exactly the set of all nonconvergent minimal elements of $\mathcal{C}$.

We will make use of the following technical lemmas which we give without proof.

Lemma 1. Let $\mathfrak{X}$ be a Hausdorff space and γ a filter on it. If $K \subset \mathfrak{X}$ is a compact such that ad $\gamma \cap K = \emptyset$, then there exists a set $M \in \gamma$ such that $M \cap K = \emptyset$.

Corollary. In the conditions of Lemma 1, if the filter γ is connected (i.e. has a base of connected sets), there exists a unique connected component, U_K, of $\mathfrak{X} \setminus K$ which belongs to γ.

Lemma 2. Let $\mathfrak{X}$ be a connected, locally connected, and locally compact space. If γ is a connected filter on $\mathfrak{X}$ such that its adherence consists of exactly one point, x_0, then γ is convergent to x_0.

Conversely, if a filter γ converges to a point x_0, then ad $\gamma = \{x_0\}$.

Remark. For the converse to be true, $\mathfrak{X}$ needs only to be Hausdorff's space.

Lemma 3. Let $\mathfrak{X}$ be a connected, locally connected, and locally compact space, and let K be a compact subset of $\mathfrak{X}$. Then:

1) There is only a finite number of connected components of $\mathfrak{X} \setminus K$ which are not relatively compact,

2) K^*, the saturation of K, defined to be reunion of K with the relatively compact connected components of $\mathfrak{X} \setminus K$, is compact.

Hint. Let U_i, $i \in I$, be the connected components of $\mathfrak{X} \setminus K$, such that

$$\mathfrak{X} \setminus K = \bigcup_{i \in I} U_i,$$

and take L a compact neighbourhood of K. One can prove that there exists only a finite number of connected components which meet the frontier ∂L, of L, and that no connected component can be included in $\mathfrak{X} \setminus L$. The set of connected components of $\mathfrak{X} \setminus K$ can, consequently, be splitted into two subsets: $\mathcal{U}_1$, consisting of those U_i's which do not meet ∂L, and for which it follows that $U_i \subset L$, and $\mathcal{U}_2$, consisting of those U_i's which meet ∂L, and about which we know that there is only a finite number of them. The first part of the lemma follows from the fact that the U_i's which are not relatively compact cannot belong to $\mathcal{U}_1$, and the second part follows from the inclusion

$$K^* \subset L \cup \bigcup_{U_i \in \mathcal{U}_2^*} \bar{U}_i ,$$

where $\mathcal{U}_2^* \subset \mathcal{U}_2$ is the set of relatively compact connected components of $\mathfrak{X} \setminus K$.

Proof of Theorem 2: 1) To prove that $\mathcal{C}$ is a Cauchy structure on $\mathfrak{X}$, let us first construct the minimal elements of $\mathcal{C}$. Any $\gamma \in \mathcal{C}$ is a connected filter of one of the following two types:

(a) γ has a unique adherence point (ad $\gamma = \{x_0\}$),

(b) γ has a void adherence (ad $\gamma = \emptyset$).

If γ is of the type (a), then, by Lemma 2, it follows that γ converges to x_0, which means that $\gamma \supseteq \mathcal{J}_{x_0}$. The filters of neighbourhoods are connected and each of them has a unique adherence point, so they belong to $\mathcal{C}$. Later we will prove that they are indeed minimal in $\mathcal{C}$. If $\gamma \in \mathcal{C}$ is of the type (b), then for any compact set $K \subset \mathfrak{X}$ we have ad $\gamma \cap K = \emptyset$; consequently, according to the corollary of Lemma 1, there exists a unique connected component U_K, of $\mathfrak{X} \setminus K$ which belongs to γ. The set of all such U_K s

$$\beta = \left\{ U_K \mid U_K \in \gamma,\ K \subset \mathfrak{X} \ ,\ K \text{ compact} \right\}$$

forms a filter base, because $U_{K \cup L} \subset U_K \cap U_L$ for any two compacts K and L. The filter γ_0 generated by β is connected and has a void adherence, so it is an element of $\mathcal{C}$, and $\gamma_0 \subseteq \gamma$.

To show that γ_0 is minimal, let $\gamma' \in \mathcal{C}$ be such that $\gamma' \subseteq \gamma_0$. We have ad $\gamma' = \emptyset$, because, otherwise, γ' would be convergent to some x_0 and, consequently, would contain both a compact neighbourhood K of x_0 and U_K, a connected component of $\mathfrak{X} \setminus K$. Because ad $\gamma' = \emptyset$, applying the same arguments as for γ above, we conclude that for any compact $K \subset \mathfrak{X}$, γ' must contain a connected component of $\mathfrak{X} \setminus K$, and, because $\gamma' \subseteq \gamma_0$, it must be the same connected component U_K which belongs to γ_0, so that the other inclusion $\gamma' \supseteq \gamma_0$, also holds.

To show that a filter of neibhbourhoods $\mathcal{J}_x$ is minimal in $\mathcal{C}$, let $\gamma \in \mathcal{C}$ be such that $\gamma \subseteq \mathcal{J}_x$. Then ad $\gamma \neq \emptyset$, because, otherwise, γ would contain a connected component of $\mathfrak{X} \setminus K$ for some compact neighbourhood K of x which would lead to contradiction. So there is a point y such that $\mathrm{ad}\,\gamma = \{y\}$. But from the inclusions $\mathcal{T}_y \subseteq \gamma \subseteq \mathcal{T}_x$ and the separation of $\mathfrak{X}$ it follows y = x and $\gamma = \mathcal{T}_x$.

One can easily see that axioms (C1)-(C4) hold. All that remains to be checked is (C5). Take first a minimal element of $\mathcal{C}$ of the form $\mathcal{T}_{x_0}$, and let M be such that $M \in \mathcal{T}_{x_0}$. Then, there exists an open relatively compact neighbourhood V of x_0, such that $x_0 \in V \subset \overline{V} \subset M$, and V has the property specified by (C5) because: if we take another minimal element of $\mathcal{C}$ of the form $\mathcal{T}_{x_1}$, such that $M' \cap V \neq \emptyset$ for every $M' \in \mathcal{T}_{x_1}$, then $x_1 \in \overline{V}$, and it follows that $M \in \mathcal{T}_{x_1}$. For a minimal element of $\mathcal{C}$ such that $\mathrm{ad}\,\gamma' = \emptyset$, there exists a connected component $U_{\overline{V}}$ of $\mathfrak{X} \setminus \overline{V}$ which belongs to γ', and we have $U_{\overline{V}} \cap V = \emptyset$. Let us check now axiom (C5) for a minimal element $\gamma \in \mathcal{C}$ of the form $\mathrm{ad}\,\gamma = \emptyset$. Let $M \in \gamma$. Then, there exist a compact $K \subset \mathfrak{X}$, and U_K a connected component of $\mathfrak{X} \setminus K$ such that $U_K \subset M$, $U_K \in \gamma$. If we take $V = U_K$, it will have the property required by (C5), because: if $M' \cap V \neq \emptyset$ for any $M' \in \mathcal{T}_x$, then $x \in \overline{V} = \overline{U}_K = U_K$ and we conclude that $U_K \in \mathcal{T}_x$ and, consequently, $M \in \mathcal{T}_x$; for some other minimal element $\gamma' \in \mathcal{C}$ such that $\mathrm{ad}\,\gamma' = \emptyset$, we have, either $U_K \in \gamma'$, which implies $M \in \gamma'$, or γ' contains another connected component V_K, $V_K \neq U_K$, of $\mathfrak{X} \setminus K$, and then $V_K \cap V = \emptyset$. The proof that $\mathcal{C}$ is a Cauchy structure on $\mathfrak{X}$ is thus conluded.

2) Let us consider now the canonical completion $\hat{\mathfrak{X}}$ of $\mathfrak{X}$, the set of all minimal elements of $\mathcal{C}$, and the structural morphism $\rho : \mathfrak{X} \to \hat{\mathfrak{X}}$ defined by $\rho(x) = \mathcal{T}_{\mathfrak{X},x}$. The set $\hat{\mathfrak{X}}$ is composed of $\rho(\mathfrak{X})$, the filters of neighbourhoods, and its complementary $\hat{\mathfrak{X}} \setminus \rho(\mathfrak{X})$, the nonconvergent minimal elements of $\mathcal{C}$ which have the property that they contain a connected component U_K of $\mathfrak{X} \setminus K$, for any compact set $K \subset \mathfrak{X}$. Because ρ is open, $\hat{\mathfrak{X}} \setminus \rho(\mathfrak{X})$ is closed. We remind that the topology on $\hat{\mathfrak{X}}$ is defined in §3.

To prove that $\hat{\mathfrak{X}}$ is compact we will show that any ultrafilter on $\hat{\mathfrak{X}}$ is convergent. Let α be an ultrafilter on $\hat{\mathfrak{X}}$. Then we are in one of the following two cases:

(a) there exists a compact $K \subset \mathfrak{X}$, such that $\rho(K) \in \alpha$. Then, α induces on $\rho(K)$ an ultrafilter $\alpha_K = \{M \in \alpha \mid M \subset \rho(K)\}$, which will have an adherence point because $\rho(K)$ is compact. So there exists an x, $x \in K$, such that $\mathcal{T}_{\mathfrak{X},x} \in \mathrm{ad}\,\alpha_K$. But this implies that $\mathcal{T}_{\mathfrak{X},x} \in \mathrm{ad}\,\alpha$ and, because α is an ultrafilter, α will converge to $\mathcal{T}_{\mathfrak{X},x}$.

(b) For any compact $K \subset \mathfrak{X}$, we have $\rho(K) \notin \alpha$. Then, $\hat{\mathfrak{X}} \setminus \rho(K) \in \alpha$. By Lemma 3, we know that if K is a saturated compact in $\mathfrak{X}$ (saturated meaning $K = K^*$) his complementary, $\mathfrak{X} \setminus K$, is of the form $\mathfrak{X} \setminus K = \bigcup_{i=1}^{n_k} U_K^i$, where U_K^i are the connected components of $\mathfrak{X} \setminus K$ which are not relatively compact. Then

$$\hat{\mathfrak{X}} \setminus \rho(K) = \bigcup_{i=1}^{n_k} c(U_K^i)$$

and, using again the fact that α is an ultrafilter, there exists an index $i \in \{1,2,\ldots,n_K\}$, such that $c(U_K^i) \in \alpha$. The set of all connected components U_K of $\mathfrak{X} \setminus K$, which are not relatively compact and such that $c(U_K) \in \alpha$, for all saturated compacts K of $\mathfrak{X}$, forms a filter base, and let γ be the filter generated by it. γ is an element of $\hat{\mathfrak{X}}$, and we will prove that α converges to γ. Indeed, for any $V \in \mathcal{T}_{\hat{\mathfrak{X}},\gamma}$, there exists an open set U in $\mathfrak{X}$, $U \in \gamma$, such that $c(U) \subseteq V$. But for this U there exist a saturated compact $K \subset \mathfrak{X}$ and a connected component U_K of $\mathfrak{X} \setminus K$, which is not relatively compact, such that $U_K \subset U$. By the monotony of c it follows that $c(U_K) \subset c(U) \subset V$ and, because $c(U_K)$ belongs to α, so does V, which concludes our proof.

Next, we have to prove that $\hat{\mathfrak{X}} \setminus \rho(\mathfrak{X})$ is totally disconnected, that is that its only connected subsets are the points. Let γ_1 and γ_2 be two points of $\hat{\mathfrak{X}} \setminus \rho(\mathfrak{X})$ such that $\gamma_1 \neq \gamma_2$.

Then, there exists a compact $K\subset\mathfrak{X}$, and U_K^1, U_K^2 two different connected components of $\mathfrak{X}\setminus K$ such that $U_K^1\in\gamma_1$, and $U_K^2\in\gamma_2$. Then, $c(U_K^1)$ and $\bigcup\limits_{\substack{i\in I\\ i\neq 1}} c(U_K^i)$ are two open, nonvoid, disjoint sets which cover $\hat{\mathfrak{X}}\setminus\rho(\mathfrak{X})$.

Last, we have to show that $\hat{\mathfrak{X}}\setminus\rho(\mathfrak{X})$ is nowhere disconnecting, that is that each point of it has a fundamental system of open neighbourhoods which remain connected if we substract $\hat{\mathfrak{X}}\setminus\rho(\mathfrak{X})$. But for any $\gamma\in\hat{\mathfrak{X}}$, a fundamental system of open and connected neighbourhoods of γ is provided by the set $\{c(u)\mid U \text{ open, connected, } U\in\gamma\}$ and we have

$$c(U)=\rho(U)\cup\{\gamma'\in\hat{\mathfrak{X}}\setminus\rho(\mathfrak{X})\mid U\in\gamma'\}$$

which implies $c(U)\setminus(\hat{\mathfrak{X}}\setminus\rho(\mathfrak{X}))=\rho(U)$ which is connected because ρ is continuous. This completes the proof of Theorem 2.

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REFERENCES

[1] N. Boboc, Gh. Siretchi: Sur la compactification d un espace topologique, Bull. Math., de la Soc. Sci. Math. Phys. R.P.R. 5(53), 3-4,(1961), 155-167.

[2] C. Constantinescu, C. Cornea: Ideale Ränder Riemannscher Flächen, Springer, (1963).

[3] H. Freundenthal: Uber Enden topologischer Räume und Gruppen, Math. Z., 33, (1931) 692.

[4] M. Jurchescu: Spatii de acoperire ramificate si proprietatea I, in "Lucrarile Consfatuirii de geometrie si topologie", Iasi, 2-5 iunie (1958). Edit. Acad. R.S.R.

[5] M. Jurchescu : Bordered Riemann Surfaces, Math. Ann., 143 Band, 3 Heft (1961).

[6] M. Jurchescu : An Axiomatic for Cauchy Filters (unpublished).

[7] B. de Kerékjártó : Verlesungen über Topologie, Berlin, (1923).

[8] N. Radu : Inele locale, vol. 1, Edit. Acad. R.S.R., Buc., (1968).

[9] S. Stoilow : Lecons sur les principes topologiques de la theorie des fonctions analytiques, Paris (1956).

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ON QUASI-CONTINUOUS MULTIVALUED MAPS

Introduction

Various results on quasi-continuous multivalued maps appeared in the literature. Usually the upper and lower quasi-continuity and their suitable combinations are used in the assumptions of stated theorems. A natural notion of a quasi-continuous multivalued map [15] is not equivalent to simultaneous upper and lower quasi-continuity. However it is an extension of the quasi-continuity of a single-valued map. The main aim of the present paper is to show that, under general assumptions on topological spaces in question, some well known results for single-valued quasi-continuous maps may be extended to multivalued maps. The most interesting of them seem to be Theorems 1, 3, 7.

Some results and their proofs are analogous to those obtained in [6], [10], [11]. But various new aspects seem to be of interest. On the other hand we also show that the results cannot be obtained from those for single-valued maps via the Vietoris topology. We show that the Vietoris topology leads to a more restrictive notion of quasi-continuity.

1. Preliminaries and comparison with simultaneous upper and lower quasi-continuity

A subset A of a topological space X is said to be

- semi-open [8], if $A \subset \mathrm{Int}\, A$,
- semi-closed [3], [4], if $X \setminus A$ is semi-open.

The union of all semi-open sets contained in A is called the semi-interior of A [3], [4] and it is denoted by s-Int A. The intersection of all semi-closed sets containing A is called the semi-closure of A and we denote it by $\underline{A}$ [3], [4].

Let F be a multivalued map which to every point $x \in X$ assigns a nonempty subset of Y. We write simply $F: X \to Y$ instead of $F: X \to \mathcal{A}(Y)$, where $\mathcal{A}(Y)$ denotes the set of all nonempty subsets of Y. If $G \subset Y$, then $F^+(G) = \{x \in X : F(x) \subset G\}$, $F^-(G) = \{x \in X : F(x) \cap G \neq \emptyset\}$, [1].

In what follows X, Y will be topological spaces. Following [6], [11], [14] a multivalued map $F: X \to Y$ is called:

- upper quasi-continuous at $x_0 \in X$ if, for each open set $G \subset Y$ such that $F(x_0) \subset G$, we have $x_0 \in$ s-Int $F^+(G)$,
- lower quasi-continuous at x_0, if, for each open set $G \subset Y$ such that $F(x_0) \cap G \neq \emptyset$, we have $x_0 \in$ s-Int $F^-(G)$. The set of all points at which a map F is upper (lower) quasi-continuous will be denoted $E_u(F)$, $(E_l(F))$.

A map $F: X \to Y$ is called upper (lower) quasi-continuous if $E_u(F) = X$, $(E_l(F) = X)$. Any single-valued map $f: X \to Y$ can be considered as a multivalued map which assigns to any $x \in X$ the singleton $\{f(x)\}$.

Applying our definitions of upper (lower) quasi-continuity to a single-valued map, we see that they coincide with the definition of quasi-continuity of a single-valued map given by Kempisty [7].

A multivalued map F is called quasi-continuous [15] at $x_0 \in X$ if for arbitrary open sets $G_1, G_2 \subset Y$ such that $F(x_0) \subset G_1$ and $F(x_0) \cap G_2 \neq \emptyset$, and for any neighbourhood U of x_0 there exists a nonempty open set $V \subset U$ such that $F(x) \subset G_1$ and $F(x) \cap G_2 \neq \emptyset$ for each $x \in V$. We use $E^*(F)$ to denote the set of all points at which F is quasi-continuous.

A map F is quasi-continuous iff $E^*(F) = X$. Obviously in case of a single-valued map the above definition also reduces to the Kempisty definition of quasi-continuity.

In [15] mamy equivalent conditions of the quasi-continuity of a multivalued map are given.

Proposition 1. For a multivalued map F the following conditions are equivalent:

(a) F is quasi-continuous at x,

(b) for any open sets $G_1, G_2 \subset Y$ such that $x \in F^+(G_1) \cap F^-(G_2)$ we have $x \quad s\text{-Int}(F^+(G_1) \cap F^-(G_2))$,

(c) for each neighbourhood U of x and for arbitrary open sets $G_1, G_2 \subset Y$ such that $x \in F^+(G_1) \cap F^-(G_2)$ we have

$$U \cap \mathrm{Int}(F^+(G_1) \cap F^-(G_2)) \neq \emptyset.$$

Proposition 2. For a multivalued map $F: X \to Y$ the following conditions are equivalent:

(a) F is quasi-continuous,

(b) for arbitrary sets $B_1, B_2 \subset Y$ we have

$$F^-(B_1) \cup F^+(B_2) \subset F^-(\bar{B}_1) \cup F^+(\bar{B}_2),$$

(c) for arbitrary sets $B_1, B_2 \subset Y$ we have

$$F^+(\mathrm{Int}\, B_1) \cap F^-(\mathrm{Int}\, B_2) \subset s\text{-Int}(F^+(B_1) \cap F^-(B_2)).$$

We omit the standard proofs of Propositions 1, 2.

It follows from definitions that any quasi-continuous multivalued map is both upper and lower quasi-continuous. The converse is not true as the following example shows.

Example 1. Let R be the set of all real numbers with its natural topology. Define $F: R \longrightarrow R$ as

$$F(x) = \begin{cases} [0,1] & \text{if } x < 0 \\ [-1, -\frac{1}{2}] \cup [0,1], & \text{if } x = 0 \\ [-1,0], & \text{if } x > 0. \end{cases}$$

It is easy to verify that F is both lower and upper quasi-continuous. Choosing $G_1 = (-2, -\frac{1}{3}) \cup (-\frac{1}{2}, 2)$ and $G_2 = (-1, -\frac{1}{2})$ one has $F^+(G_1) \cap F^-(G_2) = \{0\}$. So (Proposition 1(c)) F is not quasi-continuous.

The set of discontinuity points of a compact-valued upper (lower) quasi-continuous multivalued map with values in a second countable topological space is of the first category [6]. The following result is of similar type but for quasi-continuous multivalued maps.

Theorem 1. Let X be a topological space and Y a second countable one. If $F: X \longrightarrow Y$ is a compact valued map, then $E_u(F)\cap E_1(F)\setminus E^*(F)$ is of the first category.

Proof. Let $\{V_n: n \geq 1\}$ denote a base of open sets in Y. Let $\mathcal{J}$ denote the set of all one-to-one finite sequences of natural numbers. Then

$$\mathcal{J} = \left\{(n_{k,1}, n_{k,2}, \ldots, n_{k,j(k)}): k \geq 1\right\}.$$

Put $W_k = \bigcup\left\{V_{n_{k,i}} : i = 1,2,\ldots,j(k)\right\}$ for $k \geq 1$. If $x_0 \in (E_u(F)\cap E_l(F))\setminus E^*(F)$, then there exists open sets $G_1, G_2 \subset Y$ such that

$$x_0 \in (\text{s-Int } F^+(G_1)\cap \text{s-Int } F^-(G_2))\setminus \text{s-Int}(F^+(G_1)\cap F^-(G_2)).$$

Since $F(x_0)$ is compact we can choose W_k and V_n such that $F(x_0)\subset W_k\subset G_1$ and $F(x_0)\cap V_n\subset F(x_0)\cap G_2$. Hence we have

$$x_0 \in (\text{s-Int } F^+(W_k)\cap \text{s-Int } F^-(V_n))\setminus \text{s-Int}(F^+(W_k)\cap F^-(V_n)).$$

Consequently

$$(E_u(F)\cap E_l(F))\setminus E^*(F)\subset \bigcup_{n,k=1}^{\infty} (\text{s-Int } F^+(W_k)\cap \text{s-Int } F^-(V_n)\setminus \text{s-Int}(F^+(W_k)\cap F^-(V_n))).$$

But we have

$$(\text{s-Int } F^+(W_k)\cap \text{s-Int } F^-(V_n))\setminus \text{s-Int}(F^+(W_k)\cap F^-(V_n))\subset$$
$$\subset(\text{s-Int } F^+(W_k)\cap \text{s-Int } F^-(V_n))\setminus \text{Int}(\text{s-Int } F^+(W_k)\cap \text{s-Int } F^-(V_n))\subset$$
$$\subset \text{Fr}(\text{s-Int } F^+(W_k)\cap \text{s-Int } F^-(V_n))\subset$$
$$\subset \text{Fr}(\text{s-Int } F^+(W_k))\cup \text{Fr}(\text{s-Int } F^-(V_n)).$$

Thus

$$E_u(F) \cap E_l(F) \setminus E^*(F) \subset \bigcup_{k=1}^{\infty} \mathrm{Fr}(s\text{-Int } F^+(W_k)) \cup$$

$$\cup \bigcup_{n=1}^{\infty} \mathrm{Fr}(s\text{-Int } F^-(V_n)).$$

Since the boundary of a semi-open set is nowhere dense [6], the proof is completed.

2. Quasi-continuity and somewhat continuity

A useful tool in the study of upper and lower quasi-continuous multivalued maps is the notion of upper and lower somewhat continuity [11].

A multivalued map $F\colon X \to Y$ is said to be upper (lower) somewhat continuous if for every open set $G \subset Y$ such that $F^+(G) \neq \emptyset$ ($F^-(G) \neq \emptyset$) we have $\mathrm{Int}\, F^+(G) \neq \emptyset$ ($\mathrm{Int}\, F^-(G) \neq \emptyset$). The upper (lower) quasi-continuous multivalued map $F\colon X \to Y$ may be characterized [11] as a map for which there exists a basis $\mathcal{B}$ of open sets in X such that the restriction F/B is upper (lower) somewhat continuous for every $B \in \mathcal{B}$.

To characterize our notion of a quasi-continuous multivalued map in the same way, we first introduce the notion of somewhat continuity of a multivalued map.

A multivalued map $F\colon X \to Y$ is said to be somewhat continuous if for any two open sets $G_1, G_2 \subset Y$ such that $F^+(G_1) \cap F^-(G_2) \neq \emptyset$ we have $\mathrm{Int}(F^+(G_1) \cap F^-(G_2)) \neq \emptyset$.

Note that in case of a single-valued map the upper somewhat continuity, lower somewhat continuity and somewhat continuity coincide and become the somewhat continuity introduced for single-valued maps in [2].

Further, note that somewhat continuity of a multivalued map implies both the upper and the lower somewhat continuity. The converse is not true. The reader may use Example 1 for proving this fact. Finally note that quasi-continuity of a multivalued map implies its somewhat continuity.

Proposition 3. A multivalued map $F: X \to Y$ is quasi-continuous if and only if there exists a basis $\mathcal{B}$ of open sets in X such that $F|B$ is somewhat continuous for every $B \in \mathcal{B}$.

Proof. To prove the necessity it is sufficient to prove that the quasi-continuity of F implies the somewhat continuity of $F|B$ for any open set $B \subset X$. So let $B \subset X$ be open and $G_1, G_2 \subset Y$ open sets such that $(F|B)^+(G_1) \cap (F|B)^-(G_2) \neq \emptyset$. Then we have

$$\mathrm{Int}((F|B)^+(G_1) \cap (F|B)^-(G_2)) = \mathrm{Int}(F^+(G_1) \cap F^-(G_2) \cap B) =$$

$$= B \cap \mathrm{Int}(F^+(G_1) \cap F^-(G_2)) \neq \emptyset,$$

according to Proposition 1(c).

The proof of the sufficiency runs in a similar way.

The somewhat continuity itself may be characterized by means of dense collection of subsets and by means of dense set. The situation again resembles the characterization of upper and lower somewhat continuity by means of upper and lower dense collections of subsets.

We recall the known notions and introduce the notion of density of a collection of subsets.

A collection $\mathfrak{U}$ of nonempty subsets of Y is said to be upper (lower) dense in a collection $\mathcal{V}$ of nonempty subsets of Y [12] if for any $V \in \mathcal{V}$ and any open set $G \subset Y$ such that $V \subset G$, $(V \cap G \neq \emptyset)$ there exists a set $U \in \mathfrak{U}$ such that $U \subset G$, $(U \cap G \neq \emptyset)$.

A collection $\mathfrak{U}$ of nonempty subsets of Y is said to be dense in a collection $\mathcal{V}$ of nonempty subsets of Y if for any $V \in \mathcal{V}$ and any two open sets $G_1, G_2 \subset Y$ such that $V \subset G_1$ and $V \cap G_2 \neq \emptyset$ there exists $U \in \mathfrak{U}$ such that $U \subset G_1$, $U \cap G_2 \neq \emptyset$.

Proposition 4. A multivalued map $F: X \to Y$ is somewhat continuous if and only if for any dense set $S \subset X$ the collection $\{F(x): s \in S\}$ is dense in the collection $\{F(x): x \in X\}$.

Proof. Let S be dense in X. Suppose that F is somewhat continuous. Let us take $F(x)$ for some $x \in X$. Let $G_1, G_2 \subset Y$ be open sets such that $F(x) \subset G_1$ and $F(x) \cap G_2 \neq \emptyset$. Then $F^+(G_1) \cap F^-(G_2) \neq \emptyset$ and since F is somewhat continuous we have $\mathrm{Int}(F^+(G_1) \cap F^-(G_2)) \neq \emptyset$. From the density of S we have a point $s \in \mathrm{Int}(F^+(G_1) \cap F^-(G_2))$. Thus $F(s) \subset G_1$, $F(s) \cap G_2 \neq \emptyset$ and the density of $\{F(s): s \in S\}$ in $\{F(x): x \in X\}$ is proved. Conversely, let F be not somewhat continuous. Then there exist two open sets $G_1, G_2 \subset Y$ such that $F^+(G_1) \cap F^-(G_2) \neq \emptyset$ and $\mathrm{Int}(F^+(G_1) \cap F^-(G_2)) = \emptyset$. The set $S = X \setminus F^+(G_1) \cap F^-(G_2)$ is dense in X. Take $x_0 \in F^+(G_1) \cap F^-(G_2)$ and the corresponding set $F(x_0)$. We have $F(x_0) \subset G_1$, $F(x_0) \cap G_2 \neq \emptyset$ but for any $s \notin S$ either $s \notin F^+(G_1)$ or $s \notin F^-(G_2)$. Thus $\{F(s): s \in S\}$ is not dense in $\{F(x): x \in X\}$.

From Proposition 3 and Proposition 4 we have the following result which is well known for single-valued functions.

Corollary. A multivalued map $F: X \longrightarrow Y$ is quasi-continuous if and only if for any dense set $S \subset X$ and any open set $G \subset X$ the collection $\{F(s): s \in G \cap S\}$ is dense in the collection $\{F(x): x \in G\}$.

3. Quasi-continuity and product spaces

If a single-valued map $f: X \times Y \longrightarrow Z$ is given, then the well known theorem of Kempisty [7] guarantees the quasi-continuity of f, under the assumptions that the sections f_x and f^y are quasi-continuous. The result of Kempisty concerns the case of real functions of real variables, but the generalizations to various types of general topological spaces are known [9], [10]. For multivalued maps on product spaces the product quasi-continuity was studied in [11] from such a point of view that the upper or lower quasi-continuity was examined under certain conditions on the sections. A natural question arises, if the quasi-continuity of the sections of a multivalued map $F: X \times Y \longrightarrow Z$ gives the quasi-continuity of F. The following results give the answer to some questions of mentioned type covering from a new point of view the Kempisty theorem and some related results. Remark that for a multivalued map $F: X \times Y \rightarrow Z$, $x \in X$ and $y \in Y$, we define the sections F_x, F^y such that $F_x: Y \rightarrow Z$ and $F^y: X \rightarrow Z$, where $F_x(y) = F(x,y) = F^y(x)$.

Theorem 2. Let X be a Baire space, Y second countable and Z a normal T_1 one. Let $M \subset X$ be of first category and $F: X \times Y \rightarrow Z$ a closed-valued multivalued map such that:

(i) F_x is somewhat continuous for every $x \in X \setminus M$,

(ii) F^y is both upper and lower quasi-continuous for every $y \in Y$,

(iii) F^y is somewhat continuous for every $y \in Y$.

Then F is somewhat continuous.

Proof. Suppose that F is not somewhat continuous. Then there exist open sets $G_1, G_2 \subset Z$ such that $F^+(G_1) \cap F^-(G_2) \neq \emptyset$ and

$$X \times Y \setminus F^+(G_1) \cap F^-(G_2) \text{ is dense in } X \times Y. \tag{1}$$

Denote $\{V_n: n \geqslant 1\}$ the open basis of Y. Let $(x_0,y_0) \in F^+(G_1) \cap F^-(G_2)$ and let $z \in F(x_0,y_0) \cap G_2$. Using the normality of Z we can choose open sets $W_1, W_2 \subset Z$ such that the following relations are:

$$F(x_0,y_0) \subset W_1 \subset \overline{W}_1 \subset G_1 \quad \text{and} \quad z \in W_2 \subset \overline{W}_2 \subset G_2. \tag{2}$$

Since F^{y_0} is somewhat continuous, we obtain from (2) that

$$H = \operatorname{Int}((F^{y_0})^+(W_1) \cap (F^{y_0})^-(W_2)) \neq \emptyset.$$

Put $A_n = \{x \in H \setminus M: V_n \subset (F_x)^+(W_1) \cap (F_x)^-(W_2)\}$. We show that $\bigcup_{n=1}^{\infty} A_n = H \setminus M$. The inclusion $\bigcup_{n=1}^{\infty} A_n \subset H \setminus M$ is obvious. Now let $x \in H \setminus M$. We have $F_x(y_0) = F(x,y_0) \subset W_1$ and simultaneously $F_x(y_0) \cap W_2 = F(x,y_0) \cap W_2 \neq \emptyset$. According to somewhat continuity of F_x there exists an open set $E \subset Y$ such that $F_x(y) \subset W_1$, $F_x(y) \cap W_2 \neq \emptyset$ for any $y \in E$. Thus $F_x(y) \subset W_1$, $F_x(y) \cap W_2 \neq \emptyset$

for any y belonging to a suitable $V_n \subset E$. So $x \in A_n$ and $\bigcup_{n=1}^{\infty} A_n = H \setminus M$ is proved.

Now we show that any A_n is nowhere dense. Let n be fixed. Choose an open set $U \subset X$, $U \neq \emptyset$. According to (1), there exists a point $(x_1,y_1) \in U \times V_n$ such that $(x_1,y_1) \notin F^+(G_1) \cap F^-(G_2)$. So at least one of the following assertions is true:

$$F(x_1,y_1) \cap (Y \setminus \overline{W}_1) \neq \emptyset, \tag{3}$$

$$F(x_1,y_1) \subset Y \setminus \overline{W}_2. \tag{4}$$

If (3) is satisfied, then for a point $z' \in F(x_1,y_1) \cap (Y \setminus \overline{W}_1)$ there exists an open set $W \subset Y$ such that $z' \in W \subset \overline{W} \subset Y \setminus \overline{W}_1$. Because F^{y_1} is lower quasi-continuous at x_1, there exists a nonempty open set $U_1 \subset U$ such that $F(x_1,y_1) \cap W \neq \emptyset$ for any $x \in U_1$. Then $y_1 \notin (F_x)^+(W_1)$ for any $x \in U_1$. Consequently $x \notin A_n$. Hence $U_1 \cap A_n = \emptyset$. If (4) holds, then an open set $W_o \subset Y$ can be taken in such a way that $F(x_1,y_1) \subset W_o \subset \overline{W}_o \subset Y \setminus \overline{W}_2$. Using the upper quasi-continuity of F^{y_1} at x_1 we obtain a nonempty open set $U_2 \subset U$ such that $F(x,y_1) \subset W_o$ for any $x \in U_2$. Thus $y_1 \notin (F_x)^-(W_2)$ for any $x \in U_2$. So $U_2 \cap A_n = \emptyset$. The nowhere density of A_n is proved and as a consequence we have that $\bigcup_{n=1}^{\infty} A_n = H \setminus M$ is of first category. It is a contradiction, because X is a Baire space.

Theorem 3. Let X be a Baire space, Y locally second countable and Z normal T_1. Let $M \subset X$ be a first category set and $F: X \times Y \longrightarrow Z$ a closed-valued map such that:
(i) F_x is quasi-continuous for every $x \in X \setminus M$,
(ii) F^y is quasi-continuous for every $y \in Y$.
Then F is quasi-continuous.

Proof. Choose a base $\mathcal{B}$ of $X \times Y$ consisting of the elements $A \times B$, where A runs over a base of X and B over a base of Y. We may suppose that each B is second countable.

Then the restriction $F|A\times B$ satisfies the assumptions (i), (ii), (iii) of Theorem 2. So $F|A\times B$ is somewhat continuous and, by Proposition 3, F is quasi-continuous.

Looking carefully at the proof of Theorem 2 one can see that the following theorem is true.

Theorem 4. Let X be a Baire space, Y locally second countable and Z normal T_1. Let $M\subset X$ be a first category set and $F: X\times Y \to Z$ a closed-valued map such that:

(i) F_x is quasi-continuous for every $x\in X\setminus M$,

(ii) F^y is both upper and lower quasi-continuous and somewhat continuous for every $y\in Y$.

Then F is quasi-continuous.

One can also see that under the assumption of compact-valued map the assumption of normality of the space may be weakened. We will formulate it explicitely for Theorem 3.

Theorem 5. Let X be a Baire space, Y locally second countable and Z regular. Let $M\subset X$ be a first category set and $F: X\times Y \to Z$ a compact-valued map such that (i) and (ii) of Theorem 3 is satisfied. Then F is quasi-continuous.

The somewhat continuity of sections does not imply the somewhat continuity of the multivalued map F even if X,Y,Z coincide with R. This follows from an example of a function $f: R \to R$ which has somewhat continuous sections, without being somewhat continuous [10].

The following question occurs if we compare (ii) of Theorem 3 and (ii) of Theorem 4. Is $F:X \to Y$ quasi-continuous, being both upper and lower quasi-continuous and moreover somewhat continuous? The answer is negative even in the case of $X = Y = R$ with the natural topology.

Example 2. Let $F: R \to R$ be defined as follows

$$F(x) = \begin{cases} [0,1], & \text{if } x<0,\\ [-1,-\frac{1}{2}]\cup[0,1], & \text{if } x=0,\\ [-1,0], & \text{if } 0<x<1,\\ \{0\}, & \text{if } x\geqslant 1.\end{cases}$$

Then F is both upper and lower quasi-continuous, somewhat continuous but not quasi-continuous.

4. Comparison with the quasi-continuity in Vietoris topology

Given a finite collection $E_i \subset Y$, $i = 1,2,\dots,n$, of nonempty sets, denote by $\langle E_1,E_2,\dots,E_n\rangle$ the collection of all such sets $A \subset Y$ for which $A \subset \bigcup_{i=1}^{n} E_i$ and $A \cap E_i \neq \emptyset$ for $i = 1,2,\dots,n$. Denote by $\mathcal{A}(Y)$ the collection of all nonempty subsets of Y. As it is well known [1], the collection of all $\langle G_1,G_2,\dots,G_n\rangle$, where G_i are open for $i \leqslant n$, $n \geqslant 1$, is a base for a topology in $\mathcal{A}(Y)$. Such topology is called Vietoris topology. The following notion of quasi-continuity of a multivalued map $F\colon X \to Y$ is quite natural.

A multivalued map $F\colon X \to Y$ is said to be $\mathcal{V}$-quasi-continuous at $x_0 \in X$ if it is quasi-continuous as a single-valued map $F\colon X \to \mathcal{A}(Y)$ at x_0, i.e. if for any $\langle G_1,G_2,\dots,G_n\rangle$ containing $F(x_0)$ and any neighbourhood U of x_0 there exists a nonempty open set $V \subset U$ such that $F(x) \in \langle G_1,G_2,\dots,G_n\rangle$ for any $x \in V$. We say that F is $\mathcal{V}$-quasi-continuous if it is $\mathcal{V}$-quasi-continuous at any $x \in X$.

Proposition 5. If $F\colon X \to Y$ is $\mathcal{V}$-quasi-continuous, then it is quasi-continuous. The converse is not true.

Proof. Let F be $\mathcal{V}$-quasi-continuous. Let $x_0 \in X$ and U be a neighbourhood of x_0. Let $G_1,G_2 \subset Y$ be open sets such that $F(x_0) \subset G_1$, $F(x_0) \cap G_2 \neq \emptyset$. Putting $H_1 = G_1$, $H_2 = G_1 \cap G_2$, we have $F(x_0) \in \langle H_1,H_2\rangle$. Hence there exists a nonempty open set $V \subset U$ such that $F(x) \in \langle H_1,H_2\rangle$ for any $x \in V$. Thus, $F(x) \subset H_1 \cup H_2 = G_1$ and $F(x) \cap H_2 \neq \emptyset$ implies $F(x) \cap G_2 \neq \emptyset$ for any $x \in V$. The quasi-continuity at x_0 is proved.

To show that the convergence is not true consider the set $X = A \cup \{0\}$, where $A = \left\{1, \frac{1}{2},\dots,\frac{1}{n},\dots\right\}$. Consider the natural topology on X. Let A_i, $i = 1,2,3$, be disjoint subsets of A such that $A = A_1 \cup A_2 \cup A_3$ and $0 \in \bar{A}_i$, $i = 1,2,3$. We will define $F\colon X \to R$ in the following way:

$$F(0) = \{-1,0,1\}$$

and

$$F(x) = \begin{cases} \{1\}, & \text{if } x \in A_1, \\ \{0\}, & \text{if } x \in A_2, \\ \{-1\}, & \text{if } x \in A_3. \end{cases}$$

Let G_1, G_2 be open sets such that $F(0) \subset G_1$, $F(0) \cap G_2 \neq \emptyset$. Let U be any neighbourhood of 0. Evidently $F(\frac{1}{n}) \subset F(0) \subset G_1$ for $n = 1,2,\ldots$. Since $F(0) \cap G_2 \neq \emptyset$, we have that G_2 contains at least one of the elements of the set $\{-1,0,1\}$. Without loss of generality we may suppose that $0 \in F(0) \cap G_2$. Choosing n_0 sufficiently large and such that $\frac{1}{n_0} \in A_2$, we obtain a nonempty open subset $V = \left\{\frac{1}{n_0}\right\} \subset U$ and evidently $F\left(\frac{1}{n_0}\right) \subset F(0)$, $F\left(\frac{1}{n_0}\right) \cap G_2 \neq \emptyset$. Thus F is quasi-continuous at 0. The quasi-continuity at any point $x \in A$ is evident because these points are isolated.

Choosing $G_1 = \left(-\frac{3}{2}, -\frac{1}{2}\right)$, $G_2 = \left(-\frac{1}{2}, \frac{1}{2}\right)$, $G_3 = \left(\frac{1}{2}, \frac{3}{2}\right)$, we have $F(0) \in \langle G_1,G_2,G_3\rangle$. But if $E \subset X$ is any subset such that $F(x) \in \langle G_1,G_2,G_3\rangle$ for any $x \in E$, then $E = \{0\}$ which is not an open set. Thus F is not $\mathcal{V}$-quasi-continuous at 0.

Note that many theorems for $\mathcal{V}$-quasi-continuous multivalued maps can be derived from these which are valid for single-valued quasi-continuous maps. Evidently this is true for any theorem in which no restriction in the single-valued case is given on the space Y. Of course if in a theorem for single-valued case some assumption on Y is made, then it is not necessarily true for the space $\mathcal{A}(Y)$ and such theorem cannot be transformed immediately without further considerations. We will not examine in this paper the details of possibility of such transfering. At the end we discuss the case in which simultaneous upper and lower quasi-continuity, quasi-continuity and $\mathcal{V}$-quasi-continuity coincide.

Note that a space is called extremally disconnected [5], if for any open set $U \subset X$ the closure $\bar{U}$ is an open set.

Theorem 6. Let X be an extremally disconnected space. Then for any multivalued map $F: X \to Y$ and any topological space Y the simultaneous upper and lower quasi-continuity, quasi-continuity and $\mathcal{V}$-quasi-continuity coincide.

Proof. In an extremally disconnected space the intersection of a finite number of semi-open sets is semi-open [13]. Suppose that F is simultaneously upper and lower quasi-continuous. The sets $F^+(G)$, $F^-(G)$ are semi-open for any open set $G \subset Y$. Let $x_0 \in X$ and $F(x_0) \in \langle G_1, G_2, \ldots, G_n \rangle$, where $G_i \subset Y$, $i = 1, 2, \ldots, n$, are open sets. Let U be any neighbourhood of x_0. Then

$$Z = F^+\left(\bigcup_{i=1}^{n} G_i\right) \cap \bigcap_{i=1}^{n} F^-(G_i) \cap U$$

is a nonempty semi-open set. Hence the set $V = \operatorname{Int} Z \subset U$ is nonempty and open, and $F(x) \in \langle G_1, G_2, \ldots, G_n \rangle$ for any $x \in V$. So F is $\mathcal{V}$-quasi-continuous at x_0. The rest of the proof is evident.

In what follows we say that q_1 holds for a multivalued map $F: X \to Y$ if F is simultaneously upper and lower quasi-continuous. We say that q_2 holds for F, if F is quasi-continuous. Finally we say that q_3 holds, if F is $\mathcal{V}$-quasi-continuous.

Theorem 7. Let X be a topological space dense in itself. If for any space Y and for any multivalued map $F: X \to Y$ the property q_i for F implies q_j for F, where $i < j$, $i = 1, 2$ and $j = 2, 3$, then X is extremally disconnected.

Proof. Let $i = 1$, $j = 3$. Suppose that the assumptions of the theorem for $i = 1$, $j = 3$ are satisfied and X is not extremally disconnected. Then there is an open set U such that $\bar{U}$ is not an open set. Choose $x_0 \in \operatorname{Fr} U$ and define $F: X \to R$ such that

$$F(x) = \begin{cases} \{1\}, & \text{if } x \in U \setminus \{x_0\}, \\ \{1, 0, -1\}, & \text{if } x = x_0, \\ \{-1\}, & \text{if } x \in X \setminus \bar{U}. \end{cases}$$

Careful examination shows that F is simultaneously upper and lower quasi-continuous but not $\mathcal{V}$-quasi-continuous. So the theorem is true for $i = 1$, $j = 3$.

If $i = 2$, $j = 3$ and the assumptions of the theorem are satisfied, then they are satisfied also for $i = 1$, $j = 3$. So X is extremally disconnected.

For the case $i = 1$, $j = 2$ the proof runs in the same way as in the case $i = 1$, $j = 3$. The symbols x_0 and U have the same meaning but instead of the multivalued map F we use the map $G: X \longrightarrow R$ defined in the following way

$$G(x) = \begin{cases} [0,1], & \text{if } x \in \bar{U} \setminus \{x_0\}, \\ \left[-1, -\frac{1}{2}\right] \cup [0,1], & \text{if } x = x_0, \\ [-1,0], & \text{if } x \in X \setminus \bar{U}. \end{cases}$$

REFERENCES

[1] C. Berge: Espaces topologiques. Fonctions multivoques. Paris 1966.

[2] K.R. Gentry, H.B. Hoyle: Somewhat continuous functions, Czechosl. Math. J. 21, 96(1971), 5-12.

[3] S.G. Crossley, S.K. Hildebrand: Semi-closure, Texas J. Sci. 22 (1971), 99-112.

[4] S.G. Corssley, S.K. Hildebrand: Semi-closed sets and semi-continuity in topological spaces, Texas J. Sci. 22 (1971), 123-126.

[5] R. Engelking: General topology. Warszawa 1977.

[6] J. Ewert, T. Lipski: Quasi-continuous multivalued mappings. Math. Slovaca 33 (1983), 69-74.

[7] S. Kempisty: Sur les fonctions quasicontinues, Fund. Math. 19 (1932), 184-197.

[8] N. Levine: Semi-open sets and semi-continuity in topological spaces, Amer. Math. Monthly 70 (1963), 36-41.

[9] N.F.G. Martin : Quasi-continuous functions on product spaces, Duke Math. J. 28 (1961), 39-44.

[10] T. Neubrunn : A generalized continuity and product spaces, Math. Slovaca 26 (1976), 97-99.

[11] T. Neubrunn : On quasi-continuity of multifunctions, Math. Slovaca 32 (1982), 147-154.

[12] T. Neubrunn : On Blumberg sets for miltifunctions, Bull. Acad. Polon. Sci. Math. 30 (1982), 109-113.

[13] O. Njåstad : On some classes of nearly open sets, Pac. J. Math. 15 (1965), 961-970.

[14] V. Popa : Asupra unei descompuneri a cvasicontinuității multifuncțiilor. Ștud. Cerc. Math. 27 (1975), 323-328.

[15] V. Popa : Certaines caracterisations des fonctions multivoques quasicontinues et faiblement continues, Ștud. Cerc. Mat. 37 (1985), 77-82.

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Ivan Ginchev

A THEOREM ON THE SUBDIFFERENTIAL CALCULATION

Let X, Y be Banach spaces, $a \in Y$, $\Lambda \in \mathcal{L}(X,Y)$, Im Λ closed in Y. Let $F: Y \to \bar{R}$ be a proper convex function, where $\bar{R}$ is the extended real line $\bar{R} = R \cup \{-\infty, +\infty\}$. If F is continuous in at least one point of Im $\Lambda + a$, and $G(x) = F(\Lambda x+a)$, then we prove that $\partial G(x) = \Lambda^* F(\Lambda x+a)$, where Λ^* is the conjugate of Λ. The symbol ∂ stands for the subgradient of the corresponding function. Remind that, for a convex function $f: X \to \bar{R}$, the subgradient $\partial f(x)$ is defined by $\partial f(x) = \{x^* \in X^*;\ f(y)-f(x) \geqslant \langle x^*, y-x \rangle$ for arbitrary $y \in X\}$. For $x \in X$ and $x^* \in X^*$ the notation $\langle x^*, x \rangle$ means as usually the value of x^* on x. For a Fréchet differentiable function $F: Y \to R$ we have $G'(x) = \Lambda^* F'(\Lambda x+a)$, where "the prime" stands for the Frechet derivative [1]. Our main result, established in Theorem 2, is therefore a generalization of this property to convex functions. This result can be useful in the approximation theory. Such an application is given to the finite-dimensional approximation in $C[\alpha, \beta]$, where $[\alpha, \beta] \subset R$, and to C(S), where S is a compact topological space.

Theorem 1. Let X, Y be linear topological spaces, $a \in Y$ and $\Lambda \in \mathcal{L}(X,Y)$ be a continuous linear operator. If $F: Y \to \bar{R}$ is a proper convex function and $G: X \to \bar{R}$ is given by $G(x) = F(\Lambda x+a)$, then G is convex and

(1)
$$\partial G(x) \supset \Lambda^* F(\Lambda x+a).$$

Proof. The convexity of G is evident. Let $y^* \in \partial F(\Lambda x+a)$. Then $G(x+h) - G(x) = F(\Lambda x+\Lambda h+a) - F(\Lambda x+a) \geqslant$ $\geqslant \langle y^*, \Lambda h\rangle = \langle \Lambda^* y^*, h\rangle$ for each $h \in X$. Hence $\Lambda^* y^* \in \partial G(x)$ and we obtain (1).

Now we shall show that, under some additional assumptions, in (1) an equality holds. In the proof we check the inclusion

$$G(x) \subset \Lambda^* \partial F(\Lambda x+a). \tag{2}$$

Theorem 2. Let X, Y be Banach spaces, $a \in Y$ and $\Lambda \in \mathcal{L}(X,Y)$ be a bounded linear operator from X to Y for which $\operatorname{Im}\Lambda$ is closed in Y. Let $F\colon Y \to \bar{R}$ be a proper convex function, continuous in at least one point $\tilde{y} \in \operatorname{Im}\Lambda + a$. Then the subdifferential $\partial G(x)$, $x \in X$, for the function $G\colon X \to \bar{R}$, $G(x) = F(\Lambda x+a)$, is obtained by the equality

$$\partial G(x) = \Lambda^* F(\Lambda x+a). \tag{3}$$

Proof. The proof is made in consecutive steps:

1°. If Λ is a linear isomorphism, i.e. $\operatorname{Im}\Lambda = Y$, $\operatorname{Ker}\Lambda = \{0\}$, then (3) is true.

Let $x^* \in \partial G(x)$. Then for each $h \in X$ holds $F(\Lambda x+\Lambda h+a) - F(\Lambda x+a) \geqslant \langle x^*, h\rangle$. For $y \in Y$ one has $F(\Lambda x+a+y) - F(\Lambda x+a) = F(\Lambda x+a+\Lambda\Lambda^{-1}y) - F(\Lambda x+a) \geqslant \langle x^*, \Lambda^{-1}y\rangle = \langle (\Lambda^{-1})^* x^*, y\rangle$. Hence $(\Lambda^{-1})^* x^* \in \partial F(\Lambda x+a)$. From $(\Lambda^{-1})^* = (\Lambda^*)^{-1}$ (see Lemma 1 below) one gets $(\Lambda^*)^{-1} x^* \in \partial F(\Lambda x+a)$ and $x^* \in \Lambda^* \partial F(\Lambda x+a)$.

2°. If Λ is an epimorphism, i.e. $\operatorname{Im}\Lambda = Y$, then (3) is true.

The subspace $L = \operatorname{Ker}\Lambda \subset X$ is closed and hence $\tilde{X} = X/L$ is a Banach space. Let $\pi\colon X \to \tilde{X}$, $\pi(x) = x + L$, be the natural epimorphism. The function $\tilde{G}\colon \tilde{X} \to \bar{R}$, $\tilde{G}(\tilde{x}) = F(\Lambda x+a)$, $x \in \tilde{x} = \pi(x)$, is correctly defined. From $\operatorname{Ker}\Lambda = L$ there follows the existence of $\tilde{\Lambda} \in \mathcal{L}(\tilde{X}, Y)$ for which $\Lambda = \tilde{\Lambda} \circ \pi$ and $\operatorname{Ker}\tilde{\Lambda} = \{0\}$, $\operatorname{Im}\tilde{\Lambda} = Y$, i.e. $\tilde{\Lambda}$ is a linear isomorphism of $\tilde{X}$ onto Y.

We have $\tilde{G}(\tilde{x}) = F(\tilde{\Lambda}\tilde{x}+a)$ and from the previous case $\partial\tilde{G}(\tilde{x}) = \tilde{\Lambda}^* \partial F(\tilde{\Lambda} x+a)$.

The further reasonings are based on the commutative diagrams

$$\begin{tikzcd} X \arrow[r, "\pi"] \arrow[dr, "\Lambda"'] & X/L = \tilde{X}, \arrow[d, "\tilde{\Lambda}"] \\ & Y \end{tikzcd} \qquad \begin{tikzcd} X^* \arrow[r, "\pi^*"] & (X/L)^* = \tilde{X}^* \\ & Y^* \arrow[ul, "\Lambda^*"] \arrow[u, "\tilde{\Lambda}^*"'] \end{tikzcd}$$

Let $x^* \in \partial G(x)$. Then $G(x+h) - G(x) \geqslant \langle x^*, h\rangle$, which for $h \in \operatorname{Ker}\Lambda = L$ gives $0 \geqslant \langle x^*, h\rangle$. Since L is a subspace of X, then $\langle x^*, h\rangle = 0$ for each $h \in L$, i.e. $L \subset \operatorname{Ker} x^*$. This inclusion implies the existence of a functional $\tilde{x}^* \in \tilde{X}^*$, such that $x^* = \pi^*(\tilde{x}^*)$. The functional $\tilde{x}^*$ can be defined by the equality $\langle x^*, x\rangle = \langle \tilde{x}^*, \pi x\rangle$, $x \in X$. The latter equality defines $\tilde{x}^*$ uniquely, since $\operatorname{Ker}\pi^* \subset (\operatorname{Im}\pi)^{\perp} = \tilde{X}^{\perp}$.

The inequality $G(x+h) - G(x) \geqslant \langle x^*, h\rangle$ implies $\tilde{G}(\pi x + \pi h) - \tilde{G}(\pi x) \geqslant \langle x^*, h\rangle = \langle \pi^* \tilde{x}^*, h\rangle = \langle \tilde{x}^*, h\rangle$. Let $\tilde{x} = \pi x$ and $\tilde{h} = \pi h$. Since $\operatorname{Im}\pi = \tilde{X}$, then $\tilde{G}(\tilde{x} + \tilde{h}) - \tilde{G}(\tilde{x}) \geqslant \langle \tilde{x}^*, \tilde{h}\rangle$ is satisfied for each $\tilde{h} \in \tilde{X}$, thus $\tilde{x}^* \in \partial \tilde{G}(\tilde{x})$, consequently $\pi^* \tilde{x}^* \in \partial G(x)$, $\pi x = \tilde{x}$. Indeed, from $\tilde{G}(\tilde{x} + \tilde{h}) - \tilde{G}(\tilde{x}) \geqslant \langle \tilde{x}, \tilde{h}\rangle$, $\tilde{h} = \pi h$, one has $G(x+h) - G(x) \geqslant \langle \tilde{x}^*, \pi h\rangle = \langle \pi^* \tilde{x}^*, h\rangle$, i.e. $\pi^* \tilde{x}^* \in \partial G(x)$.

We proved that $\pi^* \partial \tilde{G}(\pi x) = \partial G(x)$ and the correspondence $\pi^*\colon \partial G(\pi x) \longrightarrow \partial G(x)$ is one-to-one. Therefore $\partial G(x) = \pi^* \partial G(\pi x) = \pi^* \tilde{\Lambda}^* \partial F(\tilde{\Lambda}\pi x + a) = \Lambda^* \partial F(\Lambda x + a)$, Q.E.D. We have used $\Lambda = \tilde{\Lambda} \circ \pi$ and $\Lambda^* = \pi^* \circ \tilde{\Lambda}^*$, seen in the diagramms.

3^0. Let $H(x) = F(\Lambda x)$. Then

$$\partial H(x) = \Lambda^* \partial (F + \delta\{\operatorname{Im}\Lambda\})(\Lambda x), \tag{4}$$

where $\delta\{\operatorname{Im}\Lambda\}$ is the indicator function for $\operatorname{Im}\Lambda$.

Denote $\tilde{Y} = \operatorname{Im}\Lambda$. Since $\tilde{Y}$ is closed in Y, then $\tilde{Y}$ is a Banach space. Denote by $\tilde{\Lambda}$ the operator Λ as a mapping in $\mathcal{L}(X, \tilde{Y})$ and by $i\colon \tilde{Y} \longrightarrow Y$, $i(y) = y$, the inclusion mapping. Let $\tilde{F}\colon \tilde{Y} \to \bar{R}$, $\tilde{F}(\tilde{y}) = F(y)$, $y = i(\tilde{y})$. Since $H(x) = \tilde{F}(\tilde{\Lambda} x)$ and $\tilde{\Lambda}\colon X \to \tilde{Y}$ is an epimorphism, then accordingly to the previous case $\partial H(x) = \tilde{\Lambda}^* \partial \tilde{F}(\tilde{\Lambda} x)$. Therefore we must show that

$$\tilde{\Lambda}^* \tilde{F}(\tilde{\Lambda} x) = \Lambda^* \partial (F + \delta\{\operatorname{Im}\Lambda\})(\Lambda x). \tag{5}$$

Let $\tilde{y}^*\in\partial\tilde{F}(y)$ and $y^*\in{}^*Y$ be such that $\langle y^*,h\rangle = \langle\tilde{y}^*, i^{-1}h\rangle$, $h\in\operatorname{Im}\Lambda$. Then $F(y+h) - F(y) = \tilde{F}(\tilde{y}+i^{-1}h) - \tilde{F}(\tilde{y}) \geq \langle\tilde{y}^*, i^{-1}h\rangle = \langle y^*,h\rangle$, $h\in\operatorname{Im}\Lambda$. Therefore $y^*\in\partial(F+\delta\{\operatorname{Im}\Lambda\})(y)$, moreover $\Lambda^*y^* = \tilde{\Lambda}^*\tilde{y}$. Indeed, $\langle\Lambda^*y^*,x\rangle = \langle\tilde{y}^*,\Lambda x\rangle = \langle\tilde{y}^*, i^{-1}\Lambda x\rangle = \langle\tilde{y}^*,\Lambda x\rangle = \langle\tilde{\Lambda}^*\tilde{y}^*,x\rangle$, $x\in X$. This reasoning proves the inclusion $\subset$ in (5).

Let $y^*\in\partial(F+\delta\{\operatorname{Im}\Lambda\})(y)$, $y\in i(y)$. Then $F(y+h) - F(y) \geq \langle y^*,h\rangle$, $h\in\operatorname{Im}\Lambda$. Let $\tilde{y}^* = i^*(y^*)$. We have $\tilde{F}(\tilde{y}+\tilde{h}) - \tilde{F}(\tilde{y}) = F(y+i\tilde{h}) - F(y) \geq \langle y^*, i\tilde{h}\rangle = \langle\tilde{y}^*,\tilde{h}\rangle$, $\tilde{h}\in\tilde{Y}$. Hence $\tilde{y}^*\in\partial\tilde{F}(\tilde{y})$. On the other hand $\langle\tilde{\Lambda}^*\tilde{y}^*,x\rangle = \langle\tilde{y}^*,\tilde{\Lambda}x\rangle = \langle i^*y^*,\tilde{\Lambda}x\rangle = \langle y^*, i\tilde{\Lambda}x\rangle = \langle y^*,\Lambda x\rangle = \langle\Lambda^*y^*,x\rangle$, $x\in X$. Therefore $\Lambda^*y = \Lambda^*y$ and we get the inclusion $\supset$ in (5).

4°. The theorem is true in the case $a = 0$.

For the reason of convenience we write H instead of G, i.e. $H: X\to\bar{R}$, $H(x) = F(\Lambda x)$. We must show that

$$\partial H(x) = \Lambda^*\partial F(\Lambda x). \tag{6}$$

In the point $\tilde{y}\in\operatorname{Im}\Lambda$ the function F is continuous and $\delta\{\operatorname{Im}\Lambda\}$ is bounded, so the hypothesis of the Moreau-Rockafellar theorem is satisfied, which applied to (4) gives

$$\begin{aligned}\partial H(x) &= \Lambda^*\partial(F+\delta\{\operatorname{Im}\Lambda\})(\Lambda x) =\\ &= \Lambda^*(\partial F+\partial\delta\{\operatorname{Im}\Lambda\})(\Lambda x) =\\ &= \Lambda^*(\partial F+(\operatorname{Im}\Lambda)^{\perp})(\Lambda x) = \Lambda^*\partial F(\Lambda x).\end{aligned} \tag{7}$$

The equality $\partial\delta\{\operatorname{Im}\Lambda\}(\Lambda x) = (\operatorname{Im}\Lambda)^{\perp}$ is established in Lemma 2. If $y^*\in\operatorname{Im}\Lambda$, then $\langle\Lambda^*y^*,x\rangle = \langle y^*,\Lambda x\rangle = 0$, $x\in X$, whence the latter equality in (7) follows.

5°. The theorem is true in the general case.

Let $H: Y\to\bar{R}$, $H(y) = F(y+a)$. Evidently H is a convex function and from the equivalence of $H(y+h) - H(y)\geq\langle y^*,h\rangle$ with $F(y+a+h) - F(y+a)\geq\langle y^*,h\rangle$, $h\in Y$, follows $\partial H(y) = \partial F(y+a)$.

Now for $G(x) = F(\Lambda x+a) = H(\Lambda x)$ from the previous case follows $\partial G(x) = \Lambda^*\partial H(\Lambda x) = \Lambda^*\partial F(\Lambda x+a)$.

The following lemmas complete the proof.

Lemma 1. Let X, Y be Banach spaces and $\Lambda \in \mathcal{L}(X,Y)$ be a linear isomorphism. Then $\Lambda^* \in \mathcal{L}(Y^*,X^*)$ is a linear isomorphism and $(\Lambda^{-1})^* = (\Lambda^*)^{-1}$.

Proof. Accordingly to the Annulator Lemma ([1], p. 130), $\operatorname{Im} \Lambda^* = (\operatorname{Ker}\Lambda)^{\perp} = \{0\}^{\perp} = X^*$.

The condition $\operatorname{Ker} \Lambda^* = \{0\}$ is also satisfied. Indeed, from $y^* \in \operatorname{Ker} \Lambda^*$ we have $0 = \langle \Lambda^* y^*, x \rangle = \langle y^*, \Lambda x \rangle$ and, since Λ is an isomorphism, we get $y^* = 0$.

Thus $\Lambda^* \in \mathcal{L}(Y^*,X^*)$ is a linear isomorphism. Therefore there exists $(\Lambda^*)^{-1} \in \mathcal{L}(X^*,Y^*)$ and for each $x^* \in X^*$ we have $\langle x^*, x \rangle = \langle \Lambda^*(\Lambda^*)^{-1}x^*, x \rangle = \langle (\Lambda^*)^{-1}x^*, \Lambda x \rangle$. Let $y \in Y$ be arbitrary. We choose x to satisfy $\Lambda x = y$, whence $\langle (\Lambda^*)^{-1}x^*, y \rangle = \langle x^*, \Lambda^{-1}y \rangle = \langle x^*, \Lambda^{-1}y \rangle = \langle (\Lambda^{-1})^* x^*, y \rangle$, $y \in Y$. Hence $(\Lambda^*)^{-1} = (\Lambda^{-1})^*$.

Lemma 2. Let Y be a Banach space and $L \subset Y$ be a closed subspace in Y. Then

$$\partial\delta\{L\}(y) = L^{\perp}, \quad y \in L. \tag{8}$$

Proof. The condition $y^* \in \partial\delta\{L\}(y)$ is equivalent to $0 \geq \langle y^*, h \rangle$, $h \in L$. Since L is a subspace, hence we get $0 = \langle y^*, h \rangle$, $h \in L$, i.e. $y^* \in L^{\perp}$.

Let us make the following remarks.

Remark 1. For the cases 1°, 2° and 3° in the proof of Theorem 2 the corresponding statements are true without the assumption that F is continuous at a point of $\Lambda x + a$. The latter condition is necessary to apply the Moreau-Rockafellar Theorem.

Remark 2. In Lemma 1 it is established that, if $\Lambda \in \mathcal{L}(X,Y)$ is a linear isomorphism, then $\Lambda^* \in \mathcal{L}(Y^*,X^*)$ is a linear isomorphism. More general, it is true that, if $\Lambda \in \mathcal{L}(X,Y)$ is an epimorphism, then Λ^* is a linear isomorphism from Y onto $(\operatorname{Ker}\Lambda)^{\perp}$.

The following theorem is a corollary of Theorem 2.

Theorem 3. Let X, Y be Banach spaces, $\Lambda \in \mathcal{L}(X,Y)$ be a linear operator such that $\operatorname{Im} \Lambda$ is closed in Y. Let $b \in Y$ and $b \notin \operatorname{Im}\Lambda$. Then $\hat{x} \in X$ is the solution of the problem

$$(9) \qquad \|b - \Lambda x\|_Y \to \min$$

if and only if there exists a linear continuous functional $y^* \in Y^*$ such that

$$(10) \qquad \langle y^*, \Lambda x\rangle = 0 \quad \text{for each } x \in X,$$

$$(11) \qquad \|y^*\| = 1,$$

$$(12) \qquad -\langle y^*, b\rangle = \|b - \Lambda\hat{x}\|_Y.$$

Proof. Let $a = -b$ in Theorem 2, $F: Y \to R$, $F(y) = \|y\|_Y$. Then $F(\Lambda x + a) = \|b - \Lambda x\|_Y$. The element $\hat{x} \in X$ is a solution of (9) if and only if $0 \in \partial G(\hat{x})$, where $G(x) = F(\Lambda x + a)$. From Theorem 2, $\partial G(x) = \Lambda^* \partial F(\Lambda x + a)$ and from ([1], p.230)

$$\partial F(y) = \begin{cases} B^* = \{y^* \in Y^*;\ \|y^*\| < 1\} & \text{for } y = 0, \\ \{y^* \in Y^*;\ \|y^*\| = 1,\ \langle y^*, y\rangle = \|y\|_Y\} & \text{for } y \neq 0. \end{cases}$$

Since $a + \Lambda x = -b + \Lambda x \neq 0$ for each $x \in X$, then $\partial G(x) = \Lambda^*\{y^* \in Y^*;\ \|y^*\| = 1,\ \langle y^*, a + \Lambda x\rangle = \|a + \Lambda x\|_Y\}$. Thus the condition $0 \in \partial G(x)$ means that there exists $y^* \in Y^*$ such that $\|y^*\| = 1$, $\langle y^*, a + \Lambda x\rangle = \|a + \Lambda x\|_Y$ and $0 = \Lambda^* y^*$. The latter equality is equivalent to $0 = \langle \Lambda^* y^*, x\rangle = \langle y^*, \Lambda x\rangle$ for each $x \in X$, i.e. to (10).

Under condition (10) the equality $\langle y^*, a + \Lambda x\rangle = \|a + \Lambda x\|_Y$ is equivalent to $\langle y^*, a\rangle = \|a + \Lambda x\|_Y$, i.e. to (12).

Observe that problem (9) is convex, so $\hat{x}$ is an absolute minimum.

The following well known result follows from Theorem 3.

Corollary 1. Let H be a Hilbert space and $L \subset H$ be a closed subspace of H. Then $\hat{x} \in L$ realizes the minimal distance from b to L, i.e.

$$(13) \qquad \|b - \hat{x}\| = \min_{x \in L} \|b - x\|$$

if and only if $b - \hat{x} \perp L$, i.e.

$$(14) \qquad (b - \hat{x}, x) = 0 \quad \text{for each } x \in L.$$

Proof. Missing the trivial case $b \in L$, suppose that $b \notin L$. In the previous theorem put $X = Y = H$ and let $\Lambda = P_L$ be the orthogonal projector of H onto L. Let y^* satisfy (10) - (12) and $\langle y^*, y\rangle = (y,c)$, where $(.,.)$ is the scalar product in H. From the Cauchy-Buniakowski inequality

$-\langle y^*, b\rangle = \langle y^*, \Lambda x - b\rangle \leqslant \|y^*\| \|\Lambda x - b\| = \|\Lambda x - b\| = -\langle y^*, b\rangle$.

The extreme terms are equal, which is possible if and only if $c = k(\Lambda\hat{x} - b)$, $k \geqslant 0$. From (11) $k \neq 0$ and from (10) $(\Lambda x, \Lambda\hat{x} - b) = 0$. Since $\Lambda x = x$ for $x \in L$, then (15) is satisfied.

In converse, if condition (14) is satisfied, then the functional $\langle y^*, y\rangle = (y, (\hat{x} - b)/\|\hat{x} - b\|)$ satisfies (10)-(12).

Corollary 2. Let Y be a Banach space and $\Lambda \in \mathcal{L}(R^n, Y)$, $\Lambda x = \sum_{r=1}^{n} x_i y_i$, where $y_i \in Y$ are fixed elements in Y. Let $b \in Y$, $b \notin \mathrm{Im}\,\Lambda$. Then $\hat{x}$ is a solution of the approximation problem (9) if and only if there exists a continuous linear functional $y^* \in Y^*$ satisfying the conditions (11), (12) and

(10') $$\langle y^*, y_i\rangle = 0, \quad i = 1, 2, \ldots, n.$$

Using Corollary 2 we obtain a theorem of Chebyshevian type for the finite dimensional approximation problem in $C([\alpha, \beta])$.

Theorem 4. Let $Y \in C([\alpha, \beta])$, $b = b(.) \in C([\alpha, \beta])$ and $\Lambda \in \mathcal{L}(R^n, Y)$, $\Lambda x = \sum_{i=1}^{n} x_i y_i(.)$, where the functions $y_i(\cdot) \in C([\alpha, \beta])$ satisfy the condition

(15) $$\begin{vmatrix} y_1(t_1) & y_1(t_2) & \ldots & y_1(t_n) \\ y_2(t_1) & y_2(t_2) & \ldots & y_2(t_n) \\ \ldots & \ldots & \ldots & \ldots \\ y_n(t_1) & y_n(t_2) & \ldots & y_n(t_n) \end{vmatrix} \neq 0$$

for arbitrary $\alpha \leqslant t_1 < t_2 < \ldots < t_n \leqslant \beta$. Then $\hat{x} \in R$ is the solution of the approximation problem (9) if and only if there exist n+1 points $\alpha \leqslant t_1 < t_2 < \ldots < t_{n+1} \leqslant \beta$ such that for some $s = \pm 1$ the condition for Chebyshevian alternance is satisfied

$$(16)\qquad -b(t_j) + (\Lambda\hat{x})(t_j) = s(-1)^j\|b-\Lambda\hat{x}\|_{C([\alpha,\beta])},\quad j=1,2,\dots,n+1.$$

Proof. 1°. Let $\alpha \leqslant t_1 < t_2 < \dots < t_{n+1} \leqslant \beta$ and $\Delta_j = \Delta_j(t_1,t_2,\dots,t_{n+1})$ denote the algebraic completion $\Delta_{n+1,j}$ of the determinant

$$\begin{vmatrix} y_1(t_1) & y_1(t_2) & \dots & y_1(t_{n+1}) \\ \dots & \dots & \dots & \dots \\ y_n(t_1) & y_n(t_2) & \dots & y_n(t_{n+1}) \\ * & * & \dots & * \end{vmatrix}.$$

Then there exists a number $\delta = \pm 1$ such that sign $\Delta_j = \delta(-1)^j$, $j = 1,2,\dots,n+1$.

Indeed, each two neighbouring elements in the sequence $\Delta_1, \Delta_2, \dots, \Delta_{n+1}$ have different signs. Let

$$\varphi(t) = \begin{vmatrix} y_1(t_1) & \dots & y_1(t_{i-1}) & y_1(t) & y_1(t_{i+2}) & \dots & y_1(t_{n+1}) \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ y_n(t_1) & \dots & y_n(t_{i-1}) & y_n(t) & y_n(t_{i+2}) & \dots & y_n(t_{n+1}) \end{vmatrix}.$$

We must show that sign $\varphi(t_i)$ = sign $\varphi(t_{i+1})$. If the converse is true, then $\varphi(t) = 0$ for some $t \in [t_i, t_{i+1}]$, which contradicts the condition (15).

2°. Sufficiency. Let condition (16) be fulfilled. A linear continuous functional on $C([\alpha,\beta])$ can be identified with a function with bounded variation. We look for a functional $\mu(t) = \sum_{j=1}^{n+1} \mu_j \theta(t-t_j)$ satisfying (10′), (11), (12). Here μ_j are constants and $\theta(\cdot)$ is the Heaviside function, i.e. $\theta(t) = 1$ for $t \geqslant 0$ and $\theta(t) = 0$ for $t < 0$. A functional of this type exists, if the constant μ_j satisfy

$$(17)\qquad \sum_{j=1}^{n+1} \mu_j\, y_i(t_j) = 0,\quad i = 1,2,\dots,n,$$

$$\sum_{j=1}^{n+1} |\mu_j| = 1, \tag{18}$$

$$-\sum_{j=1}^{n+1} b(t_j) = \sum_{j=1}^{n+1} \mu_j(\Lambda\hat{x}(t_j)-b(t_j)) = (-1)^s \sum_{j=1}^{n+1} (-1)^j \mu_j r, \tag{19}$$

where $r = \|b - \Lambda\hat{x}\|_{C([\alpha,\beta])}$.

Putting $\mu_j = \varkappa\Delta_j$, $j = 1,2,\dots,n+1$, we make condition (17) true. Condition (18) is satisfied for $\varkappa = \varepsilon \Big/ \sum_{j=1}^{n+1} |\Delta_j|$, $\varepsilon = \pm 1$. From the previous reasoning we have $|\Delta_j| = \delta(-1)^j \Delta_j$, $\delta = \pm 1$, $j = 1,2,\dots,n+1$. Hence $-\sum_{j=1}^{n+1} \mu_j\, b(t_j) = s \sum_{j=1}^{n+1} (-1)^j \varkappa\Delta_j r =$ $= s\varkappa\delta r \sum_{j=1}^{n+1} |\Delta_j| = \delta s \varepsilon r = r$, if ε is chosen so that $\varepsilon = s\delta$.

3°. Necessity. Let $\hat{x}$ is a solution of (9) and let (16) be not satisfied. Then there exist at most $k \leqslant n$ sets $Q_1, Q_2, \dots, Q_k$ satisfying the conditions:

(a) $Q_j \subset [\alpha,\beta]$, $j = 1,2,\dots,k$,

(b) $\sup Q_j < \inf Q_{j+1}$, $j = 1,2,\dots,k-1$,

(c) there exists a number $s = \pm 1$, such that $\Lambda\hat{x}(t) - b(t) = = s(-1)^j$, $t \in Q_j$, $j = 1,2,\dots,k$, where $r = \|\Lambda\hat{x} - b\|_{C([\alpha,\beta])}$.

Let μ be a linear functional satisfying conditions (10), (11), (12). Then the support of μ is in $Q = = Q_1 \cup \dots \cup Q_k$, i.e. $\int_Q |d\mu(t)| = 1$. Condition (10′) implies

$$\sum_{j=1}^{k} \int_{Q_j} y_i(t)\, d\mu(t) = 0, \quad i = 1,2,\dots,n. \tag{20}$$

Condition (12) implies $\operatorname{sign} \int_{Q'} d\mu(t) = s(-1)^j$ for each measurable $Q' \subset Q_j$, $j = 1,2,\dots,k$, whence $\int_{Q_j} |d\mu(t)| = s(-1)^j \int_{Q_j} d\mu(t)$.

Using the mean-value theorem for the integrals in (20) we conclude that for some $t_j \in Q_j$, $j = 1,2,\dots,k$, we have

$$(21) \qquad \sum_{j=1}^{k} y_i(t_j) \int_{Q_j} d\mu(t) = 0, \quad i = 1,2,\dots,n.$$

From $\alpha \leqslant t_1 < t_2 < \dots < t_k \ \beta$ and condition (15) we see that $\int_{Q_j} d\mu(t) = 0$, for (21) can be considered as a homogeneous linear system of rank k in respect to $\int_{Q_j} d\mu(t)$, $j = 1,2,\dots,k$, whence a contradiction is obtained, for $1 = \int_Q |d\mu(t)| = \sum_{j=1}^{k} \int_{Q_j} |d\mu(t)| = \sum_{j=1}^{k} s(-1)^j \int_{Q_j} d\mu(t) = 0$.

The best approximation in the space of continuous functions with a function in a finite dimensional subspace is well studied. Close in spirit to Corollary 2 is Theorem 3.3.7 in [2] which is given below as Theorem 5. The applied proof is based on the developped method and differs from that one, given in J.P. Laurent's monograph [2].

Theorem 5. Let S be a compact topological space. Then the element $g \in V$ is the best approximation of $f \in C(S)$, $f \notin V$, in the n-dimentional subspace $V \subset C(S)$ if and only if there exist points $s_1, s_2, \dots, s_k \in S$, $k \leqslant n+1$, integers $\varepsilon_j = \pm 1$ and positive reals ρ_j, $j = 1,2,\dots,k$, satisfying the conditions

$$(22) \qquad \varepsilon_j\, e(s_j) = \|e\|, \quad j = 1,\dots,k, \text{ where } e = g - f,$$

$$(23) \qquad \sum_{j=1}^{k} \rho_j \varepsilon_j h(s_j) = 0 \quad \text{for each } h \in V.$$

Proof. We apply Theorem 3 putting $X = R^n$, $Y = C(S)$, $\Lambda \in \mathcal{L}(X,Y)$, $\Lambda x = \sum_{i=1}^{n} x_i y_i$, where $y_1, y_2, \dots, y_n$ form a base in V, $b = f$. The best approximation for f in V is $g = \sum_{i=1}^{n} \hat{x}_i y_i$.

Sufficiency. Let conditions (22) and (23) be satisfied. We check that the functional $y^* \in C^*(S)$, $\langle y^*, y\rangle = \varkappa \sum_{j=1}^{k} \rho_j \varepsilon_j y(s_j)$ with $\varkappa = \left[\sum_{j=1}^{k} \rho_j\right]^{-1}$ satisfies conditions (10)-(12).

Condition (10) is true, since for $h \in V$, accordingly to (23), we have $\langle y^*, h\rangle = \varkappa \sum_{j=1}^{k} \rho_j \varepsilon_j h(s_j) = 0$.

Condition (11) follows from the equalities $\|y^*\| = \sup_{\|y\| \leq 1} \varkappa \sum_{j=1}^{k} \rho_j \varepsilon_j y(s_j) = |\varkappa| \sum_{j=1}^{k} \rho_j = 1$.

Condition (12) is satisfied, since $-\langle y^*, b\rangle = \langle y^*, -f\rangle = \langle y^*, g-f\rangle = \langle y^*, e\rangle = \varkappa \sum_{j=1}^{k} \rho_j \varepsilon_j e(s_j) = \varkappa \sum_{j=1}^{k} \rho_j \|e\| = \|e\| = \|f - \Lambda\hat{x}\|$.

Necessity. Let g be the best approximation for f in V. Then, accordingly to Theorem 3, there exists $y^* \in C^*(S)$ such that

(24) $$\langle y^*, h\rangle = 0 \quad \text{for each} \quad h \in V,$$

(25) $$\|y^*\| = 1,$$

(26) $$-\langle y^*, f\rangle = \langle y^*, g-f\rangle = \langle y^*, e\rangle = \|e\|.$$

Let $\langle y^*, y\rangle = \int_S y(s)\, d\mu(s)$. Define $Q^+ = \{s \in S;\ e(s) = \|e\|\}$, $Q^- = \{s \in S;\ e(s) = -\|e\|\}$ and $Q = Q^+ \cup Q^-$. The set Q is compact, moreover Q^+ and Q^- are non-intersecting compacts. From (25), (26) it follows that the Radon decomposition (Y^+, Y^-) for the measure μ satisfies $Y^+ \subset Q^+$, $Y^- \subset Q^-$, whence condition (24) is written in the form

(27) $$\int_{Q^+} h(s)\, d\mu(s) + \int_{Q^-} h(s)\, d\mu(s) = 0 \quad \text{for each} \quad h \in V.$$

Introduce the functions $\varepsilon(\circ): Q \to \{-1, 1\}$, $\varepsilon(s) = 1$ for $s \in Q^+$, $\varepsilon(s) = -1$ for $s \in Q^-$ and $\Phi : Q \to R^n$, $\Phi(s) = (\varepsilon(s) y_1(s), \varepsilon(s) y_2(s), \ldots, \varepsilon(s) y_n(s))$. The function Φ is continuous on Q.

Condition (27) applied for y_i, $i = 1,2,\dots,n$, together with (25), give $0 \in \overline{\operatorname{conv}}\{\Phi(s);\ s \in Q\} = \operatorname{conv}\{\Phi(s);\ s \in Q\} \subset R^n$. The equality follows from the compactness of Q and the continuity of Φ. From the Caratheodory theorem $0 = \sum_{j=1}^{n+1} \rho_j \Phi(s_j)$, $\rho_j \geqslant 0$. Removing these s_j for which $\rho_j = 0$ and eventually renumbering the remaining s_j, we have shown that there exist a positive number $k \leqslant n+1$, real numbers $\rho_j > 0$ and elements $s_j \in Q$, $j = 1,2,\dots,k$, such that

$$\sum_{j=1}^{k} \rho_j\, \varepsilon(s_j) y_i(s_j) = 0, \quad i = 1,2,\dots,n. \tag{28}$$

Put $\varepsilon_j = \varepsilon(s_j)$. The relation $s_j \in Q$ means that $\varepsilon_j\, e(s_j) = \|e\|$ which is condition (23). From condition (28) it follows that $\sum_{j=1}^{k} \rho_j \varepsilon_j h(s_i) = 0$ for each $h \in \operatorname{lin}\{y_1, y_2, \dots, y_n\} = V$ which is the condition (23).

The following theorem is a general case analogue of Theorem 4.

Theorem 6. Let S be a compact topological space and the n-dimensional subspace $V \subset C(S)$ is the linear span of $y_1, y_2, \dots, y_n$. Suppose that for arbitrary $s_1, s_2, \dots, s_n$ in S, $s_i \neq s_j$ for $i \neq j$, it is true that

$$\det(y_i(s_j)) \neq 0. \tag{29}$$

Then the number k in Theorem 5 is equal to n+1.

Proof. Suppose $k \leqslant n$. Then (23) gives $\sum_{j=1}^{k} \rho_j \varepsilon_j y_i(s_j) = 0$, $i = 1,2,\dots,n$. This is a linear homogeneous system of rank k with respect to $\rho_j \varepsilon_j$. Hence $\rho_j \varepsilon_j = 0$, $j = 1,2,\dots,k$, which is a contradiction.

REFERENCES

[1] V.M. Alekseev, V.M. Tikhomirov, S.V. Fomin: Optimal control. Moscow 1979 (in Russian).

[2] P.J. Laurent: Approximation et optimisation, Paris 1972.

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GLOBAL SOLUTION AND STABILITY FOR SOME NONLINEAR HYPERBOLIC EQUATIONS

1. Introduction

We shall consider the nonlinear hyperbolic equation of the following form

$$u_{tt} + 2bu_t + cu = Au + f(u, u_t), \tag{1.1}$$

$$u(0) = u_0, \tag{1.2}$$

$$u_t(0) = u_1, \tag{1.3}$$

where u: t → X and X is the Hilbert space. The operator A is unbounded selfadjoint positive definite linear operator with the domain D(A) dense in X, b and c are constants. The problem of the existence of the global solution and its stability will be investigated. The results will be applied in the last section for some partial differential equations. We shall assume that the linear part of Eq. (1.1) describes the damped oscilation, or in other words we introduce the following assumptions.

Assumption 1.

(i) $$b > 0, \quad c \geqslant 0,$$

(ii) $$(Ax, x) > 0.$$

Let us define a new unknown function v by the relations $u = e^{\omega t}v$. The Eq. (1.1) in terms of the function v can be written as follows.

Case 1. For $b^2 > c$, the equation takes the form

$$v_{tt} + \gamma\cdot v_t = -Av + e^{-\omega t}\tilde{F}(e^{\omega t}v, e^{\omega t}v_t) \tag{1.4}$$

where: $\omega = -b + \sqrt{b^2 - c}$, $\gamma = \sqrt{b^2 - c}$.

Case 2. For $b^2 < c$ Eq. (1.1) can be transformed to the following equation

$$v_{tt} + \alpha v = -Av + e^{-\omega t}\,\tilde{F}(e^{\omega t}v, e^{\omega t}v_t), \tag{1.5}$$

where $\omega = -b$, $\alpha = c - b^2$.

The equation of the form (1.4) and (1.5) with $\tilde{F} = 0$ was considered by several authors [3], [4], [5] and for nonlinear case with some nonlinearity by Lions [6].

Now the basic properties of the linear equation corresponding to (1.4) and (1.5) will be summarized. Let us define $v_t = z$ and $\varphi = (v,z)^T$. The Eq. (1.4) and (1.5) can be written in the form

$$\varphi_t = \mathcal{A}_1\varphi + \begin{pmatrix} 0 \\ e^{-\omega t}\,\tilde{F}(e^{\omega t}z) \end{pmatrix} \tag{1.6}$$

where

$$\mathcal{A}_1 = \begin{pmatrix} 0 & I \\ -A, & \gamma \end{pmatrix}, \tag{1.7}$$

$$\mathcal{A}_2 = \begin{pmatrix} 0\ , & I \\ -(A-\alpha), & 0 \end{pmatrix}. \tag{1.8}$$

We have

Lemma 1. If $(Ax,x) > 0$ then the operators $\mathcal{A}_i$ $(i = 1,2)$ are generators of strongly continuous groups in the space $\mathcal{H}_0 = D(A^{1/2})\times X$ with the domain $D(\mathcal{A}_i) = D(A)\times D(A^{1/2}) = \mathcal{H}_1$. Moreover

$$\|e^{\mathcal{A}_i t}\|_{\mathcal{H}} \le 1, \quad t > 0. \tag{1.9}$$

For sake of completeness we shall give the proof.

Proof. Let

$$\mathcal{A}_0 = \begin{pmatrix} 0, & I \\ -A, & 0 \end{pmatrix}.$$

We have $\mathrm{Re}(\mathcal{A}_0\varphi,\varphi) = 0$ and for $\mathrm{Re}\,\lambda \neq 0$ the equation

$$(\lambda I + \mathcal{A}_0)\varphi = (f_1, f_2)^T$$

has the solution $\varphi \in \mathcal{H}_1$ for each $(f_1,f_2)^T \in \mathcal{H}_0$, what follows from the straightforward calculation

$$v = (A + \lambda^2)^{-1} f_1 - \lambda(A + \lambda^2)^{-1} f_2,$$

$$z = \lambda(A + \lambda^2)^{-1} f_2 - \lambda^2(A + \lambda^2)^{-1} f_1 + f_1.$$

In other words the operators $\mathcal{A}_0$ and $-\mathcal{A}_0$ are maximal dissipative operators and $\mathcal{A}_0$ generates the strongly continuous group. The operator $\mathcal{A}_i$ are also generators of the strongly continuous group as bounded perturbation of the operator $\mathcal{A}_0$. The estimation (1.9) follows from the inequalities $\mathrm{Re}(\mathcal{A}_i\varphi,\varphi)_{\mathcal{H}} \leq 0$ for $i = 1,2$.

Let us define $\mathcal{H}_\alpha = D(A^{1/2+\alpha/2}) \times D(A^{\alpha/2})$ for $0 \leq \alpha \leq 2$. The spaces $\mathcal{H}_\alpha$ are Hilbert spaces with the scalar product

$$(\varphi_1,\varphi_2)_{\mathcal{H}_\alpha} = (A^{1/2+\alpha/2}u_1, A^{1/2+\alpha/2}u_2)_X + (A^{\alpha/2}z_1, A^{\alpha/2}z_2).$$

The spaces $\mathcal{H}_\alpha$ can be defined as the interpolation spaces between $\mathcal{H}_0$, $\mathcal{H}_1$ and $\mathcal{H}_1$, $\mathcal{H}_2$, where $\mathcal{H}_1 = D(\mathcal{A}_0)$ and $\mathcal{H}_2 = D(\mathcal{A}_0^2)$, ([5], p.23).

We shall introduce

Assumption 2. There exists $0 \leq \alpha \leq 2$ such that

(i) $$\|F(\varphi)\|_{D(A^{\alpha/2})} \leq M_1 \|\varphi\|^p_{\mathcal{H}_\alpha}, \quad p > 1,$$

(ii) $$\|F(\varphi) - F(\varphi')\|_X \leq M_2 \|\varphi - \varphi'\|_{\mathcal{H}_0} \quad \text{for } \|\varphi\|_{\mathcal{H}_\alpha} \leq C_1,\ \|\varphi'\|_{\mathcal{H}_\alpha} \leq C_1.$$

2. Existence of the local solution

The further reasoning is the same for both cases (Case 1 and Case 2). Let $\mathcal{A}$ denote the one of the operators $\mathcal{A}_1, \mathcal{A}_2$. We shall consider the equation

$$(2.1)\qquad \varphi(t) = e^{\mathcal{A}t}\varphi(0) + \int_0^t e^{\mathcal{A}(t-\tau)} \begin{Bmatrix} 0 \\ e^{\omega\tau} F(e^{-\omega\tau}\varphi) \end{Bmatrix} d\tau .$$

Following the idea developed by Kato [2] we can show the following lemma.

Lemma 2. For each $\varphi(0) \in \mathcal{H}_\alpha$ such that $\|\varphi(0)\|_{\mathcal{H}_\alpha} < C_1$ there exists $T > 0$ such that the solution $\varphi(t)$ to (2.1) exists on the interval $[0,T]$. Moreover

$$\sup_{0\leq t\leq T} \|\varphi(t)\|_{\mathcal{H}_\alpha} \leq C_1 .$$

Proof. Let us denote by K the operator defined by

$$(2.2)\qquad (K\varphi)(t) = e^{At}\varphi(0) + \int_0^t e^{A(t-\tau)} e^{\omega\tau} F(e^{-\omega\tau}\varphi(\tau))\,d\tau .$$

Using Assumption 2i one gets

$$(2.3)\qquad \|(K\varphi)(t)\|_{\mathcal{H}_\alpha} \leq$$

$$\leq \|\varphi(0)\|_{\mathcal{H}_\alpha} + \int_0^t M_1 \|\varphi\|^p_{\mathcal{H}_\alpha}\, d\tau \leq \|\varphi(0)\|_{\mathcal{H}_\alpha} + t\cdot M_1 \cdot \sup_{0\leq\tau\leq t} \|\varphi(\tau)\|^p_{\mathcal{H}_\alpha} .$$

The inequality (2.3) implies that there exists such a $T_1 > 0$ that for each φ, such that $\sup\limits_{0\leq t\leq T_1} \|\varphi(t)\|_{\mathcal{H}_\alpha} \leq C_1$ one has

$$\sup_{0\leq t\leq T_1} \|K\varphi\|_{\mathcal{H}_\alpha} \leq C_1 .$$

Moreover from Assumption 2ii it follows that

$$(2.4)\qquad \|(K\varphi)(t) - (K\varphi')(t)\|_{\mathcal{H}_0} \leq C \sup_{0\leq\tau\leq t} \|\varphi(t) - \varphi'(\tau)\|_{\mathcal{H}_0} \cdot t .$$

The above inequality implies that there exists $0 < T_2 \leqslant T_1$ such that

$$(2.5)\qquad \sup_{0\leqslant t\leqslant T_2} \|(K\varphi)(t) - (K\varphi')(t)\|_{\mathcal{H}_0} \leqslant N\cdot \sup_{0\leqslant t\leqslant T_2} \|\varphi - \varphi'\|_{\mathcal{H}_0}$$

and $N < 1$.

Now let us consider the sequence $\varphi^0 = e^{At}\varphi(0)$ and $\varphi^n = K\varphi^{n-1}$. The sequence φ^n is convergent in the norm $\mathcal{H}_0$ on the interval $[0,T_2]$, uniformly with respect to t. The limit $\varphi = \lim_{n\to\infty} \varphi^n$ belongs to $\mathcal{H}_\alpha$ for each t due to the fact that φ^n belongs to the ball in $\mathcal{H}_\alpha$ and $\mathcal{H}_\alpha$ is the Hilbert space.

The solution of (2.1) is the weak solution of (1.4) or (1.5), respectively. The function $v\colon [0,T] \longrightarrow D(A^{1/2})$ is the weak solution of (1.4) if it is twice differentiable with respect to t in the norm of the space $D(A^{1/2})^*$ and satisfies the equation

$$(2.6)\qquad (\Phi, v_{tt}) + \gamma(\Phi, v_t) = (\Phi, Av) + (\Phi, e^{-\omega t}\tilde{F}(e^{\omega t}v, e^{\omega t}v_t)$$

for each $\Phi \in D(A^{1/2})$. Indeed the function

$$\psi = \begin{pmatrix} A^{-1/2}, & 0 \\ 0, & A^{-1/2} \end{pmatrix}\varphi$$

satisfies the equation

$$(2.8)\qquad \psi = e^{\mathcal{A}t}\psi(0) + \int_0^t e^{\mathcal{A}(t-\tau)} \begin{pmatrix} 0 \\ A^{-1/2}\, e^{\omega t}\, F(e^{-\omega\tau}\varphi) \end{pmatrix} d\tau.$$

Now $\psi(0) \in D(\mathcal{A})$ and $(0, A^{-1/2}\, e^{\omega\tau}\, F(e^{-\omega\tau}\varphi(\tau)))^T \in D(\mathcal{A})$ and is continuous with respect to τ in the norm $D(A)$. From the semigroup properties it follows that $\psi \in C^1(0,T;\mathcal{H}_0) \cap C^0(0,T,\mathcal{H}_1)$. The function $\psi = \begin{pmatrix} \tilde{v} \\ \tilde{z} \end{pmatrix}$ satisfies the equation

$$(2.9) \qquad \psi_t = \mathcal{A}\psi + \begin{pmatrix} 0 \\ e^{-\omega t} A^{-1/2} F(e^{\omega t}\varphi) \end{pmatrix}$$

or for each $x \in X$ the equation (in case 1)

$$(2.10) \quad (x, \tilde{v}_{tt}) + \gamma(x, \tilde{v}_t) = (x, A\tilde{v}) + (x, e^{-\omega t} A^{-1/2} F(e^{\omega t}\varphi)).$$

Since $\tilde{v} = A^{-1/2}v$ one has

$$(2.11) \qquad (A^{-1/2}x, v_{tt}) + \gamma(A^{-1/2}x, v_t) = (A^{-1/2}x, Av) + \\ + (A^{-1/2}x, e^{-\omega t}F(e^{\omega t}v, e^{\omega t}v_t)).$$

Denoting $\Phi = A^{-1/2}x$ we see that v satisfies (1.4) in the weak sense. Similar reasoning can be used in Case 2.

3. Global solution and stability

We shall need the following lemma [1].

Lemma 3. If $\rho(t)$ satisfies

(i) $\quad 0 \leqslant \rho(t) \leqslant \rho(0) + C \int_0^t m(\tau)\psi(\rho(\tau))d\tau$, where

(ii) $\quad \psi : \mathbb{R}^+ \to \mathbb{R}^+$ is increasing function

(iii) $\quad m(t) > 0$ almost everywhere

then $G_\varepsilon(\rho(t)) \leqslant G_\varepsilon(\rho(0)) + \int_0^t m(\tau)d\tau$, where $G_\varepsilon(u) = \int_\varepsilon^u \frac{du'}{\psi(u')}$, $\varepsilon > 0$.

For the sake of completeness we give the proof in Appensix A.

Let us denote $\rho(t) = \|\varphi(t)\|_{\mathcal{H}_\alpha}$, where $\varphi(t)$ is the solution of Eq. (2.1). One has

$$(3.1) \qquad 0 < \rho(t) < \rho(0) + M \int_0^t e^{-\omega\tau(p-1)}(\rho(\tau))^p \, d\tau.$$

Applying Lemma 3 one obtains:

Theorem 1. Under Assumption 1 and 2 the Eq. (1.1) has the global weak solution u for each u(0), $u_t(0)$ satisfying inequality

$$\rho(0) = \left\| \begin{pmatrix} u(0) \\ u_t(0) + \omega u(0) \end{pmatrix} \right\|_{\mathcal{H}_\alpha} \leqslant \left(\frac{M}{\omega}\right)^{\frac{1}{1-p}}$$

and

$$(3.2) \qquad \left\| \begin{pmatrix} u(t) \\ u_t(t) + \omega u(t) \end{pmatrix} \right\|_{\mathcal{H}_\alpha} \leqslant$$

$$\leqslant \left[(\rho(0))^{1-p} + \frac{M}{\omega}\left(1 - e^{-\omega t(p-1)}\right) \right]^{\frac{1}{1-p}} \cdot e^{-\omega t} \leqslant \rho(0) \cdot e^{-\omega t}.$$

4. Application to partial differential equations

As an example we shall consider the differential equation of the second order. Let the operator A be defined by the following expression:

$$(4.1) \qquad \hat{A}u = - \sum_{i,j} \frac{\partial}{\partial x_i} a_{ij}(x) \frac{\partial}{\partial x_j} u$$

where $x \in \Omega \subset \mathbb{R}^n$ and the matrix a_{ij} is real and symmetric. The domain Ω is bounded and sufficiently regular. On the boundary $\partial\Omega$ the solution is assumed to satisfy one of the homogeneous boundary conditions of the following form:

$$(4.2) \qquad B_0 \, u\big|_{\partial\Omega} = u\big|_{\partial\Omega} = 0$$

or

$$(4.3) \qquad B_1 \, u\big|_{\partial\Omega} = n_i \cdot a_{1j}(x) \frac{\partial}{\partial x_j} u + a_0 \, u\big|_{\partial\Omega} = 0.$$

where n denotes the vector normal to $\partial\Omega$. In case of boundary condition $B_1 u = 0$ we shall assume that the operator is selfadjoint and positive definite.

If the domain Ω, the coefficients a_{ij} and a_0 are sufficiently regular we can define the selfadjoint operator A_i $(i=1,2)$ acting on the domain $D(A) \subset L^2(\Omega)$ by putting $Au = \hat{A}u$ for $u \in D(A)$, where

$$(4.4) \qquad D(A_i) = \left\{ u \in W^{2,2}(\Omega);\ B_i u \big|_{\partial\Omega} = 0 \right\}.$$

In further analysis it will be convenient to define the spaces $W^{\alpha,2}_{B_i}$ $(\alpha = i+1/2)$ as follows

$$(4.5) \qquad W^{\alpha,2}_{B_i}(\Omega) = \left\{ u \in W^{\alpha,2}(\Omega);\ B_i u \big|_{\partial\Omega} = 0 \text{ if } \alpha > i+1/2 \right\}.$$

In the space $W^{s,p}$ we can introduce the following norm

$$(4.6) \qquad \|f\|^p_{W^{s,p}(\Omega)} = \|f\|^p_{L^p(\Omega)} + \sum_{|\alpha|=[s]} \iint_{\Omega\Omega} \frac{|D^\alpha f(x) - D^\alpha f(y)|^p}{|x-y|^{n+\{s\}\cdot p}}\, dx dy$$

where $s = [s] + \{s\}$, $0 < \{s\} < 1$, s = integer,

$$D^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \ldots \partial x_n^{\alpha_n}} \quad \text{and} \quad |\alpha| = \sum \alpha_i.$$

From the interpolation theorem of Grisvard [7, p.320] it follows that the spaces $\mathcal{H}_\alpha$ in the considered case are defined by

$$(4.7) \qquad \mathcal{H}_\alpha = W^{1+\alpha,2}_{B_i} \times W^{\alpha,2}_{B_i}.$$

In the further analysis one needs the imbedding of Sobolev spaces. If $0 \leqslant t \leqslant s < \infty$, $\infty > q \geqslant p > 1$ and $s - \frac{n}{p} > t - \frac{n}{q}$ then

$$(4.8) \qquad W^{s,p}(\Omega) \subset W^{t,q}(\Omega), \qquad ([7], \text{ p.328}).$$

Moreover we shall use the following properties

(4.9) $\|fg\|_{W^{t,2}} \le C\cdot \|f\|_{W^{t,2}}\cdot \|g\|_{W^{s,2}}$ for $0 \le t \le s$ and $s - \frac{n}{2} > 0$.

The proof of (4.9) will be given in Appendix B.

We shall demonstrate that Assumption 2 is valid for wide class of nonlinearity. We start by considering the function F of the form

(4.10) $$F(\varphi) = F\left(\binom{u}{z}\right) = C\cdot u^p.$$

In order to prove Assumption 2i we shall consider the following inequality obtained from the definition of the norm in $W^{s,p}$

(4.11) $$C\|u^p\|^2_{W^{\alpha,2}} \le C\|u^p\|^2_{L^2} + C\iint_{\Omega\Omega} \frac{|u(x)-u(y)|^2}{|x-y|^{n/q+2\alpha+\varepsilon}}\, \frac{\sum_{i=1}^{p} \left|(u(x))^{i-1}\cdot(u(y))^{p-i}\right|^2}{|x-y|^{n/q'-\varepsilon}}\, dxdy.$$

Applying the Hölder inequality one gets

(4.12) $$C\|u^p\|^2_{W^{\alpha,2}} \le C\|u^p\|^2_{L^2} + C\left\{\iint_{\Omega\Omega} \frac{|u(x)-u(y)|^{2q}}{|x-y|^{n+(\alpha+\varepsilon/2)2q}}\, dxdy\right\}^{\frac{1}{q}} \times$$
$$\times \left\{\iint_{\Omega\Omega} \frac{\sum_{i=1}^{p} \left|\left(u(x)\right)^{2q'(i-1)}\cdot(u(y))^{2q'(p-1)}\right|}{|x-y|^{n-\varepsilon q'}}\, dxdy\right\}^{\frac{1}{q'}}$$

and by using the Hölder inequality again one obtains

(4.13) $$C\|u^p\|^2_{W^{\alpha,2}} \le C\|u\|^{2p}_{L^{2p}} + C\|u\|^2_{W^{\alpha+\varepsilon/2,2q}}\cdot \|u\|^{2(p-1)}_{L^{4q'(p-1)}}.$$

Providing that the following inequalities are satisfied

(4.14)

(4.15) $$\alpha + 1 - \frac{n}{2} > -\frac{n}{2p},$$

$$\alpha + 1 - \frac{n}{2} > \alpha + \frac{\varepsilon}{2} - \frac{n}{2q},$$

$$(4.16) \qquad \alpha + 1 - \frac{n}{2} > - \frac{n}{4q'(p-1)} ,$$

where $1/q + 1/q' = 1$, and using imbeddings theorem one gets

$$(4.17) \qquad C\|u^p\|_{W^{\alpha,2}} \leqslant C\|u\|^p_{W^{\alpha+1,2}}.$$

Now we shall consider the inequality

$$(4.18) \qquad \|u^p - v^p\|_{L^2} \leqslant \left\{ \int_\Omega |u-v|^2 \sum_{i=1}^{p} |u^{i-1} \cdot v^{p-i}|^2 \, dx \right\}^{1/2} \leqslant$$

$$\leqslant \left\{ \int_\Omega |u-v|^{2q} \, dx \right\}^{1/2q} \left\{ \int_\Omega \sum_{i=1}^{p} |u^{i-1} \cdot v^{p-i}|^{2q'} dx \right\}^{1/2q'}$$

where $1/q + 1/q' = 1$.

From Hölder's inequality one obtains

$$(4.19) \qquad \left\{ \int_\Omega |u^{i-1} \cdot v^{p-i}|^{2q'} \, dx \right\}^{1/2q'} \leqslant$$

$$\leqslant \left\{ \int_\Omega |u|^{2q'(i-1)r} \, dx \right\}^{1/2q'r} \cdot \left\{ \int_\Omega |v|^{2q'(p-i)r'} dx \right\}^{1/2q'r'}$$

where $1/r + 1/r' = 1$.

Putting $2q'(i-1)r = 2q'(p-i)r'$ one can show that

$$(4.20) \qquad \|u^p - v^p\|_{L^2} \leqslant C\|u-v\|_{L^{2q}} \cdot \sum_{i=1}^{p} \left(\|u\|^{p-1}_{L^{2q'(p-1)}} \cdot \|v\|^{\frac{p-1}{p-i}}_{L^{2q'(p-1)}} \right).$$

The inequality $\|u^p - v^p\|_{L^2} \leqslant C\|u-v\|_{W^{\alpha,2}}$ is valid under the assumptions

$$(4.21) \qquad \alpha - \frac{n}{2} > - \frac{n}{2q} ,$$

$$(4.22) \qquad \alpha + 1 - \frac{n}{2} > - \frac{n}{2q'(p-1)} .$$

The inequalities (4.14)-(4.16) and (4.21)-(4.22) are satisfied if

(4.23) $$1>\alpha>\frac{n}{2}-1-\min\left(\frac{n}{2p}\ ,\ \frac{n}{2(p-1)}\right).$$

In the similar way one can consider the nonlinearity of the form

$$F(\varphi) = f(u)$$

under the assumption that f(u) is regular and $|f(u)| < C|u|^p$, $\left|\frac{df}{du}\right| \leqslant C|u|^{p-1}$.

For this nonlinearity we shall assume that $\alpha + 1 - \frac{n}{2} > 0$. Thus we have the imbedding $W^{\alpha+1,2}(\Omega) \subset C^0(\Omega)$, where C^0 is the space of bounded continuous functions.

One has for $0<\alpha<1$

(4.24) $$\|f(u)\|^2_{W^{\alpha,2}} \leqslant \|f(u)\|^2_{L^2} + \iint_{\Omega\,\Omega} \frac{\left|\frac{df}{du}\left(\varepsilon u(x)+(1-\varepsilon)u(y)\right)\right|^2 |u(x)-u(y)|^2}{|x-y|^{n+2\alpha}}\,dxdy \leqslant$$

$$\leqslant C\cdot\|u\|^{2p}_{L^{2p}} + \sup_{\xi\in\Omega}|u(\xi)|^{2(p-1)}\cdot\|u\|^2_{W^{\alpha,2}}.$$

One can easily estimate the difference f(u)-f(v). One gets

(4.25) $$\|f(u)-f(v)\|_{L^2} \leqslant$$

$$\leqslant \left\{\int_\Omega \left|\frac{df}{du}\left(\varepsilon u(x)+(1-\varepsilon)v(x)\right)\right|^2 |u(x)-v(x)|^2 dx\right\}^{1/2} \leqslant$$

$$\leqslant C\cdot\sup_{x\in\Omega}\max_{u,v}\left(|u(x)|^{p-1}, |v(x)|^{p-1}\right)\|u-v\|_{L^2}.$$

Now we shall consider the nonlinearities of the form

$$F(\varphi) = f(u)u_t.$$

Using (4.9) one gets

$$(4.26)\qquad \|f(u)u_t\|_{W^{\alpha,2}} \leq C\|u_t\|_{W^{\alpha,2}}\,\|f(u)\|_{W^{1+\alpha,2}}.$$

From the definition of the norm it follows that

$$(4.27)\qquad \|f(u)\|^2_{W^{1+\alpha,2}} = \|f(u)\|^2_{L^2} + \sum_{|\alpha|=1}\iint_{\Omega\,\Omega}\frac{\left|\frac{df}{du}(u(x))D^{\alpha}u(x) - \frac{df}{du}(u(y))D^{\alpha}u(y)\right|^2}{|x-y|^{n+2\alpha}}\,dxdy = I_1 + I_2.$$

The second term can be estimated as follows

$$(4.28)\qquad I_2 \leq \sum_{|\alpha|=1}\iint_{\Omega\,\Omega}\frac{\left|\frac{df}{du}(u(x)) - \frac{df}{du}(u(y))\right|^2\cdot\left|D^{\alpha}u(x)\right|^2}{|x-y|^{n+2\alpha}}\,dxdy + \sum_{|\alpha|=1}\iint_{\Omega\,\Omega}\frac{\left|\frac{df}{du}(u(y))\right|^2\cdot\left|D^{\alpha}u(x) - D^{\alpha}u(y)\right|^2}{|x-y|^{n+2\alpha}}\,dxdy.$$

Using the Hölder inequality one gets

$$(4.29)\qquad I_2 \leq C\sup_{x\in\Omega}|u(x)|^{2(p-2)}\|u\|^2_{W^{\alpha+\varepsilon/2,2q}}\|u\|^2_{W^{1,2q'}} + C\sup_{x\in\Omega}|u(x)|^{2(p-1)}\|u\|^2_{W^{\alpha,2}}.$$

We shall assume the following conditions

$$(4.30)\qquad 1+\alpha-\frac{n}{2} > \alpha+\frac{\varepsilon}{2}-\frac{n}{2q},$$

$$(4.31)\qquad 1+\alpha-\frac{n}{2} > 1-\frac{n}{2q'},$$

$$(4.32)\qquad 1+\alpha-\frac{n}{2} > 0,$$

where $1/q + 1/q' = 1$.

Using the imbedding theorem one gets

$$(4.33)\quad \|f(u)u_t-f(v)v_t\|_{L^2} \leqslant \|f(u)(u_t-v_t)\|_{L^2}+\|v_t(f(u)-f(v))\|_{L^2} \leqslant$$

$$\leqslant \sup_{x\in\Omega}|u(x)|^{p}\|u_t-v_t\|_{L^2} + \sup_{x\in\Omega}|u(x)|^{p-1}\|v_t\|_{L^{2q}}\|u-v\|_{L^{2q'}}$$

In order to have the proper imbedding one has to assume that

$$(4.34)\quad 1-\frac{n}{2} > -\frac{n}{2q},$$

$$(4.35)\quad \alpha-\frac{n}{2} > -\frac{n}{2q'}.$$

The condition (4.30)-(4.32) and (4.34)-(4.35) are satisfied if

$$(4.36)\quad 1 > \alpha > \frac{n-2}{2}.$$

One has to remember that the function $F(\varphi)$ belongs to $W_B^{\alpha,2}(\Omega)$ and thus it satisfies the boundary condition $B_0F(\varphi)|_{\partial\varphi} = 0$ or $B_1F(\varphi)|_{\partial\Omega} = 0$ for α sufficiently large. One observes that for $n \leqslant 2$ we can always choose α small and satisfies the Assumption 2. for all types of nonlinearities. For $n = 3$ the nonlinearities of the type $f(u)$ and $f(u)u_t$ can be included into consideration for boundary condition of the form $B_0F(\varphi)|_{\partial\Omega} = 0$. In this case one has $f(u) = 0$ for $u = 0$.

Appendix A. The proof of Lemma 3.

Since ρ is increasing function, one has

$$\psi(\rho(t)) \leqslant \psi\Big(C+\int_0^t m(s)\,(\rho(s))\,ds\Big)$$

or introducing $v = \psi\circ\rho$

$$v(t) \leqslant \psi\Big(C+\int_0^t m(s)\,v(s)\,ds\Big).$$

One obtains

$$\frac{m(t)\ v(t)}{\psi\left(C + \int\limits_0^t m(s)\ v(s)\ ds\right)} \leqslant m(t)$$

and integrating with respect to t

$$G_\varepsilon\left(C + \int\limits_0^t m(s)\ v(s)\ ds\right) \leqslant G_\varepsilon(C) + \int\limits_0^t m(s)\ ds.$$

From assumption (i) of Lemma 3 the desired result follows.

Appendix B. We shall use the following results ([7], p. 324).

Definition. The bounded domain $\Omega \subset \mathbb{R}^n$ is said to be a domain of cone-type if there exist domains $U_1 \ldots U_n$ and cones $C_1 \ldots C_M$, which may be carried over by rotations into the cone K_n, such that

$$U_j \subset \partial\Omega_j, \quad (U_j \cap \Omega) + C_j, \quad j = 1, \ldots, M$$

where

$$K_n = \left\{ x\colon x = (x', x_n) \subset \mathbb{R}^n, \quad 0 < x_n < n, \quad |x'| < dx_n \right.$$

Theorem. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain of cone-type, then in $W^{s,p}(\Omega)$ one can introduce the norm (4.6).

We shall prove the inequality

$$\|fg\|_{W^{t,p}(\Omega)} \leqslant C\|f\|_{W^{t,p}(\Omega)} \cdot \|g\|_{W^{s,p}(\Omega)}, \quad 0 \leqslant t \leqslant s, \quad s > \frac{n}{p}.$$

Step 1. $f \in L^p(\Omega)$, $g \in W^{s,p}(\Omega)$

$$\|fg\|_{L^p(\Omega)} = \int\limits_\Omega |f|^p \cdot |g|^p\ dx \leqslant C \sup |g|^p \int\limits_\Omega |f|^p\ dx \leqslant$$

$$\leqslant C\|g\|^p_{W^{s,p}(\Omega)} \cdot \|f\|^p_{L^p(\Omega)}.$$

Step 2. $f \in W^{s,p}(\Omega)$, $g \in W^{s,p}(\Omega)$

$$\|fg\|^p_{W^{s,p}(\Omega)} = \|fg\|^p_{L^p(\Omega)} +$$

$$+ \sum_{|\alpha|=[s]} \iint_{\Omega\,\Omega} \frac{|(D^\alpha fg)(x) - (D^\alpha fg)(y)|^p}{|x-y|^{n+\{s\}p}}\, dxdy = J_1 + J_2.$$

The second term J_2 can be estimated as follows

$$J_2 \leqslant \sum_{\substack{\alpha',\alpha'' \\ |\alpha'|+|\alpha''|=[s]}} CN(\alpha',\alpha'') \times$$

$$\times \iint_{\Omega\Omega} \frac{|D^{\alpha'} f(x)|^p \cdot |D^{\alpha''} g(x) - D^{\alpha''} g(y)|^p + |D^{\alpha'} f(x) - D^{\alpha'} f(y)|^p \cdot |D^{\alpha''} g(y)|^p}{|x-y|^{n+\{s\}\cdot p}}\, dxdy =$$

$$= \sum_{\alpha',\alpha''} CN(\alpha',\alpha'') \left[J_1(\alpha',\alpha'') + J_2(\alpha',\alpha'') \right],$$

$$J_1(\alpha',\alpha'') = \iint_{\Omega\,\Omega} \frac{|D^{\alpha'} f(x)|^p}{|x-y|^{n/r-\varepsilon}} \cdot \frac{|D^{\alpha''} g(x) - D^{\alpha''} g(y)|}{|x-y|^{n/r+\{s\}p+\varepsilon}}\, dxdy$$

where $1/r + 1/r' = 1$. Since one has for $|\alpha''| < [s]$

$$J_1(\alpha',\alpha'') \leqslant C \left\{ \iint_{\Omega\,\Omega} \frac{|D^{\alpha'} f(x)|^{pr}}{z^{n-\varepsilon r}}\, dzdx \right\}^{1/r} \cdot \|g\|^p_{W^{|\alpha''|+\{s\}+\varepsilon/p,r'}},$$

$$C\|f\|^p_{W^{|\alpha'|,pr}} \cdot \|g\|^p_{W^{|\alpha''|+\{s\}+\varepsilon/p,pr}}.$$

One can choose ε, r, r' such that

$W^{|\alpha'|,pr}(\Omega) \subset W^{s,p}(\Omega)$ and $W^{|\alpha''|+\{s\}+\varepsilon/p,pr}(\Omega) \subset W^{s,p}(\Omega)$.

The above is equivalent to the following inequalities

$$s - \frac{n}{p} > |\alpha'| - \frac{n}{pr},$$

$$s - \frac{n}{p} > |\alpha''| + \{s\} + \frac{\varepsilon}{p} - \frac{n}{pr'}.$$

Thus $I_1(\alpha',\alpha'') \leqslant C\|f\|_{W^{s,p}}$.

In case $|\alpha''| = [s]$ one has

$$I_1(\alpha',\alpha'') \leqslant C \iint\limits_{\Omega\,\Omega} |f(x)|^p \cdot \frac{|D^{\alpha''}g(x) - D^{\alpha''}g(y)|^p}{|x-y|^{n+\{s\}p}}\, dxdy \leqslant$$

$$\leqslant C \sup |f(x)|^p \, \|g\|_{W^{s,p}}.$$

Similar procedure can be applied for second term

$$I_2(\alpha',\alpha'') = \iint\limits_{\Omega\,\Omega} \frac{|D^{\alpha'}f(y) - D^{\alpha'}f(y)|^p \cdot |D^{\alpha''}g(y)|^p}{|x-y|^{n+\{s\}\cdot p}}\, dxdy.$$

Finally one has

$$I_2 \leqslant C\|f\|_{W^{s,p}} \cdot \|g\|_{W^{s,p}}.$$

Step 3. Let $P_g f = gf$ for $g \in W^{s,p}$ one has

$$\|P_g f\|_{L^p(\Omega)} \leqslant C\|g\|_{W^{s,p}(\Omega)} \cdot \|f\|_{L^p(\Omega)},$$

$$\|P_g f\|_{W^{s,p}(\Omega)} \leqslant C\|g\|_{W^{s,p}(\Omega)} \cdot \|f\|_{W^{s,p}(\Omega)}.$$

The operator P_g is bounded in $L^p(\Omega)$ and $W^{s,p}(\Omega)$. The interpolation theorem implies

$$\|P_g f\|_{W^{t,p}} \leqslant C\|g\|_{W^{s,p}} \, \|f\|_{W^{t,p}} \quad \text{for } f \in W^{t,p}.$$

REFERENCES

[1] M. Crouzeix, P. Raviart: Approximation des problemes d'evolution (to appear).

[2] T. Kato: Quasi-linear equations of evolution, with applications to partial differential equations, Proceedings of the Symposium at Dundee, 1974, Lecture Notes in Mathematics, Springer 1975, pp.25-70.

[3] J. Kisyński: On cosine operator functions and one-parameter groups of operator, Studia Math., 44 (1978), 93-105.

[4] S.G. Krein: Linear Differential Equations in Banach Space, Moscow 1967 (in Russian).

[5] I.L. Lions, E. Magenes: Problèmes aux limites non homogènes et applications, Paris 1968.

[6] I.L. Lions: Quelques méthodes de résolution des problèmes aux limites non linéaires. Paris 1969.

[7] M. Triebel: Interpolation Theory, Function Space, Differential Operators. Berlin 1978.

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INVERSE THEOREMS FOR A CERTAIN CLASS OF OPERATORS

1. Introduction

The object of this paper is to establish some inverse theorems concerning a certain class of positive linear smoothing operators which are built up by both integration and summation processes. It will be shown that the function class $\mathrm{Lip}^*\alpha\ (0<\alpha<2)$ can be characterized by the degree of approximation of the above-mentioned operators.

As is known in the literature [4] and [6] have dealt with approximation properties of the following operators, respectively,

$$F_n(f,x) = \sum_{k=0}^{n} p_{nk}(x)(n+1)\int_0^1 p_{nk}(t)f(t)dt, \tag{1.1}$$

$$p_{nk}(x) = \binom{n}{k}x^k(1-x)^{n-k}.$$

$$M_n(f,x) = \sum_{k=0}^{\infty} b_{nk}(x)(n-1)\int_0^{\infty} b_{nk}(t)f(t)dt, \tag{1.2}$$

$$b_{nk}(x) = \binom{n+k-1}{k}x^k(1+x)^{-(n+k)}.$$

These may be called modified Bernstein-Kantorovich type and modified Baskakov type operators. Surely one can construct many such operators by using various well-known kernels.

Notice that both (1.1) and (1.2) (with $L_n := F_n$ and $L_n := M_n$) have the property

$$|L_n(t-x),x)| \asymp L_n((t-x)^2,x), \ (n \to \infty).$$

This causes some difficulty for investigating the inverse theorems about these operators. Thus we need to introduce slight modifications of (1.1) and (1.2). More precisely, we will be concerned with the following four operators:

$$(1.3) \qquad A_n(f,x) = \sum_{k=0}^{n-2} p_{n,k+1}(x)(n-1)\int_0^1 p_{n-2,k}(t)f(t)dt,$$

$$(1.4) \qquad B_n(f,x) = \sum_{k=0}^{\infty} b_{n,k+1}(x)(n+1)\int_0^{\infty} b_{n+2,k}(t)f(t)dt,$$

$$(1.5) \qquad C_n(f,x) = \sum_{k=0}^{\infty} S_{n,k+1}(x)\ n\int_0^{\infty} S_{n,k}(t)f(t)dt,$$

$$(1.6) \qquad D_n(f,x) = \sum_{k=0}^{\infty} b_{n,k+1}(x)\ n\int_0^{\infty} S_{n,k}(t)f(t)dt,$$

where $S_{nk}(x) = e^{-nx}(nx)^k/_{k!}$, and $x \in (0,1)$ for (1.3), and $x \in (0,\infty)$ for other operators.

For convenience, operators given by (1.3)-(1.6) may be written in the unified form

$$(1.7) \qquad L_n(f,x) = \sum_{k=0}^{N} \lambda_{n,k+1}(x)\int_0^{B} \omega_{n,k}(t)f(t)dt,$$

where N and B denote n-2 and 1 respectively for (1.3), and ∞ for (1.4)-(1.6). Evidently (1.5) is generated from Szasz-Mirakyan operator, and (1.6) is of compound type.

Similarly, for periodic functions of period 2π we will investigate the following three operators:

$$(1.8)\quad F_n(f,x) = \sum_{k=0}^{2n} V_{nk}(x)\,\frac{2n+1}{2\pi}\int_0^{2\pi} V_{nk}(t)f(t)dt,$$

$$(1.9)\quad G_n(f,x) = \sum_{k=0}^{4n-1} J_{nk}(x)\,\frac{2n}{\pi}\int_0^{2\pi} J_{nk}(t)f(t)dt,$$

$$(1.10)\quad H_n(f,x) = \sum_{k=0}^{2n-1} K_{nk}(x)\,\frac{n}{\pi}\int_0^{2\pi} K_{nk}(t)f(t)dt,$$

where $x \in [0,2\pi]$, and $V_{nk}(x)$, $J_{nk}(x)$ and $K_{nk}(x)$ are defined respectively by the following

$$V_{nk}(x) = \frac{2^{2n}(n!)^2}{(2n+1)!}\cos^{2n}\frac{1}{2}\left(x - \frac{2k\pi}{2n+1}\right),$$

$$J_{nk}(x) = \frac{3}{4n^2(2n^2+1)}\left(\frac{\sin\frac{n}{2}\left(x - \frac{k\pi}{2n}\right)}{\sin\frac{1}{2}\left(x - \frac{k\pi}{2n}\right)}\right)^4,$$

$$K_{nk}(x) = \left(\frac{1}{n}\sin\frac{\pi}{2}\right)^2\left(\frac{\cos\frac{n}{2}\left(x - \frac{k\pi}{n}\right)}{\cos\left(x - \frac{k\pi}{n}\right) - \cos\frac{\pi}{n}}\right)^2.$$

The above kernels are known as Rappoport´s discrete Vallée-Poussin kernel and Bojanic-Shisha´s discrete Jackson kernel, respectively (cf. e.g. [8], p.216, p.220). We shall also obtain direct and inverse theorems for these operators.

2. Statement of main results

Consider the operators L_n defined by (1.7). Using some simple calculations one may verify the following

$$(2.1)\quad \begin{cases} \frac{1}{2}L_n((t-x)^2,x) = \frac{\varphi(x)}{n} + o\left(\frac{\varphi(x)}{n}\right), \\ L_n((t-x),x) = o\left(\frac{\varphi(x)}{n}\right), \\ L_n(1,x) = 1 + o\left(\frac{\varphi(x)}{n}\right), \end{cases}$$

where $\varphi(x) = x(1-x)$ for (1.3), $\varphi(x) = x(1+x)$ for (1.4), $\varphi(x) = x$ for (1.5), and $\varphi(x) = \frac{1}{2}x(2+x)$ for (1.6).

Let us introduce the usual notations

$$\Delta_h^2 f(x) = f(x+h) - 2f(x) + f(x-h), \quad x \in [h, B-h],$$

$$\omega_2(f,\delta) = \sup_{0 \le h \le \delta} \ \sup_{x \in [h,B-h]} \left|\Delta_h^2 f(x)\right|,$$

$$\mathrm{Lip}^*\alpha := \left\{f \in C[0,B],\ \omega_2(f,\delta) = O(\delta^\alpha),\ \delta \to 0+\right\},$$

$\|f\| = \sup\limits_{x \in [0,B]} |f(x)|$, when $B = \infty$, $[0,B]$ is replaced by $[0,\infty)$.

Throughout, M is used to denote a positive constant, independently of n and x, and it may represent different values at different occurrences. A basic result obtained in this paper is the following

Theorem. For given $f \in C[0,B]$, $\alpha \in (0,2)$, the following two assertions are equivalent

(i) $\forall n$, $x \in (0,B)$

$$|L_n(f,x) - f(x)| \le M\left(\frac{\varphi(x)}{n}\right)^{\alpha/2}, \tag{2.2}$$

$$\text{(ii)} \quad f \in \mathrm{Lip}^*\alpha, \tag{2.3}$$

where L_n represents any of the operators defined by (1.3)-(1.6).

This theorem is implied by a direct theorem and an inverse theorem, namely the following

Theorem 1. For $x \in (0,B)$, there exists an $M > 0$ such that

$$|L_n(f,x) - f(x)| \le M\,\omega_2\left(f, \sqrt{\frac{\varphi(x)}{n}}\right) + o\left(\frac{1}{n}\right). \tag{2.4}$$

In particular, if $f \in \mathrm{Lip}^*\alpha$, $0 < \alpha < 2$, then

$$|L_n(f,x) - f(x)| \le M\left(\frac{\varphi(x)}{n}\right)^{\alpha/2}. \tag{2.5}$$

Theorem 2. If there exists an $M > 0$ such that for $f \in C[0,B]$ and all $x \in (0,B)$ we have

$$\left|L_n(f,x) - f(x)\right| \leqslant M \left(\frac{\varphi(x)}{n}\right)^{\alpha/2},$$

then $f \in \mathrm{Lip}^*\alpha$, where $0 < \alpha < 2$.

For operators F_n and G_n we have the following results:

Theorem 3. If $f \in C_{2\pi}$, $\alpha \in (0,2)$, then the following relation

$$\left|F_n(f,x) - f(x)\right| \leqslant M\, n^{-\alpha/2}$$

is equivalent to $f \in \mathrm{Lip}^*\alpha$.

Theorem 4. If $f \in C_{2\pi}$, $\alpha \in (0,2)$, then the relation

$$\left|G_n(f,x) - f(x)\right| \leqslant M\, n^{-\alpha}$$

is equivalent to $f \in \mathrm{Lip}^*\alpha$.

There is also a result completely parallel to Theorem 4 for the operator H_n.

3. Proof of Theorem 1

Lemma 1. Let $g \in C^2[0,B] := \{f \in C[0,B],\ f'' \in C[0,B]\}$. Then for $x \in (0,B)$ we have

$$\left|L_n(g,x) - g(x)\right| \leqslant \|g''\| \frac{\varphi(x)}{n} + o\left(\frac{1}{n}\right).$$

Proof. In the first place,

$$g(t) - g(x) = (t-x)g'(x) + \int_x^t \int_x^s g''(u)\,du\,ds,$$

$$(x \in (0,B),\quad t \in [0,B]),$$

$$\left|\int_x^t \int_x^s g''(u)\,du\,ds\right| \leqslant \|g''\| \left|\int_x^t \int_x^s du\,ds\right| = \frac{1}{2}(t-x)^2 \|g''\|.$$

By (2.1) we have

$$|L_n(g,x) - g(x)| \leqslant \frac{1}{2} L_n(t-x)^2,x) \, \|g''\| + o\,(\tfrac{1}{n}) =$$

$$= \|g''\| \, \frac{\varphi(x)}{n} + o\,(\tfrac{1}{n}).$$

This is what we desired.

For the latter use we need Steklov's mean, for $h > 0$,

$$f_h(y) = h^{-2} \int_{-h/s}^{h/2} \int_{-h/2}^{h/2} f(y+s+t)\,ds\,dt, \quad (y \in [0,B]).$$

It is known that (cf. [7])

$$|f(y) - f_h(y)| \leqslant \frac{5}{2}\,\omega_2(f,h),$$

$$|f_h''(y)| \leqslant 5\,h^{-2}\,\omega_2(f,h).$$

For proving Theorem 1, let $f \in C[0,B]$, $x \in (0,B)$, $h > 0$. Then by Lemma 1 we find

$$|L_n(f,x) - f(x)| \leqslant$$

$$\leqslant |L_n(f-f_h,x)| + |L_n(f_h,x) - f_h(x)| + |f_h(x) - f(x)| + o\,(\tfrac{1}{n}) \leqslant$$

$$\leqslant \|f - f_h\|\,(2 + o\,(\tfrac{1}{n})) + \|f_h''\|\,\frac{\varphi(x)}{n} + o\,(\tfrac{1}{n}) \leqslant$$

$$\leqslant M\,\omega_2(f,h)\left[1 + \frac{\varphi(x)}{nh^2}\right] + o\,(\tfrac{1}{n}).$$

Thus (2.4) is obtained by taking $h = \sqrt{\varphi(x)/n}$, and consequently (2.5) is proved.

4. Proof of Theorem 2

We shall require three lemmas for the proof of the inverse theorem.

Lemma 2. If $f \in C^2[0,B]$, $x \in (0,B)$, then

(4.1) $$|(L_n(f,x))''| \leqslant M\|f''\|.$$

Proof. Simple calculations yield

$$(p_{nk}(x))'' = n(n-1)\sum_{j=0}^{2}\binom{2}{j} p_{n-2,k-j}(x)(-1)^{2-j}, \tag{4.2}$$

$$(b_{nk}(x))'' = n(n+1)\sum_{j=0}^{2}\binom{2}{j} b_{n+2,k-j}(x)(-1)^{2-j}, \tag{4.3}$$

$$(S_{nk}(x))'' = n^2\sum_{j=0}^{2}\binom{2}{j} S_{n,k-j}(x)(-1)^{2-j}. \tag{4.4}$$

Let us now give a proof of (4.1) for the operator defined by (1.6). In fact, for the other operators the proofs are entirely similar. It is easily seen that

$$(D_n(f,x))'' =$$

$$= n(n+1)\sum_{j=0}^{2}\binom{2}{j}\sum_{k=0}^{\infty} b_{n+2,k+1}(x)(-1)^{2-j}\, n\int_0^{\infty} S_{n,k+j}(t)f(t)dt =$$

$$= n(n+1)\sum_{k=0}^{\infty} b_{n+2,k+1}(x)\; n\int_0^{\infty}\sum_{j=0}^{2}\binom{2}{j} S_{n,k+j}(t)(-1)^{2-j}f(t)dt.$$

According to (4.4) we have

$$\frac{d^2}{dx^2} S_{n,k+2}(t) = n^2\sum_{j=0}^{2}\binom{2}{j} S_{n-k+j}(t)\,(-1)^{2-j}.$$

Thus integration by parts gives

$$(D_n(f,x))'' = (n+1)\sum_{k=0}^{\infty} b_{n+2,k+1}(x)\int_0^{\infty} S''_{n-k+2}(t)\; f(t)\; dt =$$

$$= (n+1)\sum_{k=0}^{\infty} b_{n+2,k+1}(x)\int_0^{\infty} S_{n,k+2}(t)\; f''(t)\; dt.$$

Consequently, we obtain

$$\left|(D_n(f,x))''\right| \leqslant \|f''\| \frac{n+1}{n} \sum_{k=0}^{\infty} b_{n+2,k+1}(x)\, n \int_0^{\infty} S_{n,k+2}(t)\,dt \leqslant M\|f''\|.$$

Since the cases (1.3)-(1.5) can be treated in the same manner, (4.1) is obtained.

Lemma 3. For $f \in C[0,B]$, $x \in (0,B)$, we have

$$(4.5) \qquad \left|(L_n(f,x))''\right| \leqslant M\|f\| \frac{n}{\psi(x)} ,$$

where $\psi(x) = \varphi(x)$ for (1.3)-(1.5), and $\psi(x) = x(1+x)$ for (1.6).

Proof. Notice that

$$\int_0^B \omega_{nk}(t)\,dt = 1.$$

We have

$$\left|(L_n(f,x))''\right| = \left| \sum_{k=0}^{N} \lambda''_{n,k+1}(x) \int_0^B \omega_{nk}(t)\, f(t)\, dt \right| \leqslant$$

$$\leqslant \|f\| \sum_{k=0}^{N} \left|\lambda''_{n,k+1}(x)\right|.$$

Thus it suffices to show that

$$(4.6) \qquad \sum_{k=0}^{N} \left|\lambda''_{n,k+1}(x)\right| \leqslant M \frac{n}{\psi(x)} .$$

Now direct computations give

$$\sum_{k=0}^{n} \left| p''_{n+2,k+1}(x) \right| =$$

$$= \frac{n+2}{x(1-x)} \sum_{k=0}^{n} \left| -\frac{(x(1-x))'}{x(1-x)} \left(\frac{k+1}{n+2} - x\right) + \frac{n+2}{x(1-x)} \left(\frac{k+1}{n+2} - x\right)^2 - 1 \right| p_{n+2,k+1}(x),$$

$$\sum_{k=0}^{\infty} \left| b''_{n,k+1}(x) \right| =$$

$$= \frac{n}{x(x+1)} \sum_{k=0}^{\infty} \left| - \frac{(x(1+x))'}{x(1+x)} \left(\frac{k+1}{n} - x \right) + \frac{n}{x(1+x)} \left(\frac{k+1}{n} - x \right)^2 - 1 \right| b_{n,k+1}(x),$$

$$\sum_{k=0}^{\infty} \left| S''_{n,k+1}(x) \right| =$$

$$= \frac{n}{x} \sum_{k=0}^{\infty} \left| - \frac{1}{x} \left(\frac{k+1}{n} - x \right) + \frac{n}{x} \left(\frac{k+1}{n} - x \right)^2 - 1 \right| S_{n,k+1}(x).$$

By a comparison with (13) of [7] we see that the righ-hand sides of the above three summations are all bounded by a positive constant M. Hence (4.6) is valid, and (4.5) is thereby proved.

Lemma 4. For $x \in [h, B-h]$, we have

$$\int_{-h/2}^{h/2} \int_{-h/2}^{h/2} \frac{ds\, dt}{\psi(x+s+t)} \leq \frac{M h^2}{m_\psi(x,h)}, \tag{4.7}$$

where $m_\psi(x,h) = \max\{\psi(x-h), \psi(x), \psi(x+h)\}$, and $0<h<\min\{B/4,1\}$, $\psi(x)$ being given by $x(1-x)$, x, $x(1+x)$, respectively.

This is a known result (cf. [7], Lemma 2, (10)).

For proving the theorem we need to make use of the idea due to Berens and Lorentz [3]. It suffices to show that for $f \in C[0,B]$, $0<\alpha<2$, we have (cf. [1], [2], [7])

$$\omega_2(f,h) \leq M \left[\delta^\alpha + \left(\frac{h}{\delta} \right)^2 \omega_2(f,\delta) \right], \tag{4.8}$$

where $0<h<\min\{B/4, 1\}$, $0<\delta<\min\{\sup_x \sqrt{\varphi(x)}, 1\}$.

Let us prove (4.8). For $x \in (h, B-h)$, we have

$$(4.9) \qquad \left|\Delta_h^2 f(x)\right| = \left|f(x+h) - 2f(x) + f(x-h)\right| \leqslant$$

$$\leqslant \left|f(x+h) - L_n(f,x+h)\right| + 2\left|f(x) - L_n(f,x)\right| + \left|f(x-h) - L_n(f,x-h)\right| +$$

$$+ \left|L_n(f,x+h) - 2L_n(f,x) + L_n(f,x-h)\right| \leqslant$$

$$\leqslant M\left\{\left[\frac{\varphi(x+h)}{n}\right]^{\alpha/2} + 2\left[\frac{\varphi(x)}{n}\right]^{\alpha/2} + \left[\frac{\varphi(x-h)}{n}\right]^{\alpha/2}\right\} +$$

$$+ \left[\int_{-h/2}^{h/2}\int_{-h/2}^{h/2}\left[L_n(f,x+s+t)\right]'' \, ds \, dt\right| .$$

Moreover,

$$(L_n(f,x))'' = (L_n(f-f_\delta, x))'' + (L_n(f_\delta, x))'' .$$

Thus by Lemmas 2 and 3, and (3.2) we get

$$\left|(L_n(f-f_\delta, x))''\right| \leqslant M\|f-f_\delta\| \frac{n}{\psi(x)} \leqslant M \frac{n}{\psi(x)} \omega_2(f,\delta),$$

$$\left|(L_n(f_\delta, x))''\right| \leqslant M\|f_\delta''\| \leqslant M \, \delta^{-2} \, \omega_2(f,\delta).$$

Consequently

$$(4.10) \qquad \left|(L_n(f,x))''\right| \leqslant M \, \omega_2(f,\delta)\left[\frac{n}{\psi(x)} + \frac{1}{\delta^2}\right].$$

Denote $m_\varphi(x,h) = \max\{\varphi(x-h), \varphi(x), \varphi(x+h)\}$. From (4.9), (4.10) and (4.7) we can infer that

$$\left|\Delta_h^2 f(x)\right| \leqslant M\left(\frac{m_\varphi(x,h)}{n}\right)^{\alpha/2} + M\omega_2(f,\delta)\left[n\int_{-h/2}^{h/2}\int_{-h/2}^{h/2}\frac{ds\,dt}{\psi(x+s+t)} + \left(\frac{h}{\delta}\right)^2\right] \leqslant$$

$$\leqslant M\left\{\left[\frac{m_\varphi(x,h)}{n}\right]^{\alpha/2} + \omega_2(f,\delta)\left[\frac{nh^2}{m_\psi(x,h)} + \left(\frac{h}{\delta}\right)^2\right]\right\}.$$

For (1.3)-(1.5) we have $\varphi(x) = \psi(x)$, so that $m_\varphi(x,h) = m_\psi(x,h)$. Now choose n such that

$$\sqrt{\frac{m_\varphi(x,h)}{n}} \leqslant \delta < \sqrt{\frac{m_\varphi(x,h)}{n-1}} \leqslant \sqrt{2}\,\sqrt{\frac{m_\varphi(x,h)}{n}}\,.$$

Thus we obtain

$$(4.11)\qquad \left|\Delta_h^2\, f(x)\right| \leqslant M\left\{\delta^\alpha + \omega_2(f,\delta)\,\left(\frac{h}{\delta}\right)^2\right\}.$$

Moreover, for (1.6) we have $\varphi(x) = \frac{1}{2}\,x(2+x)$, $\psi(x) = x(1+x)$, so that

$$m_\varphi(x,h) = \frac{1}{2}\,(x+h)(2+x+h),$$

$$m_\psi(x,h) = (x+h)(1+x+h).$$

Consequently,

$$\frac{nh^2}{m_\psi(x,h)} = \frac{nh^2}{m_\varphi(x,h)}\cdot\frac{m_\varphi(x,h)}{m_\psi(x,h)} \leqslant M\,\frac{nh^2}{m_\varphi(x,h)}\;.$$

Thus for (1.6) we can also obtain, in a similar manner,

$$(4.12)\qquad \left|\Delta_h^2\, f(x)\right| \leqslant M\left\{\delta^\alpha + \omega_2(f,\delta)\,\left(\frac{h}{\delta}\right)^2\right\}.$$

Now (4.8) is implied by (4.11) and (4.12). Hence the theorem is proved.

5. Proof of Theorems 3 and 4

We will make use of the following basic lemma (cf. [5], p.20).

Lemma 5. Let x_0 be any point, and let $x_k = x_0 + 2k\pi/n$, $(k = 0,1,\ldots,n-1)$. If $T(x)$ is a trigonometric polynomial of degree not exceeding $(n-1)$, then

$$\frac{1}{2\pi}\int_0^{2\pi} T(x)\,dx = \frac{1}{n}\sum_{k=0}^{n-1} T(x_k).$$

Let us prove Theorem 3. Using Lemma 5, we easily find

$$(5.1)\qquad \begin{cases} F_n(1,x) = 1, \quad F_n(\sin(t-x),x) = 0, \\ F_n\left(\sin^2\left(\frac{t-x}{2}\right),x\right) = \frac{1}{n+1}\left(\frac{2n+1}{2n+2}\right) \leqslant \frac{1}{n}\,. \end{cases}$$

Notice that for every function $f(t) \in C^2_{2\pi}$ we have

$$f(t) = f(x)+f'(x)\,\sin(t-x) + 2f''(x)\,\sin^2\left(\frac{t-x}{2}\right) + \alpha(t)\,\sin^2\left(\frac{t-x}{2}\right)$$

with $\alpha(t) \to 0$ ($|t-x| \to 2k\pi$, $k = 0,1,2,\dots$). Thus it is not difficult to obtain a direct theorem by means of (5.1). In order to get an inverse theorem, we will need a pair of lemmas just corresponding to Lemmas 2 and 3 of the foregoing section (§ 4).

Lemma 6. If $f \in C^2_{2\pi}$, then

$$(5.2)\qquad \left|(F_n(f,x))''\right| \leqslant \|f''\|\,.$$

Proof. Denote $V_{nk}(x) = \hat{V}_{nk}\left(x - \frac{2k\pi}{2n+1}\right)$. Then by Lemma 5 we have

$$(F_n(f,x))'' =$$

$$= \frac{d^2}{dx^2}\left[\frac{2n+1}{2\pi}\int_0^{2\pi}\left\{\sum_{k=0}^{2n}\hat{V}_{nk}\left(x - \frac{2k\pi}{2n+1}\right)\hat{V}_{nk}\left(t - \frac{2k\pi}{2n+1}\right)\right\}f(t)\,dt = \right.$$

$$= \frac{d^2}{dx^2}\left(\frac{2n+1}{2\pi}\right)^2\int_0^{2\pi}\left\{\int_0^{2\pi}\hat{V}_{nk}(x-u)\,\hat{V}_{nk}(t-u)\,du\right\}f(t)\,dt =$$

$$= \left(\frac{2nt}{2\pi}\right)^2\int_0^{2\pi}\hat{V}''_{nk}(x-u)\left\{\int_0^{2\pi}\hat{V}_{nk}(t-u)f(t)\,dt\right\}du =$$

$$= \left(\frac{2n+1}{2\pi}\right)^2\int_0^{2\pi}\hat{V}''_{nk}(x-u)\left\{\int_0^{2\pi}\hat{V}_{nk}(t)f(t+u)\,dt\right\}du =$$

$$= \left(\frac{2n+1}{2\pi}\right)^2 \int_0^{2\pi} \hat{V}_{nk}(t) \left\{ \int_0^{2\pi} \hat{V}''_{nk}(x-u)f(t+u)\,du \right\} dt \;=$$

$$= \left(\frac{2n+1}{2\pi}\right)^2 \int_0^{2\pi} \hat{V}_{nk}(t) \left\{ \int_0^{2\pi} \hat{V}(x-u)f''(t+u)\,du \right\} dt.$$

Utilizing the normalization condition $(2n+1)/_{2\pi} \cdot \int_0^{2\pi} \hat{V}_{nk}(t)\,dt = 1$, we find

$$\left| \frac{d^2}{dx^2} F_n(f,x) \right| \leqslant \|f''\|,$$

this proving the lemma.

Lemma 7. For $f \in C^2_{2\pi}$, we have

(5.3) $$\left|(F_n(f,x))''\right| \leqslant Mn\|f\|.$$

Proof. In accordance with the procedure for proving Lemma 6 we have

$$(F_n(f,x))'' = \left(\frac{2n+1}{2\pi}\right)^2 \int_0^{2\pi} \left\{ \int_0^{2\pi} \hat{V}''_{nk}(x-u)\,\hat{V}_{nk}(t-u)\,du \right\} f(t)\,dt =$$

$$= \left(\frac{2n+1}{2\pi}\right)^2 \int_0^{2\pi} \left\{ \int_0^{2\pi} (-1)\hat{V}'_{nk}(x-u)\hat{V}'_{nk}(t-u)\,du \right\} f(t)\,dt.$$

But it is easily seen that

$$\left| \int_0^{2\pi} \hat{V}'_{nk}(t-u)f(t)\,dt \right| \leqslant$$

$$\leqslant \|f\| \int_0^{2\pi} \left|\hat{V}'_{nk}(t)\right| dt = \|f\| \frac{2^{2n}(n!)^2}{(2n+1)!} \int_0^{2\pi} \left| n\cos^{2n-1}\frac{t}{2}\sin\frac{t}{2} \right| dt =$$

$$= \|f\| \frac{2^{2n}(n!)^2}{(2n+1)!}\, n \left[\int_0^{\pi} - \int_{\pi}^{2\pi} \cos^{2n-1}\frac{t}{2}\sin\frac{t}{2}\,dt \right] = 2\|f\| \frac{2^{2n}(n!)^2}{(2n+1)!}.$$

Similarly, we have

$$\int_0^{2\pi} \left|\hat{V}'_{nk}(x-u)\right| du = 2\,\frac{2^{2n}(n!)^2}{(2n+1)!}\,.$$

Thus we can get the conclusion

$$\left|(F_n(f,x))''\right| \leqslant 4\|f\|\,\frac{1}{4\pi^2}\left(\frac{(2n)!!}{(2n-1)!!}\right) < \frac{4}{\pi^2}\,n\|f\|,$$

where $(2n)!! = 2\cdot 4\ldots(2n)$, $(2n-1)!! = 1\cdot 3\ldots(2n-1)$, and the familiar inequality $(2n)!!/(2n-1)!! < 2\sqrt{n}$ has been utilized. Consequently, making use of Lemmas 6 and 7 and applying the same procedure as used in the proof of Theorem 2, we arrive at Theorem 3.

For the operator $G_n(f,x)$, we easily find

$$(5.4)\qquad \begin{cases} G_n(1,x) = 1, \quad G_n(\sin(t-x),x) = 0, \\ G_n\left(\sin^2\left(\frac{t-x}{x}\right),\ x\right) = \frac{3}{2n^2+1}\left(1 + \frac{2(n^2-1)}{2n^2+1}\right) \leqslant \frac{3}{n^2}. \end{cases}$$

In an entirely similar manner to the proof of Lemma 6 we can obtain

$$(5.5)\qquad \left|(G_n(f,x))''\right| \leqslant \|f''\|\,.$$

A similar proposition corresponding to Lemma 7 can easily be deduced from Bernstein's inequality. More precisely, we have

$$(5.6)\qquad \left|(G_n(f,x))''\right| \leqslant M\|f\|\,n^2.$$

Thus in accordance with (5.4)-(5.6) and in a parallel manner to § 4 we can accomplish a proof of Theorem 4.

Remark. Note that the operator G_n yields the best possible order of approximation for $f \in C_{2\pi}$, so that the converse part of Theorem 4 (inverse theorem) may also be proved by means of the Bernstein inverse theorem.

The operator $H_n(f,x)$ can be treated entirely similarly as $G_n(f,x)$, and the details may be omitted.

Finally, it may be worth mentioning that all the operators $F_n(f,x)$, $G_n(f,x)$ and $H_n(f,x)$ do possess the mean convergence property for $f \in L_p(0,2\pi)$ $(1<p<\infty)$ as $n \to \infty$. Moreover, an operator of compound type consisting of both kernel V_{nk} and K_{nk} may also be investigated in a like manner without essential difficulty.

REFERENCES

[1] M. Becker, R.J. Nessel: An elementary approach to inverse approximation theorems, J. Approx. Theory, 23 (1978), 99-103.

[2] M. Becker: An elementary proof of the inverse theorem for Bernstein polynomials, Aequations Math. 19 (1979), 145-150.

[3] H. Berens, G.G. Lorentz: Inverse theorems for Bernstein polynomials, Indiana Univ. Math. J. 21 (1972), 693-708.

[4] M.M. Derriennic: Sur l´approximation de functions integ´rebles sur $[0,1]$ par des polynomes de Bernstein modifies, J. Approx. Theory, 31 (1981), 325-343.

[5] R.A. DeVore: The approximation of continuous functions by positive linear operators, Lecture Notes in Math. Springer-Verlag, New York, 1972.

[6] A. Sahal, G. Pasad: On simultaneous approximation by modified Lupas operator, J. Approx. Theory, 45 (1985), 122-128.

[7] K. Satô: Global approximation theorems for some exponential-type operators, J. Approx. Theory, 32 (1981), 32-46.

[8] R.H. Wang : Approximation of unbounded functions, Science Press, Beijing, 1983 (in Chinese).

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AN UNIVALENCE CRITERION AND THE SCHWARZIAN DERIVATIVE

1. Let H denote the class of functions analytic in the unit disk $U = \{z : |z| < 1\}$. We call the expression

$$q(z) = \{f,z\} = \left(\frac{f''(z)}{f'(z)}\right)' - \frac{1}{2}\left(\frac{f''(z)}{f'(z)}\right)^2 \tag{1.1}$$

the Schwarzian derivative of function $f \in H$.

Theorem A. (cf. [7]). If the function f is analytic and locally univalent in U and such that

$$\left|\{f,z\}\right| \leqslant \frac{2}{(1-|z|^2)^2}, \quad z \in U, \tag{1.2}$$

then f is univalent in U.

Theorem B. (cf. [1]). If the function f is analytic and locally univalent in U and there exists a constant c, $|c-1| < 1$, such that

$$\left|\frac{1}{2}\{f,z\}(1-|z|^2)^2 - c(1-c)\,\bar{z}^2\right| \leqslant |c|, \quad z \in U,$$

then f is univalent in U.

Theorem B is special case of the following result obtained in [5].

Theorem C. If the function f is analytic and locally univalent in U, and there exists an analytic function p, $\operatorname{Re} p(z) > 0$ in U, such that

$$\left|\frac{1}{4}\{f,z\}(1-|z|^2)^2 \frac{z}{\bar{z}}(1+p(z)) + \frac{zp'(z)}{p(z)}(1-|z|^2) + |z|^2 \frac{1-p(z)}{1+p(z)}\right| \leqslant 1,$$

$z \in U$, then f is univalent in U.

2. In this note we give a generalization of Theorem C. For this purpose we introduce the following notations.

Denote by $\Omega \subset H$ the class of functions φ such that $|\varphi(z)| < 1$ for $z \in U$, $\varphi(0) = 0$. Next, for the fixed complex numbers A and B, $|A| \leqslant 1$, $|B| \leqslant 1$, $1 - |A+B| + \operatorname{Re}\{A \cdot \bar{B}\} > 0$, we denote by $P(A,B)$ the class of functions

$$p(z) = 1 + b_1 z + b_2 z^2 + \dots \tag{2.1}$$

analytic in U and such that

$$p(z) = \frac{1 + A\varphi(z)}{1 - B\varphi(z)}, \quad \varphi \in \Omega, \quad z \in U. \tag{2.2}$$

$P(A,B)$, being a subclass of the class of the Caratheodory's functions (i.e. of functions with a positive real part in U), in the case $-1 \leqslant A \leqslant 1$ and $-A < B \leqslant 1$, was considered in [3], and for instance, in [4], [10], [6] in the case of complex A,B.

Suppose that the function $f(z,t)$ is analytic in U for almost all t in $I = \langle 0, +\infty)$ and that, for $p(z,t)$ in $P(A,B)$, $t \in I$, it satisfies the equation

$$\dot{f}(z,t) \doteq z p(z,t) \cdot f'(z,t), \quad z \in U, \tag{2.3}$$

$$\left(\dot{f}(z,t) = \frac{\partial f(z,t)}{\partial t}, \quad f'(z,t) = \frac{\partial f(z,t)}{\partial z}\right).$$

Following [8] and [2], we call $f(z,t)$ the subordination chain over I, and we denote by $S(A,B)$ the class of all such functions. It is well known [8] that the functions $f(z,t)$ satisfying (2.3) are univalent in U for every $t \in I$, moreover, $f(z,0) = f(z)$ is univalent in U.

In virtue of (2.2) for each fixed $t \in I$ there exists a function $\varphi(z,t) \in \Omega$ such that

$$p(z,t) = \frac{1 + A\varphi(z,t)}{1 - B\varphi(z,t)}, \quad z \in U. \tag{2.4}$$

3. We now state the following theorem.

Theorem 1. If the function f is analytic and locally univalent in U and there exists a function ω, $|\omega(z)| < 1$ in U, such that

(3.1)
$$\left| \frac{(A+B)\omega(z)}{2+(A-B)\omega(z)}\, |z|^2 - (1-|z|^2) \left[\frac{(A+B)z\,\omega'(z)}{[1+A\omega(z)]\,[2+(A-B)\omega(z)]} - \right.\right.$$
$$\left.\left. - (1-z^2)\, \frac{z}{\bar{z}} \cdot \frac{2+(A+B)\omega(z)}{4\ 1+A\omega(z)} \{f,z\} \right] - \frac{(\bar{A}-\bar{B})(A+B)}{4-|A-B|^2} \right| \leqslant$$
$$\leqslant \frac{2|A+B|}{4-|A-B|^2}, \quad z \in U,$$

where

(3.2) $$1-|A+B|+\mathrm{Re}\{A\cdot\bar{B}\} > 0, \quad |A| \leqslant 1, \quad |B| \leqslant 1,$$

then f is univalent in U.

Proof. Let $\tau(z,t)$ be analytic function in U for every $t \in I$. Moreover, let $\mathrm{Re}\{\tau(z,t)\} > \frac{1}{2}$, $z \in U$, $t \in I$ and $\tau(0,t) = 1$, $t \in I$. We put

(3.3) $$f(z,t) = \frac{u_1(ze^{-t}) + (e^t - e^{-t})\, z u_1'(ze^{-t})\tau(ze^{-t})}{u_2(ze^{-z}) + (e^t - e^{-t})\, z u_2'(ze^{-t})\tau(ze^{-t})} =$$
$$= e^t z + \ldots$$

in the disc $\{z : |z| < e^t,\ t \in I\}$. Here u_1, u_2 are linearly independent solutions of the Beltrami differential equation

(3.4)
$$u''(z) + \frac{1}{2}\, q(z)\cdot u(z) = 0 \quad \text{with} \quad q(z) = \{f,z\}.$$

Let u_1, u_2 satisfy the initial conditions

(3.5)

(3.6) $$u_1(0) = 0, \quad u_1'(0) = 1,$$
$$u_2(0) = 1, \quad u_2'(0) = 0,$$

then they satisfy the identities

(3.7)
$$u_1'(z)\cdot u_2(z) - u_1(z)\cdot u_2'(z) = 1,$$

(3.8) $$u_1''(z)\cdot u_2'(z) - u_1'(z)\cdot u_2''(z) = \frac{1}{2}\, q(z).$$

Let the function

$$w = f_1(z) = \frac{u_1(z)}{u_2(z)} = z + \frac{q(0)}{3!}\, z^3 + \dots$$

satisfy the differential equation

(3.9) $$\{w,z\} = q(z).$$

It is well known that, if f satisfies the equation (3.9), then $f(z) = (T\circ f_1)(z)$, where T is a fractional-linear transformation. Since $f(z,0) = \dfrac{u_1(z)}{u_2(z)}$, so in order to prove the univalence of f in U it is sufficient to prove that $f(z,t)$, given by (3.3), is a subordination chain. Hence, it is enough to show that, if $q(z) = \{f,z\} = \{f_1,z\}$ satisfies the condition (3.1), then $f(z,t)$ satisfies the equation (2.3). By (2.3), (2.4), we have

(3.10) $$\varphi(z,t) = \frac{\dot f(z,t) - zf'(z,t)}{Azf'(z,t) + B\dot f(z,t)}, \quad z\in U, \quad t\in I.$$

From (3.3) we compute the derivatives $\dot f$, f' and then put into (3.10). After simple calculations we obtain

(3.11) $$\varphi(z,t) = -\frac{2w}{(A+B)+(A-B)w},$$

where

(3.12) $$w = \frac{1-\tau}{\tau}\, e^{-2t} + (e^{t}-e^{-t})e^{-2t}\,\frac{z\tau'}{\tau} + (e^{t}-e^{-t})^2 e^{-2t} z^2 \tau^2 \frac{1}{2}\, q,$$

with $\tau = \tau(ze^{-t})$, $q = q(ze^{-t}) = \{f, ze^{-t}\}$.

The equation (2.3) is satisfied when the function $\varphi(z,t)$, given by (3.11), satisfies for $t\in I$ the Schwarz lemma assumptions, hence

$$\left|\frac{w}{A+B+(A-B)w}\right| \leqslant \frac{1}{2}$$

and

$$\left| w - \frac{(\bar{A}-\bar{B})(A+B)}{4-|A-B|^2} \right| \leqslant \frac{2A+B}{4-|A-B|^2}, \tag{3.13}$$

where w is given by (3.12).

By (3.12), (3.13), for $|z| = 1$ we have

$$\left| \frac{1-\tau}{\tau} |\zeta|^2 - (1-|\zeta|^2) \left[\frac{\zeta\tau'}{\tau} + (1-|\zeta|^2) \frac{\bar{\zeta}}{\zeta} \tau \frac{1}{2} q(\zeta) \right] - \right. \tag{3.14}$$

$$\left. - \frac{(\bar{A}-\bar{B})(A+B)}{4-|A-B|^2} \right| \leqslant \frac{2|A+B|}{4-|A-B|^2},$$

where $\tau = \tau(\zeta)$, $\zeta = ze^{-t}$.

The inequality (3.14) is satisfied, if (3.1) holds for suitable choosen function ω, $|\omega(z)| < 1$ in U. Putting

$$\frac{(A+B)\omega(\zeta)}{2+(A-B)\omega(\zeta)} = \frac{1-\tau(\zeta)}{\tau(\zeta)}$$

in (3.1) we obtain (3.14). Hence the chain (3.3) satisfies the equation (2.3), if the inequality (3.1) holds. The proof is now complete.

Corollary :

1. In the case $A = B = 1$, $\omega(z) = \frac{1-p(z)}{1+p(z)}$, $\operatorname{Re} p(z) > 0$ in U, Theorem 1 coincides with Theorem C.
2. Putting $A = B = 1$ and $\omega(z) = c = \text{const}$ for $z \in U$, from Theorem 1 we obtain Theorem B (see [9], p.169).
3. In the case $A = B = 1$ and $\omega(z) = 0$, $z \in U$, Theorem 1 coincides with Theorem A.

Remark. It is worth to notice that the condition (3.1) assures the univalence of an analytic function in more general case than that of Theorem A. Let the function $f(z) = z + a_2z^2 + a_3z^3 + \dots$ be locally univalent and analytic in U. It is known that $\{f,0\} = a_3 - \frac{3}{2} a_2^2$. It is easy to notice that there exist a_2, a_3 such that, if $a_3 - \frac{3}{2} a_2^2$ satisfies (3.1) with $\omega(z) \equiv c$, $|c| < 1$, then it does not satisfy the inequality (1.2).

REFERENCES

[1] L.V. Ahlfors: Sufficient conditions for quasiconformal extension, Princeton Annales of Math. Studies 79 (1974) 23-29.

[2] J. Becker: Löwnersche Differentialgleichung und quasikonform fortsetzbare schlichte Funktionen, Reine Angew. Math. 255 (1972) 23-43.

[3] W. Janowski: Some extremal problems for certain familie of analytic functions I, Ann. Polon. Math. 28 (1973) 297-326.

[4] Z.J. Jakubowski: On the coefficients of Caratheodory functions, Bull. Acad. Polon. Sci. Ser. Sci. Math. Astronom. Phys. 19 (1971) 805-809.

[5] Z. Lewandowski, J. Stankiewicz: Some sufficient conditions for univalence, Zeszyty Nauk. Politech. Rzeszow. 14 Mat. i Fiz. z.1 (1984) 11-16.

[6] T.H. MacGregor: Functions whose derivative has a positive real part, Trans. Amer. Math. Soc. 104 (1962) 532-537.

[7] Z. Nehari: The Schwarzian derivative of schlicht functions, Bull. Amer. Math. Soc. 55 (1949), 545-551.

[8] Ch. Pommerenke: Uber die Subordination analytischer Funtionen, J. Reine Angew. Math. Soc. 218 (1965) 150-173.

[9] G. Schober: Univalent functions, Springer-Verlag 1975.

[10] J. Stankiewicz, J. Waniurski: Some classes of functions subordinate to linear transformations and their applications, Ann. Univ. Mariae Curie-Skłodowska 28, (1974) 85-94.

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FUNCTIONAL INTEGRABILITY OF SOLUTIONS OF THE SYSTEM OF NONLINEAR DIFFERENTIAL EQUATIONS OF SECOND ORDER

In this paper we will consider properties of solutions of the system of nonlinear differential equations of second order in the form

$$x'' = a(t)y^{\alpha}, \tag{1a}$$

$$y'' = b(t)x^{\beta}, \tag{1b}$$

where a,b: $R \to R_+$ are continuous functions and α, β are natural numbers.

By a solution of the system (1) on $[0,\infty)$ we mean every ordered pair $\langle x(t),y(t)\rangle$ of functions $x,y\colon [0,\infty) \to R$, $x,y \in C^2[0,\infty)$, satisfying (1). A non-trivial solution $\langle x(t),y(t)\rangle$ of the system (1) is called a continuable solution, if

$$|x(t)| + |y(t)| < \infty \quad \text{for} \quad t \in R_+.$$

A function $v(t)$ belongs to the class $L(m,W(\cdot))$, if

$$0 < \int_{t_0}^{\infty} s^m\, W(|v(s)|)ds < \infty,$$

where m is a real number, $W\colon R \to R$ is a given continuous, nondecreasing function and $W(|v|) \geq 0$.

If in the above definition we put $t_0 = 0$, $m = 0$ and $W(v) = v^p$, $p > 0$, then we obtain the well-known class $L(0, |\cdot|^p) = L_p(0,\infty)$ of such functions v that

$$0 < \int_0^\infty |v(s)|^p \, ds < \infty.$$

In what follows $A, B, C, K, K_1, K_2, K_3, M, N$ mean the positive constants.

In the proofs of theorems we shall use the following lemma.

Lemma 1, [4]. If $|u^{(m)}(t)| \leqslant M$ for $t \geqslant 0$, $m \geqslant 1$, and $u \in L(m-1, W(\cdot))$, then $\lim\limits_{t \to \infty} u(t) = 0$.

Theorem 1. If

1^0 $|a(t)| \leqslant M$ for $t \geqslant t_0$,

2^0 $|b(t)| \leqslant N$ for $t \geqslant t_0$,

3^0 $x \in L(0, |\cdot|^\beta)$, $y \in L(0, |\cdot|^\alpha)$,

then the solution $\langle x, y \rangle$ of (1) satisfies the following conditions

4^0 $\lim\limits_{t \to \infty} x(t) = 0$,

5^0 $\lim\limits_{t \to \infty} y(t) = 0$.

Proof. We shall prove 4^0 using the assumptions 1^0, 3^0 (the proof of 5^0 is similar).

From (1a) we obtain

$$|x'(t) - x'(t_0)| = \left| \int_{t_0}^{t} a(s) y^\alpha(s) ds \right|,$$

$$|x'(t)| \leqslant |x'(t_0)| + M \int_{t_0}^{t} |y^\alpha(s)| ds =: A < \infty.$$

Having $|x'(t)| \leqslant A$ and $x \in L(0, |\cdot|^\beta)$, we state 4^0, by Lemma 1 for $W(v) = v^\beta$.

Theorem 2. If

1^0 $a \in L\left(0, |\cdot|^{\frac{n}{n-\alpha}}\right)$, $n > \alpha$,

2^0 $b \in L(0, |\cdot|^{\frac{m}{m-\beta}})$, $m > \beta$,

3^0 $x \in L(0, |\cdot|^{m})$, $y \in L(0, |\cdot|^{n})$,

then the solution $\langle x,y \rangle$ of (1) satisfies the conditions

4^0 $\lim\limits_{t \to \infty} x(t) = 0$,

5^0 $\lim\limits_{t \to \infty} y(t) = 0$.

Proof. We shall prove 4^0 using the assumptions 1^0, 3^0 (the proof of 5^0 is similar). From (1a) we have

$$|x'(t) - x'(t_0)| = \left|\int_{t_0}^{t} a(s)y^{\alpha}(s)ds\right|.$$

By Hölder's inequality, we obtain

$$|x'(t)| \leqslant |x'(t_0)| + \left|\int_{t_0}^{t} a(s)y^{\alpha}(s)ds\right| \leqslant$$

$$\leqslant A + \left(\int_{t_0}^{t} |a(s)|^{\frac{n}{n-\alpha}} ds\right)^{\frac{n-\alpha}{n}} \left(\int_{t_0}^{t} |y(s)|^{n} ds\right)^{\frac{\alpha}{n}} =: B < \infty .$$

Taking into account that $|x'(t)| \leqslant B$ and $x \in L(0, |\cdot|^{m})$, we have 4^0 by Lemma 1 for $W(v) = v^m$.

Problems of the tending to zero and the proportionality of two solutions of differential equation has been considered in [2], [4].

Theorem 3. If

$$|a(t)| \leqslant M, \ |b(t)| \leqslant N \quad \text{for} \quad t \geqslant t_0,$$

$$y_k \in L(0, |\cdot|^{\alpha}), \quad x_k \in L(0, |\cdot|^{\beta}), \quad k \in \{1,2\},$$

then two arbitrary solutions of (1), $\langle x_1, y_1 \rangle$ and $\langle x_2, y_2 \rangle$, are proportional, i.e. there exist constants c_1 and c_2 such that

$$x_1(t) = c_1 x_2(t), \quad y_1(t) = c_2 y_2(t).$$

Proof. From the proof of Theorem 1 it follows that $|y_k'| \leqslant B$, $|x_k'| \leqslant A$ and $\lim\limits_{t\to\infty} x_k(t) = 0$, $\lim\limits_{t\to\infty} y_k(t) = 0$.

Let us consider the following determinant

$$W(t,s) = \begin{vmatrix} x_1(t), & x_2(t) \\ x_1(s), & x_2(s) \end{vmatrix}, \quad t,s \geqslant t_0.$$

Since $W^2(t,s) \leqslant 2x_1^2(t)x_2^2(s) + 2x_2^2(t)x_1^2(s)$, so

$$\int_{t_0}^{t} W^2(t,s)ds \leqslant 2x_1^2(t)\int_{t_0}^{t} x_2^2(s)ds + 2x_2^2(t)\int_{t_0}^{t} x_1^2(s)ds$$

and

$$\lim_{t\to\infty}\int_{t_0}^{t} W^2(t,s)ds \equiv 0.$$

This implies $W(t,s) = 0$ for each s and $x_1(t) = c_1x_2(t)$.

In similar way we prove that $y_1(t) = c_2y_2(t)$.

Theorem 4. If $\langle x_1,y_1\rangle$, $\langle x_2,y_2\rangle$ are two arbitrary solutions of (1) and

1° $a \in L(0,|\cdot|^2)$,

2° $x_k \in L(0,|\cdot|^2)$, $y_k \in L(0,|\cdot|^{2\alpha})$ for $k \in \{1,2\}$,

then

$$\lim_{t\to\infty}\left[x_1(t) - x_2(t)\right] = 0.$$

Proof. Denote $u(t) = x_1(t) - x_2(t)$. Hence, by (1a),

$$u'(t) = \int_{t_0}^{t} a(s)y_1^{\alpha}(s)ds - \int_{t_0}^{t} a(s)y_2^{\alpha}(s)ds + C,$$

$$|u'(t)| \leqslant \int_{t_0}^{t} |a(s)y_1^{\alpha}(s)|\,ds + \int_{t_0}^{t} |a(s)y_2^{\alpha}(s)|\,ds + C.$$

By Hölder's inequality we obtain

$$|u'(t)| \leqslant \left(\int_{t_0}^{t} a^2(s)ds\right)^{\frac{1}{2}} \left(\int_{t_0}^{t} y_1^{2\alpha}(s)ds\right)^{\frac{1}{2}} +$$

$$+ \left(\int_{t_0}^{t} a^2(s)ds\right)^{\frac{1}{2}} \left(\int_{t_0}^{t} y_2^{2\alpha}(s)ds\right)^{\frac{1}{2}} + C =: K < \infty.$$

Furthermore note that, if $x_1, x_2 \in L(0, |\cdot|^2)$, then $u(t) \in$ $\in L(0, |\cdot|^2)$, since

$$\int_{t_0}^{t} u^2(s)ds \leqslant 2 \int_{t_0}^{\infty} x_1^2(s)ds + 2 \int_{t_0}^{\infty} x_2^2(s)ds.$$

Hence and from Lemma 1 it follows that

$$\lim_{t\to\infty} u(t) = \lim_{t\to\infty} \left[x_1(t) - x_2(t)\right] = 0.$$

Remark 1. If we take in Theorem 4 the following assumptions

1^0 $b \in L(0, |\cdot|^2)$,

2^0 $x_k \in L(0, |\cdot|^{2\beta})$, $y_k \in L(0, |\cdot|^2)$ for $k \in \{1,2\}$,

then the thesis is

$$\lim_{t\to\infty} \left[y_1(t) - y_2(t)\right] = 0.$$

From Theorem 4 and Remark 1 we obtain the following results:

Corollary 1. If $\langle x_1, y_1\rangle$, $\langle x_2, y_2\rangle$ are two arbitrary solutions to a linear system

$$(2) \qquad \begin{cases} x'' = a(t)y \\ y'' = b(t)x \end{cases}$$

with $a \in L(0, |\cdot|^2)$, $b \in L(0, |\cdot|^2)$ and $x_k \in L(0, |\cdot|^2)$, $y_k \in L(0, |\cdot|^2)$ for $k \in \{1,2\}$, then

$$\lim_{t\to\infty}\left[x_1(t) - x_2(t)\right] = 0, \quad \lim_{t\to\infty}\left[y_1(t) - y_2(t)\right] = 0.$$

Corollary 2. If $\langle x_1, y_1\rangle$, $\langle x_2, y_2\rangle$ are two arbitrary solutions to (2) and $|a(t)| \leqslant M$, $|b(t)| \leqslant N$, $x_k \in L(0, |\cdot|^2)$, $y_k \in L(0, |\cdot|^2)$ for $k \in \{1,2\}$, then

$$\lim_{t\to\infty}\left[x_1(t) - x_2(t)\right] = 0 \quad \text{and} \quad \lim_{t\to\infty}\left[y_1(t) - y_2(t)\right] = 0.$$

The convergence of solutions of differential equation has been considered in [1], [2].

Theorem 5. If

1^0 $|a(t)| \leqslant M$, $|b(t)| \leqslant N$ for $t \geqslant t_0 > 0$,

2^0 $\int\limits_{t_0}^{\infty} a^2(s)ds = \infty$ or $\int\limits_{t_0}^{\infty} b^2(s)ds = \infty$,

and the solution $\langle x, y\rangle$ of (1) satisfies the following condition

3^0 $\left|y'(s)\sqrt{x^{\beta}(s)} - x'(s)\sqrt{y^{\alpha}(s)}\right| \geqslant K_1 > 0$ for $t \geqslant t_0 > 0$,

then

(3a) $x \in L(0, |\cdot|^{\beta}) \Rightarrow y \notin L(0, |\cdot|^{\alpha})$

(3b) $y \in L(0, |\cdot|^{\alpha}) \Rightarrow x \notin L(0, |\cdot|^{\beta})$.

Proof. We shall prove (3b) by contradiction. Let $x \in L(0, |\cdot|^{\beta})$ and $y \in L(0, |\cdot|^{\alpha})$. By the assumptions of Theorem 5 and from the proof of Theorem 1, it follows that $|y'(t)| \leqslant B$, $|x'(t)| \leqslant A$. Assume $\int\limits_{t_0}^{\infty} a^2(s)ds = \infty$ (in the other case of 2^0 the proof is similar) and consider the relation

$$I(t) = \int\limits_{t_0}^{\infty}\left|y'(s)\sqrt{x^{\beta}(s)} - x'(s)\sqrt{y^{\alpha}(s)}\right|^2 a^2(s)ds \leqslant$$

$$\leqslant Q_2^2\int\limits_{t_0}^{\infty}|x^{\beta}(s)|ds + Q_1^2\int\limits_{t_0}^{\infty}|y^{\alpha}(s)|ds + 2Q_1Q_2\int\limits_{t_0}^{\infty}\left|\sqrt{x^{\beta}(s)y^{\alpha}(s)}\right|ds$$

with $Q_1 = AM$, $Q_2 = BM$. By Hölder's inequality for $p = 2$ and $q = 2$, we obtain

$$I(t) \leqslant Q_1^2 \int_{t_0}^{\infty} |y^{\alpha}(s)|\,ds + Q_2^2 \int_{t_0}^{\infty} |x^{\beta}(s)|\,ds +$$

$$+ 2Q_1Q_2 \left(\int_{t_0}^{\infty} |x^{\beta}(s)|\,ds \right)^{\frac{1}{2}} \left(\int_{t_0}^{\infty} |y^{\alpha}(s)|\,ds \right)^{\frac{1}{2}} =$$

$$= \left[Q_1 \left(\int_{t_0}^{\infty} |y^{\alpha}(s)|\,ds \right)^{\frac{1}{2}} + Q_2 \left(\int_{t_0}^{\infty} |x^{\beta}(s)|\,ds \right)^{\frac{1}{2}} \right]^2 \leqslant C < \infty .$$

On the other hand, by 3°,

$$I(t) = \int_{t_0}^{\infty} \left| y'(s)\sqrt{x^{\beta}(s)} - x'(s)\sqrt{y^{\alpha}(s)} \right|^2 a^2(s)\,ds \geqslant$$

$$\geqslant K_1^2 \int_{t_0}^{\infty} a^2(s)\,ds = \infty .$$

Thus we get the contradiction. The proof of (3a) is analogous.

Theorem 6. If

1° $\dfrac{|b(t)|}{t^m} \leqslant N$, $|a(t)| \leqslant M$ for $t \geqslant t_0$,

2° $\displaystyle\int_{t_0}^{\infty} s^m a^2(s)\,ds = \infty$,

3° $|x^{\beta}(t) y'^2(t)| \geqslant K_2$ for $t \geqslant t_0 > 0$,

then $x \notin L(m, |\cdot|^{\beta})$.

Proof. Let $x \in L(m, |\cdot|^{\beta})$. By (1b) and 1°, we get

$$|y'(t) - y'(t_0)| = \left| \int_{t_0}^{t} \frac{b(s)}{s^m} s^m x^{\beta}(s)\,ds \right| \leqslant \left| N \int_{t_0}^{t} s^m x^{\beta}(s)\,ds \right| < \infty$$

i.e. there exists a constant $C_1 > 0$ such that $|y'(t)| \le C_1$. Let us consider

$$I(t) = \int_{t_0}^{\infty} |x^{\beta}(s) y'^{2}(s) s^{m} a^{2}(s)|\, ds \le C_1^2 M^2 \int_{t_0}^{\infty} s^{m} |x(s)|^{\beta}\, ds < \infty.$$

On the other hand

$$I(t) = \int_{t_0}^{\infty} |x^{\beta}(s) y'^{2}(s) s^{m} a^{2}(s)|\, ds > K_2 \int_{t_0}^{\infty} s^{m} a^{2}(s)\, ds = \infty.$$

This contradiction proves the theorem.

Remark 2. If we take in Theorem 6 the following assumptions

1^{0} $\frac{|a(t)|}{t^{m}} \le M$, $|b(t)| \le N$ for $t \ge t_0$,

2^{0} $\int_{t_0}^{\infty} s^{m} b^{2}(s)\, ds = \infty$,

3^{0} $|x'^{2}(t) y^{\alpha}(t)| > K_3$ for $t \ge t_0 > 0$,

then $y \notin L(m, |\cdot|^{\alpha})$.

The belonging to some classes of solutions of differential equations has been investigated in [2], [4]. The boundedness of solutions of the system (1) has been considered in [3].

REFERENCES

[1] D. Bobrowski: On convergence of solutions of a nonlinear differential equation of second order, Nonlinear Vibration Problems 14 (1973) 93-97.

[2] D. Bobrowski, A. Wyrwińska: Asymptotic behaviours of square integrable solutions of certain differential equation, Fasc. Math. 11 (1979) 81-87.

[3] I. Foltyńska: Some asymptotic properties of solutions of a nonlinear differential equations system, Proc. X Intern. Confer. Nonlinear Oscillations, Varna (1984) 588-594.

[4] J. Werbowski, A. Wyrwińska: On functional integrability of solutions of differential equations with deviating argument. Colloquia Mathematica Societatis Janos Bolyai 30, Qualitative Theory of Differential Equations, Szeged - Hungary, (1979) 1045-1059.

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Jan Bochenek

NODES OF EIGENFUNCTIONS OF STURM–LIOUVILLE PROBLEM WITH AN INDEFINITE WEIGHT FUNCTION

1. In this paper we study the boundary value problem for the equation

$$\mathcal{L}u = \lambda mu \tag{1}$$

in (a,b) with boundary conditions

$$\alpha_1 u(a) - \beta_1 u'(a) = 0, \quad \alpha_2 u(b) + \beta_2 u'(b) = 0 \tag{2}$$

where a,b are real fixed numbers, and $\alpha_1, \alpha_2, \beta_1, \beta_2$ are non-negative real numbers such that $(\alpha_1 + \beta_1)(\alpha_2 + \beta_2) > 0$. Here

$$\mathcal{L}u := -pu'' + qu' + ru \tag{3}$$

is the Sturm-Liouville differential expression with $p,q,r \in C([a,b])$ such that $p(x) > 0$ and $r(x) \geq 0$ for all $x \in [a,b]$, $m \in C([a,b])$ is a given (real-valued) weight function, $\lambda \in R$ is the eigenvalue parameter; it is assumed that $r(x) > 0$ for $x \in [a,b]$ if $\alpha_1 = \alpha_2 = 0$.

The object of this paper is to study some properties of eigenvalues and eigenfunctions corresponding to equation (1) and the conditions (2) (we shall also shortly say: eigenvalues and eigenfunctions of problem (1), (2)).

From the above assumptions it follows that, if the weight function m is positive in $[a,b]$, then the problem (1), (2) has a smallest eigenvalue $\lambda_1 > 0$. It is the only eigenvalue of (1), (2) corresponding to a positive eigenfunction.

Let us denote by L the differential operator defined by the expression (3) and the boundary conditions (2). Using the method of [1] one may prove that the operator L does not lower the number of sign changes, i.e. Lu as a function of x does not change signs less often than u does on $[a,b]$. From this, in the case of $m(x) > 0$ in $[a,b]$, by [5], we have the typical spectral properties:

1) problem (1), (2) has a countably infinite number of real simple eigenvalues λ_n for which

$$0 < \lambda_1 < \lambda_2 < \dots, \ \lim \lambda_n = +\infty, \tag{4}$$

2) eigenfunction u_n corresponding to λ_n has exactly n-1 zeros in (a,b) all of which are simple,

3) every linear combination of eigenfunctions of (1), (2) of the form $c_m u_m + \dots + c_n u_n$ ($n \geqslant m \geqslant 1$, $c_m, \dots, c_n$ real constants, $c_m^2 + \dots + c_n^2 > 0$) has in (a,b) at least m-1 and at most n-1 zeros.

In this paper we assume that the weight function m in (1) may change sign in (a,b). As is well known (see for example [2], [3]), if $m(x_0) > 0$ for some $x_0 \in (a,b)$, then the problem (1), (2) has a principal eigenvalue $\lambda_1 > 0$ characterized by being the unique positive eigenvalue corresponding to a positive eigenfunction. If $\lambda \geqslant 0$ is an another eigenvalue of (1), (2), then $\lambda \geqslant \lambda_1$.

2. Properties of principal eigenvalue of problem (1), (2)

Let $I_1 \subset I_2 \subset [a,b]$ be such intervals that $I_2 \setminus I_1 \neq \phi$. Let us denote by $\lambda_1^{(1)}$ and $\lambda_2^{(2)}$ the positive principal eigenvalues of the equation (1) for the intervals I_1 and I_2, respectively, with the boundary conditions: u vanishes on these ends of I_1 or I_2 which belong to (a,b) and u satisfies the conditions (2) on the ends of I_1 or I_2 equal a or b.

Lemma 1. If there exists a principal eigenvalue $\lambda_1^{(1)} > 0$ of the equation (1), for the interval I_1, then there exists a principal eigenvalue $\lambda_1^{(2)} > 0$ of the equation (1), for the interval I_2 and

$$\lambda_1^{(2)} < \lambda_1^{(1)}. \tag{5}$$

Proof. Without loss of generality we may further assume that $|m| < 1$ in $[a,b]$. Let u_1 denote a positive eigenfunction corresponding to $\lambda_1^{(1)}$. Let us define the continuous function U_1 in interval I_2 as follows

$$U_1 := \begin{cases} u_1 \text{ in } I_1 \\ 0 \text{ in } I_2 \setminus I_1. \end{cases}$$

Analogously as in the proof of Lemma 2 in [3], we show that

$$U_1 \leq \lambda_1^{(1)}(L_2 + \lambda_1^{(1)})^{-1}(M + 1)U_1, \tag{6}$$

where L_2 denotes the operator defined by (3) in interval I_2 and by suitable boundary conditions on the ends of I_2 as above; M denotes the multiplication operator induced by the function m.

From the inequality (6), by Lemma 1 of [3], it follows that there exists $\tilde{\lambda}$, $0 < \tilde{\lambda} \leq \lambda_1^{(1)}$, and a function $\tilde{u} \in C^2(I_2)$, $\tilde{u} > 0$ in I_2, such that

$$L_2\tilde{u} = \tilde{\lambda}M\tilde{u}. \tag{7}$$

The equality (7) shows that the number $\tilde{\lambda}$ is the positive principal eigenvalue of the equation (1), for the interval I_2. This means that $\tilde{\lambda} = \lambda_1^{(2)}$, and $u_2 = \gamma\tilde{u}$, $\gamma > 0$, is positive eigenfunction corresponding to $\lambda_1^{(2)}$. Therefore $\lambda_1^{(2)} \leq \lambda_1^{(1)}$. Suppose that $\lambda_1^{(2)} = \lambda_1^{(1)}$; then we have

$$u_2 = \lambda_1^{(1)}(L_2 + \lambda_1^{(1)})^{-1}(M + 1)u_2,$$

$$U_1 \leq \lambda_1^{(1)}(L_2 + \lambda_1^{(1)})^{-1}(M + 1)U_1.$$

From this, analogously as in [4], we get

$$U_1 = \sigma u_2, \tag{8}$$

where σ is a positive number. Let $x_0 \in I_2 \setminus I_1$, then $U_1(x_0) = 0$ whereas $u_2(x_0) > 0$. This contradiction shows that (8) is impossible, and so $\lambda_1^{(2)} < \lambda_1^{(1)}$. Lemma 1 is proved.

Let us denote by $\lambda_1(t)$ the principal eigenvalue of equation (1) for the interval $[a,t]$ with boundary conditions $\alpha_1 u(a) - \beta_1 u'(a) = 0$, $u(t) = 0$, where $a < t < b$. By Lemma 1, the mapping $t \rightarrow \lambda_1(t)$ is a decreasing function in (a,b).

Lemma 2. The function $t \rightarrow \lambda_1(t)$ is continuous for $t \in (a,b)$.

Proof. By monotonicity of $t \rightarrow \lambda_1(t)$, it is continuous excepting at most countably many points. Suppose that $t_0 \in (a,b)$ is a point of discontinuity of $t \rightarrow \lambda_1(t)$. Let $(t_n) \subset (a,b)$ be a sequence with $t_n \nearrow t_0$, and $\bar{\lambda} := \lim \lambda_1(t_n)$. Further let $u_n > 0$, where $\|u_n\| = 1$, be eigenfunction corresponding to $\lambda_1(t_n)$, where

$$\|u\| := \sup \left\{ |u(x)| : x \in [a,b] \right\}. \tag{9}$$

Thus

$$u_n = \lambda_1(t_n) L_n^{-1} M u_n, \tag{10}$$

where L_n denotes the operator defined by (3) in $[a,t_n]$ and the boundary conditions $\alpha_1 u(a) - \beta_1 u'(a) = 0$, $u(t_n) = 0$. We infer the relative compactness of the sequence (u_n) in E, where $E = C^1([a,b])$ with norm defined by (9). Hence for a subsequence $u_n \rightarrow u$ in E, $u > 0$, and

$$u = \bar{\lambda} L_0^{-1} M u, \tag{11}$$

where L_0 is the same operator as above for the interval $[a,t_0]$. Of course $\|L_0^{-1} - L_n^{-1}\| \rightarrow 0$. From (11) it follows that $0 < \bar{\lambda} = \lambda_1(t_0 - 0)$ is the principal eigenvalue of equation (1) for the interval $[a,t_0]$ with boundary conditions

$\alpha_1 u(a) - \beta_1 u'(a) = 0$, $u(t_0) = 0$. Taking the sequence $(s_n) \subset (a,b)$ such that $s_n \searrow t_0$ and reasoning as above, we get

$$v = \underline{\lambda} L_0^{-1} Mv, \tag{12}$$

where $0 < \underline{\lambda} := \lim \lambda_1(s_n) = \lambda_1(t_0+0)$, and $v > 0$. By (12), we get that $\underline{\lambda}$ is also positive principal eigenvalue of the same problem as $\bar{\lambda}$. Because the positive principal eigenvalue is unique, we have $\bar{\lambda} = \underline{\lambda}$, so $\lambda_1(t_0-0) = \lambda_1(t_0+0)$. Thus the function $t \to \lambda_1(t)$ is continuous in t_0. This contradiction proves Lemma 2.

3. Existence and properties of second eigenvalue of (1), (2)

In this section we shall use the following lemma.

Lemma 3. Let u_i and u_j be two arbitrary eigenfunctions of problem (1), (2) corresponding to positive eigenvalues λ_i and λ_j, respectively, such that $\lambda_i < \lambda_j$. Then between each two zeros of u_i there is a zero of eigenfunction u_j.

This lemma is the particular case of Theorem 4, [2].

Using the results of Section 2, we shall prove the following theorems.

Theorem 1. If the problem (1), (2) has an eigenvalue $\lambda > \lambda_1 > 0$ (λ_1 being the principal eigenvalue of (1), (2)), then there exists an eigenvalue λ_2 of (1), (2) such that $\lambda \geq \lambda_2 > \lambda_1$ and such that the eigenfunction u_2, corresponding to λ_2, has exactly one zero in (a,b).

Proof. Let u be an eigenfunction of (1), (2) corresponding to the eigenvalue λ. Since λ_1 is unique positive eigenvalue of (1), (2) to which corresponds positive eigenfunction, then the function u has zeros in (a,b). If u has exactly one zero in (a,b), then putting $\lambda_2 := \lambda$ and $u_2 := u$ we get the thesis of Theorem 1.

Suppose now that the function u has at least two zeros in (a,b). Let $x_0 \in (a,b)$ be a smallest and $x_1 \in (a,b)$ a largest of zeros of the function u in (a,b). Without loss of generality we may assume that the function u is positive in

(a,x_0). This means that $\lambda > 0$ is positive principal eigenvalue of equation (1) for interval (a,x_0) with boundary conditions $\alpha_1 u(a) - \beta_1 u'(a) = 0$, $u(x_0) = 0$. Because $x_0 < x_1$, so denoting $I_1 = (a,x_0)$, $I_2 = (a,x_1)$ and using Lemma 1, we get

$$(13) \qquad \lambda_1^{(2)} < \lambda ,$$

where $\lambda_1^{(2)}$ is positive principal eigenvalue of equation (1) for interval I_2 with boundary conditions $\alpha_1 u(a) - \beta_1 u'(a) = 0$, $u(x_1) = 0$. On the other hand λ is positive principal eigenvalue of equation (1) for interval (x_1,b) with boundary conditions $u(x_1) = 0$, $\alpha_2 u(b) + \beta_2 u'(b) = 0$. Reasoning analogously as above we conclude that

$$(14) \qquad \lambda > \lambda_1^{(0)} ,$$

where $\lambda_1^{(0)}$ is positive principal eigenvalue of equation (1) for interval (x_0,b) with boundary conditions $u(x_0) = 0$, $\alpha_2 u(b) + \beta_2 u'(b) = 0$.

Let $t \in [x_0,x_1]$ be a fixed number. We denote by $\lambda_1^{(2)}(t)$ and $\lambda_1^{(0)}(t)$ the positive principal eigenvalues of equation (1) for intervals (a,t) and (t,b), respectively, with boundary conditions $\alpha_1 u(a) - \beta_1 u'(a) = 0$, $u(t) = 0$, $\alpha_2 u(b) + \beta_2 u'(b) = 0$. By Lemma 2, the mappings $t \longrightarrow \lambda_1^{(2)}(t)$ and $t \longrightarrow \lambda_1^{(0)}(t)$ are continuous in the interval $[x_0,x_1]$. From (13) and (14) we obtain

$$(15) \qquad \lambda_1^{(2)} = \lambda_1^{(2)}(x_1) < \lambda = \lambda_1^{(0)}(x_1) \text{ and } \lambda = \lambda_1^{(2)}(x_0) > \lambda_1^{(0)} = \lambda_1^{(0)}(x_0).$$

Due to the continuity and monotonicity of the mappings $t \rightarrow \lambda_1^{(2)}(t)$ and $t \rightarrow \lambda_1^{(0)}(t)$, it follows from (15) that there exists exactly one point $t_0 \in (x_0,x_1)$ such that

$$(16) \qquad \lambda_1^{(2)}(t_0) = \lambda_1^{(0)}(t_0).$$

Naturally, $\lambda_1^{(2)}(t_o)$ and $\lambda_1^{(o)}(t_o)$ are positive principal eigenvalues of equation (1) for the intervals (a,t_o) and (t_o,b), respectively, with boundary conditions $\alpha_1 u(a) - \beta_1 u'(a) = 0$, $u(t_o) = 0$, $\alpha_2 u(b) + \beta_2 u'(b) = 0$. Let w and v be positive eigenfunctions corresponding to $\lambda_1^{(2)}(t_o)$ and $\lambda_1^{(o)}(t_o)$, respectively. As we know $w \in C^1([a,t_o]) \cap C^2((a,t_o))$ and $v \in C^1([t_o,b]) \cap C^2((t_o,b))$. Let as put

$$u_2(x) := \begin{cases} w(x) & \text{for } x \in [a,t_o], \\ \delta v(x) & \text{for } x \in [t_o,b], \end{cases} \tag{17}$$

where δ is such a number that

$$w'(t_o-0) = \delta v'(t_o+0). \tag{18}$$

It follows from (17) and (18) that $u_2 : [a,b] \to R$ is a mapping of the class $C^1([a,b]) \cap C^2((a,t_o) \cup (t_o,b))$. Let us observe that

$-p(x)u_2''(x) + q(x)u_2'(x) + r(x)u_2(x) = \lambda_1^{(2)}(t_o)m(x)u_2(x)$ for $x \in (a,t_o)$,

$-p(x)u_2''(x) + q(x)u_2'(x) + r(x)u_2(x) = \lambda_1^{(o)}(t_o)m(x)u_2(x)$ for $x \in (t_o,b)$.

This implies the existence of u_2'' at t_o, by $u_2'(t_o) = 0$, $u_2'(t_o-0) = u_2'(t_o+0)$, by continuity of the coefficients p, q, r, m in $[a,b]$ and the condition $p(x) > 0$ for $x \in [a,b]$. Therefore the function u_2, defined by (17), is an eigenfunction of problem (1), (2) which corresponds to the eigenvalue $\lambda_2 := \lambda_1^{(2)}(t_o)$. Of course, the function u_2 has exactly one zero $t_o \in (a,b)$ such that $u_2'(t_o) \neq 0$, whereas the eigenvalue λ_2 satisfies the inequality

$$\lambda_1 < \lambda_2 \leqslant \lambda. \tag{19}$$

The proof of Theorem 1 is complete.

Theorem 2. Under the assumptions of Theorem 1, there is no eigenvalues of problem (1), (2) belonging to the interval (λ_1, λ_2).

This theorem is a direct consequence of Lemma 3.

4. Existence and properties of n-th eigenvalue of (1), (2)

In this section the results of Sections 2, 3 will be used in the case of another positive eigenvalues of (1), (2), provided that these exist. Denote by λ_n the n-th eigenvalue of (1), (2), i.e. this positive eigenvalue of (1), (2) which corresponds to eigenfunction u_n having exactly n-1 zeros in (a,b).

Lemmas 1, 2 refer to the case n=1; in order to generalize them on the case of arbitrary natural n, we use the induction with respect to n. Therefore let $\lambda_n^{(1)}$ and $\lambda_n^{(2)}$ denote n-th eigenvalues of equation (1) for intervals I_1 and I_2, respectively. The intervals I_1, I_2 and also the boundary conditions on the ends of I_1, I_2 are the same as in Section 2.

Assuming that Lemmas 1, 2 hold for any natural n, we shall prove the following ones.

Lemma 4. If there exists the eigenvalue $\lambda_{n+1}^{(1)}$ of equation (1) for interval I_1, then there exists an eigenvalue $\lambda_{n+1}^{(2)}$ of equation (1) for interval I_2 and

$$\lambda_{n+1}^{(2)} < \lambda_{n+1}^{(1)}. \tag{20}$$

Proof. Let $u_{n+1}^{(1)}$ denote an eigenfunction corresponding to the eigenvalue $\lambda_{n+1}^{(1)}$. By assumption, $u_{n+1}^{(1)}$ has exactly n zeros in I_1. Let $I_1 = (x_0, x_{n+1})$ and $I_2 = (\underline{x}, \bar{x})$. At first we consider the case $x_0 = \underline{x}$. Evidently $x_{n+1} < \bar{x}$, because $I_1 \subset I_2$ and $I_2 \setminus I_1 \neq \emptyset$. We denote by $x_1, x_2, \ldots, x_n$ the zeros of $u_{n+1}^{(1)}$ in I_1. Let us observe that $u_{n+1}^{(1)}$, considering in the interval (x_0, x_n), is the n-th eigenfunction of (1) for this interval with boundary conditions $u(x_0) = 0$, $u(x_n) = 0$, if $x_0 > a$, and $\alpha_1 u(x_0) - \beta_1 u'(x_0) = 0$, $u(x_n) = 0$, if $x_0 = a$. This means that $\lambda_{n+1}^{(1)}$ is the n-th eigenvalue of (1) for the interval (x_0, x_n). On the other hand $\lambda_{n+1}^{(1)}$ is the principal positive eigenvalue of (1) for the interval (x_n, x_{n+1}). Since $x_{n+1} < \bar{x}$, then, by Lemma 1, the principal positive eigenvalue $\bar{\lambda}$ of equation (1) for interval $(x_n, \bar{x})$ exists and $\bar{\lambda} < \lambda^{(1)}_{n+1}$.

Now let $x_n \nearrow \bar{x}$; then, by the induction assumption, $\lambda^{(1)}_{n+1}$ as the n-th eigenvalue of (1) for the interval (x_0, x_n) decreases continuously, while $\bar{\lambda}$ increases continuously. Because $\lim \bar{\lambda}(x_n) = +\infty$, as easily can be seen, there exists $\bar{x}_n$ such that $x_n < \bar{x}_n < \bar{x}$ and

$$\lambda^{(1)}_{n+1}(\bar{x}_n) = \bar{\lambda}(\bar{x}_n) < \lambda^{(1)}_{n+1} = \lambda^{(1)}_{n+1}(x_n). \tag{21}$$

Reasoning analogously as in the proof of Theorem 1, we infer that $\lambda^{(1)}_{n+1}(\bar{x}_n)$ is the (n+1)-th eigenvalue at (1) for interval I_2. From this, by (21), there follows the thesis of Lemma 4 in the case of $x_0 = \underline{x}$. Of course, when $\underline{x} < x_0$ and $x_{n+1} = \bar{x}$, the proof of Lemma 4 is analogous.

It remains to consider the case of $\underline{x} < x_0$ and $x_{n+1} < \bar{x}$. Repeating the first part of this proof we get $\lambda^{(1)}_{n+1} > \bar{\lambda}_{n+1}$, where $\bar{\lambda}_{n+1}$ is the (n+1)-th eigenvalue of (1) for the interval $(x_0, \bar{x})$. Using again the first part of this proof we have $\bar{\lambda}_{n+1} > \lambda^{(2)}_{n+1}$. The proof of Lemma 4 is completed.

Let us denote by $\lambda_{n+1}(t)$ the (n+1)-th positive eigenvalue of equation (1) for interval (a, t) with boundary conditions $\alpha_1 u(a) - \beta_1 u'(a) = 0$, $u(t) = 0$, where $a < t < b$. Assuming that $\lambda_{n+1}(t)$ exists for each $t \in (a, b)$, it follows from Lemma 4, that the mapping $t \longrightarrow \lambda_{n+1}(t)$ is a decreasing function in (a, b).

Lemma 5. The mapping $t \longrightarrow \lambda_{n+1}(t)$ is continuous in the interval (a, b).

We omit the proof, analogous to that of Lemma 2.

Lemmas 4, 5 imply the following result.

Theorem 3. If for the problem (1), (2) there exists an eigenvalue $\lambda > \lambda_n > 0$, then:

(a) there exists the eigenvalue λ_{n+1} of the problem (1), (2) such that $\lambda_n < \lambda_{n+1} \leqslant \lambda$,

(b) there exist no eigenvalues of the problem (1), (2) belonging to the interval $(\lambda_n, \lambda_{n+1})$.

The proof of (a) is similar to that of Theorem 1, while (b) follows from Lemma 3.

Up to now we have considered the case of the problem (1), (2) having the positive principal eigenvalue λ_1 and we were interested in these eigenvalues of (1), (2) which are greater than λ_1.

As we know (cf. [3]) that the weight function m in the equation (1) admits both positive and negative value in (a,b), there exist two principal eigenvalues $\lambda_{-1} < 0$ and $\lambda_1 > 0$ of the problem (1), (2). Moreover the eigenfunction u_{-1} corresponding to λ_{-1} is also positive in (a,b). It is easily seen that, if the problem (1), (2) has negative eigenvalues smaller than λ_{-1}, then we may arrange them in sequence

$$0 > \lambda_{-1} > \lambda_{-2} > \dots$$

such that each eigenfunction u_{-n} corresponding to λ_{-n} has exactly n-1 zeros in the interval (a,b).

REFERENCES

[1] J. Bochenek: On a certain question for the linear combinations of the eigenfunctions, in the Sturm-Liouville problem, Zeszyty Nauk. Univ. Jagiellon. 7 (1962), 43-47.

[2] J. Bochenek: On some linear eigenvalue problems with an indefinite weight function, Univ. Iag. Acta Math. 27 (1988), 315-325.

[3] P. Hess, T. Kato: On some linear and non-linear eigenvalue problems with an indefinite weight function, Comm. Partial Diff. Equations, 5 (1980), 999-1030.

[4] P. Hess: On the principal eigenvalue of a second order linear elliptic problem with an indefinite weight function, Math. Z., 179 (1982), 237-239.

[5] А.Ю. Левин, Г.Д. Степанов: Одномерные краевые задачи с операторами не понижающими числа перемен знака I, II, Сиб. Мат. Ж. 17 (1976), 606-626, 813-830.

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ON THE DEGREE OF APPROXIMATION, II

1. Introduction and results

Let f be periodic with period 2π, and integrable in the sense of Lebesgue. The Fourier series associated with f at the point x is given by

(1.1) $$f(x) \sim \frac{1}{2} a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

Let

(1.2) $$S_n(x) = \frac{1}{2} a_0 + \sum_{\nu=1}^{n} (a_v \cos vx + b_v \sin vx)$$

denote the n-th partial sum of the Fourier series (1.1). A function f(x) is said to belong to the class Lip((t),p), $p > 1$, if

$$|f(x+t) - f(x)| = O\left(\psi(t)\, t^{-\frac{1}{p}}\right), \quad t \to 0$$

where $\psi(t)$ is a positive increasing function.

A series $\sum_{n=0}^{\infty} a_n$ with the sequence of partial sum S_n is said to be almost Borel-summable to S if

$$\lim_{x \to \infty} e^{-x} \sum_{n=0}^{\infty} \frac{x^n}{n!} S_{n,r} = s,$$

uniformly with respect to r, where

(1.3) $$S_{n,r} = \frac{1}{(n+1)} \sum_{k=r}^{n+r} S_k$$

We define the operator,

$$B_{n,r}(x) = e^{-x} \sum_{n=0}^{\infty} S_{n,r}(x) \frac{x^n}{n!}$$

where $S_{n,r}(x)$ is defined by (1.3).

We write

$$\phi(t) = \{f(x+t) + f(x-t) - 2f(x)\},$$

and

$$\Theta_{B}^{\sim}(t) = \frac{1}{2x \sin^2(\frac{t}{2})} \left\{ \cos rt - \frac{\cos(rt + x \sin t)}{e^{2x \sin^2(\frac{t}{2})}} \right\}. \tag{1.4}$$

The problem of determining the degree of approximation using the infinite summability means (such as Logarithmic means, generalized Abel s means, (J, p_n) means, Borel means) as operator have already been discussed by Khan, Huzoor H. [3], [4], [5] for the functions belonging to the class $\mathrm{Lip}(\psi(t),p)$, $p > 1$, imposing certain conditions on the function involved. It was noticed that the degree of approximation obtained by these summability means are almost the same as obtained by other finite summability operators. In the present paper, we have for the first time discussed the degree of approximation of functions belonging to the class $\mathrm{Lip}(\psi(t),p)$, $p > 1$ using almost Borel summability operator (of course, it is an infinite summability operator). The idea of almost convergence of Fourier series was introduced by Dayal, S [1]. Using this idea, we introduce the "almost Borel summability operator" and determine the rapidity for convergence.

2. Theorem. If $f(x)$ be a periodic function belonging to the class $\mathrm{Lip}(\psi(t),p)$, $p > 1$ such that

$$\left\{ \int_0^{1/(x+r+1)} \left(\frac{\psi(t)}{t^{1/p}} \right)^p dt \right\}^{1/p} = O\left(\psi\left(\frac{1}{x+r+1} \right) \right) \tag{2.1}$$

and

$$(2.2)\qquad \left\{ \int_{1/(x+r+1)}^{1/(x+1)^{\delta/2}} \left(\frac{\psi(t)}{t^{1+(1/p)}} \right)^p dt \right\}^{1/p} = \left(\psi\left(\frac{1}{x+r+1}\right)(x+r+1) \right)$$

then

$$B_{n,r}(x) - f(x) = O\left(\psi\left(\frac{1}{x+r+1}\right) (x+r+1)^{1/p} \right).$$

In order to prove the theorem we need the following lemmas:

Lemma 1. If $0 < t < \frac{1}{x+r+1}$, then

$$\theta_{\tilde{B}}(t) = O\,(x+r+1).$$

Lemma 2. If $\frac{1}{x+r+1} < t < \pi$, then

$$\theta_{\tilde{B}}(t) = O(1/t).$$

The proofs of Lemmas are trivial.

Proof of the theorem. We know that

$$S_n(x) - f(x) = \frac{1}{2\pi} \int_0^{\pi} \phi(t)\, \frac{\sin(n+\frac{1}{2})\, t}{\sin(\frac{t}{2})}\, dt,$$

then

$$S_{n,r}(x) - f(x) = \frac{1}{(n+1)} \sum_{k=r}^{n+r} \left\{ S_k(x) - f(x) \right\} =$$

$$= \frac{1}{(n+1)2\pi} \int_0^{\pi} \left\{ \frac{\cos rt - \cos(x+r+1)t}{2\sin^2(\frac{t}{2})} \right\} dt.$$

Now using (1.4), we have

$$B_{n,r}(x) - f(x) = e^{-x}\sum_{n=0}^{\infty}\frac{x^n}{n!}\left\{S_{n,r}(x) - f(x)\right\} =$$

$$= \frac{1}{2\pi}\left[e^{-x}\int_0^{\pi}\phi(t)\,\frac{1}{2x\sin^2(\frac{t}{2})}\,e^{x}\left\{\cos rt - \frac{\cos(rt+x\sin t)}{e^{\left\{2x\sin^2(\frac{t}{2})\right\}}}\right\}dt\right] =$$

$$= \frac{1}{2\pi}\int_0^{\pi}\phi(t)\,\Theta_{\tilde{B}}(t)\,dt =$$

$$= \frac{1}{2\pi}\left(\int_0^{1/(x+r+1)} + \int_{1/(x+r+1)}^{1/(x+1)^{\delta/2}} + \int_{1/(x+1)^{\delta/2}}^{\pi}\right)\phi(t)\Theta_{\tilde{B}}(t)\,dt =$$

$$= B_1(x) + B_2(x) + B_3(x).$$

For evaluating $B_1(x)$ we use Hölder's inequality and the fact that $\phi(t) \in \mathrm{Lip}(\psi(t),p)$, for $p > 1$ and obtain from (2.1):

$$|B_1(x)| \leqslant \frac{1}{\pi}\left[\left\{\int_0^{1/(x+r+1)}|\phi(t)|^p\,dt\right\}^{1/p}\cdot\left\{\int_0^{1/(x+r+1)}|\Theta_{\tilde{B}}(t)|^q dt\right\}^{\frac{1}{q}}\right] =$$

$$= O(1)\left[\left\{\int_0^{1/(x+r+1)}\left(\frac{\psi(t)}{t^{1/p}}\right)^p dt\right\}^{1/p}\times\right.$$

$$\left.\times\, O(x+r+1)\left\{\int_0^{1/(x+r+1)}(dt)\right\}^{1/q}\right] =$$

$$= O\left(\psi\left(\frac{1}{x+r+1}\right)(x+r+1)\right)\cdot\frac{1}{(x+r+1)^{1/q}} =$$

$$= O\left(\psi\left(\frac{1}{x+r+1}\right)(x+r+1)^{1/p}\right).$$

Similarly for evaluating $B_2(x)$ we have by (2.2)

$$B_2(x) = O\left[\left\{\int_{1/(x+r+1)}^{1/(x+1)^{\delta/2}} \left(\frac{|\phi(t)|}{t}\right)^p dt\right\}^{1/p} \cdot \left\{\int_{1/(x+r+1)}^{1/(x+1)^{\delta/2}} |t\Theta_{\tilde{B}}(t)|^q \, dt\right\}^{1/q}\right] =$$

$$= \left[\left\{\int_{1/(x+r+1)}^{1/(x+1)^{\delta/2}} \left(\frac{\psi(t)}{t^{1+(1/p)}}\right)^p dt\right\}^{1/p} \cdot O(1) \left\{\int_{1/(x+r+1)}^{1/(x+1)^{\delta/2}} dt\right\}^{1/q}\right] =$$

$$= O\left(\psi\left(\frac{1}{(x+r+1)}\right)\right)(x+r+1) \cdot O\left(\frac{1}{(x+r+1)}\right)^{1/q} =$$

$$= O\left(\psi\left(\frac{1}{x+r+1}\right)(x+r+1)^{1/p}\right), \quad \text{where} \quad 0 < \delta < 1.$$

Finally, for $B_3(x)$ we have

$$B_3(x) = \frac{1}{2\pi}\left[\int_{1/(x+1)^{\delta/2}}^{\pi} \phi(t) \frac{1}{2x\sin^2(\frac{t}{2})} \left\{\cos rt - \frac{\cos(rt + x\sin t)}{e^{2x\sin^2(\frac{t}{2})}}\right\} dt\right] =$$

$$= \frac{1}{2\pi}\left\{\int_{1/(x+1)^{\delta/2}}^{\pi} \phi(t) \frac{\cos rt}{2x\sin^2(\frac{t}{2})}\right\} dt - \frac{1}{2\pi}\left\{\int_{1/(x+1)^{\delta/2}}^{\pi} \phi(t) \left[\frac{\cos(rt + x\sin t)}{2x\sin^2(\frac{t}{2}) \cdot e^{2x\sin^2(\frac{t}{2})}}\right] dt\right\} = B_{3,1}(x) + B_{3,2}(x).$$

Now using the Riemann-Lebesgue theorem [6] and Silverman--Toeplitz theorem [2], we have

$$|B_{3,1}(x)| = \left(2x\sin^2\left(\frac{1}{2(x+1)^{\delta/2}}\right)\right)^{-1} \int_{1/(x+1)^{\delta/2}}^{\xi} \phi(t)\cos rt \, dt =$$

$$= O\left(\frac{(x+1)}{x}\right)^{\delta} \int_{1/(x+1)^{\delta/2}}^{\xi} \phi(t)\cos rt \, dt = o(1), \text{ where } \frac{1}{(x+1)^{\delta/2}} < \xi < \pi.$$

$$B_{3,2}(x) = \left(2x \sin^2\left(\frac{1}{2(x+1)^{\delta/2}}\right) \cdot e^{2x \sin^2\left(\frac{1}{2(x+1)^{\delta/2}}\right)}\right)^{-1}$$

$$\times \int_{1/(x+1)^{\delta/2}}^{\xi} \phi(t) \cos rt \, dt =$$

$$= O\left(\frac{(x+1)}{x}\right) \cdot e^{-\left\{\frac{x}{2(x+1)^{\delta/2}}\right\}} \int_{1/(x+1)^{\delta/2}}^{\xi} \phi(t) \cos rt \, dt =$$

$$= o(1), \quad \frac{1}{(x+1)^{\delta/2}} < \xi < \pi .$$

Combining the results of $B_1(x)$, $B_2(x)$ and $B_3(x)$, we have

$$B_{n,r}(x) - f(x) = O\left(\psi\left(\frac{1}{(x+r+1)}\right)(x+r+1)^{1/p}\right),$$

which was to be proved.

It can be easily seen that the order of approximation arrived at is the best possible and is free from the mean generating sequence. It can also be noticed that the order of approximation is same as for other infinite summability operators.

REFERENCES

[1] S. Dayal : On Banach limit of Fourier series and conjugate series I, Canadian Mat.. Bull. 11 (1968) 225--262.

[2] G.H. Hardy : Divergent series, Oxford, 1949.

[3] Huzoor H. Khan, S. Umar: Rapidity of approximation, Alig. Bull. Math. 8 (1978), 51-57.

[4] Huzoor H. Khan: On the degree of approximation, Math. Chronicle (Newzealand), 10 (1981), 63-72.

[5] Huzoor H. Khan, S. Umar: On the degree of approximation I, Bull. Inst. Math. Acad. Sinica (China), 10 (3), (1982), 213-223.

[6] E.C. Titchmarsh: Theory of functions. Oxford (1958).

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ON ALMOST r-CONTACT STRUCTURE MANIFOLDS

1. Preliminaries

Almost contact metric structure manifolds have been defined and studied by Sasaki [1], Mishra [2] and others. In the present paper, we have considered r linearly independent C^∞ vector fields $\underset{x}{U}$ and r (C^∞) I-forms $\overset{x}{u}$, r some finite integer, and studied the notion of almost r-contact structure.

Let M^{2n+r} be (2n+r) dimensional differentiable manifold of class C^∞. Suppose there exists on M^{2n+r} a tensor field F of type (1,1), $r(C^\infty)$ linearly independent vector fields $\underset{x}{U}$ and $r(C^\infty)$ 1-forms $\overset{x}{u}$, $x = 1,2,\dots,r$ and satisfying the following properties:

(1.1) $F^2 = -I_n + \overset{x}{u} \otimes \underset{x}{U}$, I_n being unit tensor field

(1.2) $\bar{X} \overset{def}{=\!=} F(X)$ for arbitrary vector field X

and

$$(1.3)\quad \begin{cases} \text{(i)} & F(\underset{x}{U}) = 0, \\ \text{(ii)} & \overset{x}{u} \circ F = 0, \\ \text{(iii)} & \overset{x}{u}(\underset{y}{U}) = \delta^x{}_y, \end{cases}$$

$x,y = 1,2,\dots,r$ and $\delta^x{}_y$ denotes Kronecker delta.

Thus M^{2n+r} satisfying conditions (1.1), (1.2) and (1.3) will be said to possess an almost r-contact structure.

Suppose further that M^{2n+r} is endowed with a Riemannian metric g satisfying [1]

$$(1.4) \qquad g(\overline{X},\overline{Y}) = g(X,Y) - \overset{x}{u}(X)\overset{x}{u}(Y).$$

Then in view of equations (1.1), (1.2), (1.3) and (1.4), M^{2n+r} will be said to possess an almost r-contact metric structure.

Substituting $X = \underset{l}{U}$ in (1.4) and making use of the equations (1.3)(i) and (iii), we get

$$(1.5) \qquad g(\underset{l}{U},Y) = \overset{l}{u}(Y), \quad l = 1,2,\dots,r.$$

Again, putting $\overline{Y}$ for Y in (1.4) and using (1.1) and (1.3)(ii), we get

$$(1.6) \qquad g(\overline{X},Y) + g(X,\overline{Y}) = \overset{l}{u}(Y)g(\overline{X},\underset{l}{U}).$$

Putting $X = \underset{m}{U}$ in (1.6) and making use of (1.3)(i), we obtain

$$(1.7) \qquad g(\underset{m}{U},\overline{Y}) = 0, \quad m = 1,2,\dots,r.$$

Thus in view of (1.7), equation (1.6) takes form

$$(1.8) \qquad g(\overline{X},Y) + g(X,\overline{Y}) = 0.$$

Let us now define 2-form 'F as

$$(1.9) \qquad 'F(X,Y) = g(\overline{X},Y).$$

2. Some results

In this section we shall prove some theorems related to almost r-contact structure.

Theorem 2.1. An almost r-contact structure is not unique. If μ be a non-singular vector valued function on M^{2n+r}, let us put

$$(2.1)\quad \begin{cases} \text{(i)} & \mu \circ F' = F \circ \mu, \\ \text{(ii)} & \overset{x}{v} = \overset{x}{u} \circ \mu, \\ \text{(iii)} & \mu V_x = U_x, \quad x = 1,2,\dots,r. \end{cases}$$

Then $\{F', V_x, \overset{x}{v}\}$ gives an almost r-contact structure on M^{2n+r}.

Proof. Post multiplying (2.1)(i) by F' and making use of equations (1.1) and (2.1), we get

$$\mu \circ F'^2 = F \circ \mu \circ F' = F^2 \circ \mu = -\mu + \overset{x}{u}(\mu) U_x, \quad \mu \circ F'^2 = -\mu + \overset{x}{v}(\mu) V_x,$$

or

$$(2.2)\quad F'^2 = -I_n + \overset{x}{v} \otimes V_x.$$

Also from (2.1)(i) and (iii), we have $\mu \circ F' V_x = F \circ \mu V_x = F U_x = 0$. Thus

$$(2.3)\quad F' V_x = 0, \quad x = 1,2,\dots,r.$$

Again $\overset{x}{v} \circ F' = \overset{x}{u} \circ \mu \circ F' = \overset{x}{u} \circ F = 0$ by (1.1). Thus we have

$$(2.4)\quad \overset{x}{v} \circ F' = 0.$$

Further $\overset{x}{v}(V_y) = \overset{x}{u} \circ \mu (V_y) = \overset{x}{u}(U_y) = \delta^x{}_y$ or

$$(2.5)\quad \overset{x}{v}(V_y) = \delta^x{}_y, \quad x,y = 1,2,\dots,r.$$

By virtue of equations (2.2), (2.3), (2.4) and (2.5) we conclude that $\{F', V_x, \overset{x}{v}\}$ gives an almost r-contact structure to M^{2n+r}.

Theorem 2.2. The necessary and sufficient condition that M^{2n+r} be an almost r-contact structure manifold is that it possesses a tangent bundle π_n of dimension n tangent bundle $\tilde{\pi}_n$ conjugate to π_n and the product set $\pi_r(R^r)$ of ordered r-tuples of real numbers such that $\pi_n \cap \tilde{\pi}_n = \tilde{\pi}_n \cap \pi_r = \pi_n \cap \pi_r = \varphi$ and they span together a tangent bundle of dimension (2n+r). Projections L, M, N on π_n, $\tilde{\pi}_n$ and π_r are given by

$$(2.6)\qquad \begin{cases} \text{(i)} & 2L \overset{\text{def}}{=} -F^2 - iF, \\ \text{(ii)} & 2M \overset{\text{def}}{=} -F^2 + iF, \\ \text{(iii)} & N = F^2 + I_n = \overset{x}{u} \otimes \underset{x}{U}. \end{cases}$$

Proof. Suppose first that M^{2n+r} admits an almost r-contact structure. Hence corresponding to eigenvalue i[2], let $\underset{x}{P}$, x = 1,2,...,n be n linearly independent eigenvectors. Let $\underset{x}{Q}$ be eigenvectors conjugate to $\underset{x}{P}$. Further there are r linearly independent vector fields $\underset{x}{U}$. Thus we have

$$\overset{x}{a}\underset{x}{P} = 0 \Rightarrow \overset{x}{a} = 0,$$

$$\overset{x}{b}\underset{x}{Q} = 0 \Rightarrow \overset{x}{b} = 0,$$

$$\overset{x}{c}\underset{x}{U} = 0 \Rightarrow \overset{x}{C} = 0 \Rightarrow (\overset{x}{a},\ \overset{x}{b},\ \overset{x}{c} \text{ are scalars}).$$

Now, if

$$(2.7)\qquad \overset{x}{a}\underset{x}{P} + \overset{x}{b}\underset{x}{Q} + \overset{x}{c}\underset{x}{U} = 0$$

then

$$(2.8)\qquad \overset{x}{a}\underset{x}{\bar{P}} + \overset{x}{b}\underset{x}{\bar{Q}} + \overset{x}{c}\underset{x}{\bar{U}} = 0.$$

In view of the equation (1.2)(i) and the fact that $\underset{x}{P}, \underset{x}{Q}$ are eigenvectors corresponding to eigenvalues i and -i respectively, we have

(2.9) $$\overset{x}{a}\underset{x}{P} - \overset{x}{b}\underset{x}{Q} = 0, \quad x = 1,2,\ldots,n.$$

Baring (2.9) again and using the same fact that $\underset{x}{P}, \underset{x}{Q}$ are eigenvectors corresponding to eigenvalues i, -i, we get

(2.10) $$\overset{x}{a}\underset{x}{P} + \overset{x}{b}\underset{x}{Q} = 0.$$

Thus from (2.9) and (2.10), we have

$$\overset{x}{a} = \overset{x}{b} = 0, \quad x = 1,2,\ldots,n.$$

Thus from (2.7), it follows that $\overset{x}{C} = 0$.

Thus $\left\{\underset{x}{P}, \underset{x}{Q}, \underset{x}{U}\right\}$ is a linearly independent set. From equation (2.6), we can easily show that

(2.11) (i) $L\underset{x}{P} = \underset{x}{P}$, (ii) $L\underset{x}{Q} = 0$, (iii) $L\underset{x}{U} = 0$,

(2.12) (i) $M\underset{x}{P} = 0$, (ii) $M\underset{x}{Q} = \underset{x}{Q}$, (iii) $M\underset{x}{U} = 0$,

(2.13) (i) $N\underset{x}{P} = 0$, (ii) $N\underset{x}{Q} = 0$, (iii) $N\underset{x}{U} = \underset{x}{U}$.

Thus, there exists a tangent bundle π_n of dimension n, a tangent bundle $\tilde{\pi}_n$ conjugate to π_n and the product space π_r or ordered r-tuples of real numbers such that $\pi_n \cap \tilde{\pi}_n = \pi_n \cap \pi_r = \tilde{\pi}_n \cap \pi_r = \varphi$ and $\pi_n \cup \tilde{\pi}_n \cup \pi_r$ gives a tangent bundle of dimension (2n+r), projections on π_n, $\tilde{\pi}_n$ and π_r being L, M and N respectively.

Suppose conversely that in M^{2n+r} there exists a tangent bundle π_n of dimension n, $\tilde{\pi}_n$ conjugate to π_n and product set π_r such that they are mutually disjoint and span together a tangent bundle of dimension (2n+r).

Let $\underset{x}{P}$ be n linearly independent vectors in π_n, $\underset{x}{Q}$ in $\tilde{\pi}_n$ conjugate to $\underset{x}{P}$ and $\underset{x}{U}$ be r linearly independent vectors in product set π_r. Suppose $\left\{\underset{x}{P}, \underset{x}{Q}, \underset{x}{U}\right\}$ span a tangent bundle of dimension (2n+r). Define the inverse set $\left\{\overset{x}{p}, \overset{x}{q}, \overset{x}{u}\right\}$ as

$$\overset{x}{p}\otimes\underset{x}{P} + \overset{x}{q}\otimes\underset{x}{Q} + \overset{x}{u}\otimes\underset{x}{U} = I_n. \tag{2.14}$$

Let us now put

$$F \overset{\text{def}}{=} i\left\{\overset{x}{p}\otimes\underset{x}{P} - \overset{x}{q}\otimes\underset{x}{Q}\right\}. \tag{2.15}$$

Thus we have

$$F^2 = i\left\{\overset{x}{p}\otimes\underset{x}{\bar{P}} - \overset{x}{q}\otimes\underset{x}{\bar{Q}}\right\}. \tag{2.16}$$

In view of the equation (2.15), the above equation takes form

$$F^2 = -\left\{\overset{x}{p}\otimes\underset{x}{P} + \overset{x}{q}\otimes\underset{x}{Q}\right\}$$

which by virtue of (2.14) takes the form

$$F^2 = -I_n + \overset{x}{u}\otimes\underset{x}{U}. \tag{2.17}$$

Thus M^{2n+r} admits an almost r-contact structure.

REFERENCES

[1] S. Sasaki : On differentiable manifolds with certain structures which are closely related to almost contact structure I, Tohoku Math. J. 12 (1960) 459-476.

[2] R.S. Mishra : Structures on a differentiable manifold and their applications Chandrama Prakashan, Allahabad, India (1984).

[3] K.K. Dube, Ram Nivas: Almost r-contact hyperbolic structure in a product manifold, Demonstratio Math. 11 (1978) 887-897.

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SINGULAR FOLIATIONS OF DIFFERENTIAL SPACES

1. Introduction

The notion of the nonsingular foliation of a differentiable manifold was introduced by Ehresmann and Reeb in [2]. In [8] and [9] Stefan generalized that notion and introduced the notion of the foliation with singularities. In [11] Waliszewski introduced the concept of nonsingular foliation in the category of Sikorski differential spaces (see [3], [6] and [7]).

In the present paper we introduce the notion of singular foliations of differential spaces and study some properties of this notion. For definition of differential spaces we refer to the papers [1], [3], [4], [6], [7], [11], [12] and [14].

We introduce the following notational conventions used in the present paper.

Let M be a differential space or a topological space. Then we denote:

setM - the set of all points of M,

topM - the topology of M,

M_A for $A \subset$ setM - the subspace of M with the set of all points A.

For a differential space M we put TopM = (setM, topM). The distinction of a space from its set of all points makes possible the following convention. By $f:M \longrightarrow N$ we mean that f is a morphism in a suitable category, namely

- f is a smooth map of M into N, if M and N are differential spaces,

- f is a continuous map of M into N, if M and N denote topological spaces,

- f is any set-theoretical function from M into N, if M and N are sets.

By an imbedded differential subspace of a differential space M we mean any differential space N with $\mathrm{set}N \subset \mathrm{set}M$ such that for any point p of N there is such an open in topN neighbourhood U of p that $N_U = M_U$. It is obvious that if K and N are imbedded differential subspaces of M with TopK = = TopN then K = N.

1.1. Definition. Let M be a differential space and let X be a topological space with $\mathrm{set}X \subset \mathrm{set}M$ and $\mathrm{id}_{\mathrm{set}X}: X \longrightarrow \mathrm{Top}M$. Let C be the class of all such functions $\beta: \mathrm{set}X \longrightarrow$ that for any p setX there exist a neighbourhood U of p open in X and a smooth real function α on M satisfying the condition $\beta|_U = \alpha|_U$. Then C is a differential structure on setX. The differential space (setX,C) we denote by M_X. We have $\mathrm{Top}M_X = X$. Any space of the form M_X will be called a pre-imbedded differential subspace of M. Any imbedded subspace is a pre-imbedded subspace.

2. Definition of foliation

2.1. Definition. Let p be a point of a differential space M. By a local trivialization of M around p we mean any system (B,T,U,f), where B, T and U are subsets of setM containing p, U is open in topM and f is such a function $f: B \times T \longrightarrow U$ that $f(p,.) = \mathrm{id}_T$, $f(.,p) = \mathrm{id}_B$ and f maps diffeomorphically $M_B \times M_T$ onto M_U.

2.2. Definition. By a foliation on a differential space M we mean a differential space F such that setM = setF and for any point p of M there is such a local trivialization (B,T,U,f) of M around p that B is open in topF, $F_N = M_B$ and (B,T,U,f) is a local trivialization of F around p. We say that (B,T,U,f) is a local trivialization distinguished by F.

2.3. **Corollary.** Any foliation F of M is regularly lying (see [12]) in M.

2.4. **Corollary.** Any foliation F of M is an imbedded differential subspace of M.

2.5. **Corollary.** If E and F are foliations of M and $\mathrm{Top}E = \mathrm{Top}F$ then $E = F$.

2.6. **Definition.** By a leaf of a foliation F of M we mean any connected component of $\mathrm{Top}F$.

Following Tamura [10] we prove the following topological properties of leaves.

2.7. **Theorem.** Let F be a locally connected foliation on M and let A be an union of leaves. Then the closure $\bar{A}$ in $\mathrm{Top}M$ is an union of leaves too.

Proof. We have to prove that $L \subset \bar{A}$ for any leaf L intersecting $\bar{A}$. Let us take any point p of L and a trivialization (B,T,U,f) distinguished by F around p with connected F_B. Then for any point s of T the set $f(B\times\{s\})$ is contained in a leaf. Since then we have $B \subset \bar{A}$ or $B \cap \bar{A} = \emptyset$. Thus $\bar{A}$ is open and closed in F, hence is an union of leaves.

The following two corollaries from Theorem 2.7 hold for locally connected foliations.

2.8. **Corollary.** For any U open in $\mathrm{Top}M$ the union W of all leaves intersecting U is open in $\mathrm{top}M$.

Proof. Let $A = \mathrm{set}M \setminus W$. Then A is an union of leaves. By Theorem 2.7 $\bar{A}$ is an union of leaves too and since $\bar{A} \cap U = \emptyset$ we have $\bar{A} = \mathrm{set}M \setminus W$, what ends the proof.

Let M/F denote the topological quotient space of $\mathrm{Top}M$ by the equivalence relation determined by the decomposition of M into the leaves of F.

2.9. **Corollary.** The canonical projection from $\mathrm{Top}M$ onto M/F is open (cf. [5]).

3. Operations on foliations

3.1. **Corollary.** If F and G are foliations on differential spaces M and N respectively then their product $F\times G$ is a foliation on $M\times N$.

3.2. **Corollary.** For any foliation F on M and any set U open in topM F_U is a foliation on M_U.

3.3. **Corollary.** If F is an imbedded differential subspace of M with setF = setM and for any point p of M there is a neibhbourhood U of p open in topM such that F_U is a foliation on M_U then F is a foliation on M.

3.4. **Definition.** We say that a point p is a nonsingular point of a foliation F on M if there exists a local trivialization (B, T, U, f) around p distinguished by F with F_T being a discrete space, i.e. $F_T = (T, \mathbb{R}^T)$.

3.5. **Corollary.** The set of nonsingular points of F is open in topM.

One may think that we can introduce the notion of a nonsingular leaf of a foliation. The following example shows that a leaf may contain singular and nonsingular points simultaneously.

3.6. **Example.** Let for a differential space K and its subset $S \subset \mathrm{set}K$, $K_{\infty S}$ denote the differential space with $\mathrm{set}K_{\infty S} = \mathrm{set}K \times \{1\} \cup S \times \mathbb{N}$, $\mathbb{N}$ being the set of all natural numbers, and with the differential structure containing all functions of the form $\alpha \circ \mathrm{pr}_1$, where α is a smooth real function on K and pr_1 denote the projection onto the first coordinate. Saying roughly: $K_{\infty S}$ is the space K, where any point of the set S is considered as infinite number of points in the same place. Put $M' = (\mathbb{R}^2, C^\infty(\mathbb{R}^2))$ and let F be the differential space with $\mathrm{set}F' = \mathbb{R}^2$ and the differential structure containing all real functions α on $\mathbb{R}^2$ such that $\alpha(\cdot, x) \in C^\infty(\mathbb{R})$ for any real x. Then F′ is the decomposition of the plane into the horizontal lines. Put $S = (0,1) \times \mathbb{R} \cup (-\infty, 0] \times (\mathbb{R} \setminus \{0\})$. Then $F = F'_{\infty S}$ is a foliation on $M = M'_{\infty S}$. The set of all singular points of F is $(-\infty, 0] \times \{0\} \times \{1\}$ and is contained in the leaf $\mathbb{R} \times \{0\} \times \{1\} \cup (0,1) \times \{0\} \times \mathbb{N}$, but not equal to it.

3.7. **Definition.** We say that a foliation F on M is nonsingular iff all points of F are nonsingular.

Since Waliszewski (see [11]) assumed foliation to be locally homogeneous, the above definition of nonsingular foliation of differential space is not equivalent to that of Waliszewski.

We notice that Corollaries 3.2 and 3.3 hold when the notion "foliation" is replaced by "nonsingular foliation".

3.8. Theorem. Let $g:M \to N$ be any coregular mapping of differential spaces (see [12]). Let F be a differential space defined as follows: setF = setM and F is the disjoint sum of all spaces of the form $M_{g^{-1}(q)}$, where $q \in$ setN. Then F is a nonsingular foliation on M.

For the proof it sufficies to use the definition of coregularity and apply Corollary 3.3 for nonsingular foliations.

4. Universal local connectedness

4.1. Definition. Let X be a topological space. By a topological universally locally connected (shortly: Ulc) subspace of X we mean a locally connected topological space Y with setY ⊂ setX such that $id_{setY}:Y \to X$ and for any mapping $f:T \to X_{setY}$ of a topological locally connected space T we have $f:T \to Y$. We do not assume Y be a topological subspace of X.

4.2. Corollary. If Y and Z are topological Ulc-subspaces of X and setY = setZ then Y = Z. Indeed, the mapping $id_{setY}:Y \to Z$ is a homeomorphism.

4.3. Definition. By a differential Ulc-subspace of a differential space M we mean any pre-imbedded subspace N such that TopN is a topological Ulc-subspace of TopM.

4.4. Corollary. For any differential Ulc-subspaces N and L of M with setN = setL we have N = L. Moreover, if M is a differential space and X is a topological Ulc-subspace of TopM then there exists unique differential Ulc-subspace N of M such that TopN = X, namely $N = M_X$.

4.5. Theorem. Let K be such a differential Ulc-subspace of a product M×N, where M is locally connected, that setK is of the form setM×S, S ⊂ setN. Then we find such a differential Ulc-subspace P of N with setP = S that K = M×P. In view of Corollary 4.4 such P is unique.

Proof. Let X be the topological quotient space TopK/rel_S, where rel_S is the relation with the cosets of the form setM×{q}, q ∈ S. Identifying setM×{q} with we can assume that setX = S. Since TopK is locally connected, X is locally connected as its quotient space. Since $\mathrm{id}_{\mathrm{set}K}$:TopK ⟶ Top(M×N), we have id_S:X ⟶ TopN ≈ Top(M×N)/$\mathrm{rel}_{\mathrm{set}N}$. We intend to show that X is a topological Ulc-subspace of TopN. Let g be any continuous mapping of a topological locally connected space Y into TopN_S. Then for p setM we have continuous mappings: (p,g(.)):Y ⟶ Top(M×N_S) = Top(M×N)$_{\mathrm{set}K}$, (p,g(.)):Y ⟶ TopK, g:Y ⟶ X and hence X is a topological Ulc-subspace of TopN. In view of Corollary 4.4 we have the unique differential structure on S, which turns it into a differential Ulc-subspace P of N. It is easy to verify that M×P is a differential Ulc-subspace of M×N. Since both the spaces M×P and K are differential Ulc-subspaces of M×N with the same set of points, we have by Corollary 4.4 M×P = K.

4.6. Definition. We say that a locally connected foliation F of M is an Ulc-foliation iff any leaf of F is a differential Ulc-subspace of the foliated space M.

4.7. Theorem. Let F and M be differential spaces, F locally connected. Then F is an Ulc-foliation on M iff the following conditions hold:

(i) setF = setM.

(ii) Any connected component of F is a differential Ulc-subspace of M.

(iii) For any point p of M we find such a local trivialization (B,T,U,f) of M around p that B is open and connected in topF, $F_B = M_B$ and for any q T there is such a connected component L of F that f(B×{q}) ⊂ L.

Proof. The only difficult step of the proof is to show that any local trivialization from condition (iii) satisfies conditions required for trivialization distinguished by a foliation, i.e. that f:F_B×F_T ⟶ F_U is a diffeomorphism.

It sufficies to prove that for any connected component L of F we have a diffeomorphism $f|_{B\times(T\cap L)}: F_B\times F_{T\cap L} \longrightarrow F_{U\cap L}$. But this follows from Theorem 4.5 applied to $f^{-1}(F_{U\cap L})$, F_B, M_T instead of K, M, N, where $f^{-1}(F_{U\cap L})$ denotes the differential space with set of points $B\times(T\cap L)$ and the differential structure induced by f^{-1} from $F_{U\cap L}$.

5. The case of differentiable manifold

Now we consider the case, where the foliated space M is a differentiable manifold. We assume manifolds to be separable. In [13] Waliszewski proved that if a product of two nonempty differential spaces is a differentiable manifold then the factors are differentiable manifolds too. Therefore we have the following theorem.

5.1. Theorem. Let F be a foliation on a differential space M being a differentiable manifold. Then leaves of F are differentiable manifolds too.

5.2. Theorem. Any foliation F of a differentiable manifold M is an Ulc-foliation.

Proof. Let $g:X \longrightarrow \mathrm{Top}M_L$ be a mapping of a topological locally connected space X into a leaf L. Let x be a point of X and let us consider any local trivialization (B,T,U,f) distinguished by F around g(x) with connected M_B. In view of separability of F_L the set $L\cap U$ is of the form $f(B\times S)$, where S is a countable subset of T. Taking as V such an open connected neighbourhood of x in X that $g(V)\subset U$ we conclude that g(V) is contained in the connected component of $\mathrm{Top}M_{f(B\times S)}$ containing f(x), i.e. in B. Since $M_B = F_B$ we conclude that $g|_V:X_V \longrightarrow \mathrm{Top}F_B$. Since x was an arbitrary point of X we have $g:X \longrightarrow \mathrm{Top}F_L$.

Combining Theorems 4.7, 5.1 and 5.2 we have the following theorem.

5.3. Theorem. The notion of foliation on differentiable manifold considered in this paper is equivalent to the notion introduced by Stefan (see [5], [8] and [9]).

There is only a formal difference, since Stefan defined a foliation as a partition of a manifold into leaves.

REFERENCES

[1] G. Andrzejczak: On regular tangent covectors, regular differential forms, and smooth vector fields on a differential space, Colloq. Math. 46 (1982) 243-255.

[2] C. Ehresmann, G. Reeb: Sur les champs d'elements de contact de dimension p complement integrable dans une variete continuement differentiable, C.R. Acad. Sci. Paris 218 (1944) 955-957.

[3] S. MacLane: Differentiable spaces, Notes for Geometrical Mechanics, Winter 1970, p.1-9 unpublished.

[4] H. Matuszozyk: On the formula of Slebodziński for Lie derivative of tensor fields in a differential space, Colloq. Math. 46 (1982) 233-241.

[5] A. Piątkowski: A stability theorem for foliations with singularities, Inst. Math. Pol. Acad. Sci. Preprint 298 June 1984.

[6] R. Sikorski: Abstract covariant derivative, Colloq. Math. 18 (1967) 251-272.

[7] R. Sikorski: Wstęp do geometrii różniczkowej (Introduction to differential geometry), Warszawa 1972 (in Polish).

[8] P. Stefan: Accessibility and foliations with singularities, Proc. London Math. Soc., 29 (1974) 699-713.

[9] P. Stefan: Accessible sets, orbits, and foliation with singularities, Proc. London Math. Soc., 29 (1974) 699-713.

[10] I. Tamura: Topologija slojenij. Moskwa 1979.

[11] W. Waliszewski: Foliations of differential spaces, Demonstratio Math. 18 (1985) 347-352.

[12] W. Waliszewski: Regular and coregular mappings of differential spaces, Ann. Polon. Math. 30 (1975) 263-281.

[13] W. Waliszewski: On differential spaces the Cartesian product of which is a differentiable manifold (to appear).

[14] W. Waliszewski: On differential subspaces of Cartesian space, Colloq. Math. 45 (1981) 257-265.

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KERNELS OF VECTOR BUNDLES MORPHISMS

In this paper we define and study the category of ker-bundles consisting of objects which one may define as kernels of homomorphisms of locally trivial real (if $F = R$) or complex (if $F = C$) vector bundles. We do not assume that homomorphisms are of constant rank, therefore the dimension of fibres may depend on base points; such "bundles" are called quasibundles (see [2]). This paper is a continuation of [3] where we described the category of images of vector bundles morphisms called im-bundles. In the main theorems of [3] we gave an "inner" characterization of any im-bundle and presented a one to one correspondence between the classes of isomorphisms of im-bundles over a space B and the classes of the so called cr-homotopies of maps from B into $\mathrm{End}(F^n)$ (for large n). We proved that a quasibundle $p : E \longrightarrow B$ is an im-bundle if and only if in a neighbourhood U_x of each point $x \in B$ with k-dimensional fibre $p^{-1}(x) \cong F^k$ there exist at least k linearly independent sections $s_1, \dots, s_k : U_x \longrightarrow p^{-1}(U_x)$.

A homotopy of form $\psi : B \times I \longrightarrow \mathrm{End}(F^n)$ was called a cr-homotopy if and only if $\mathrm{rank}\psi(x,t) = \mathrm{rank}\psi(x,0)$ for any $(x,t) \in B \times I$. It was shown in [3] that to any cr-homotopy class $[\varphi]_{cr}$ one may assign bijectively the class $[\mathrm{im}\,\varphi]$ of im-bundles isomorphic with $\mathrm{im}\,\varphi$. The im-bundle $\mathrm{im}\,\varphi$ was defined as the quasibundle with $\mathrm{im}\varphi(x)$ as the fibre over each $x \in B$.

We shall establish analogues of the above facts in the category of ker-bundles. In the first part of this paper we describe the structure of ker-bundles.

In Theorem 1 we show that a quasibundle $\xi = (E,p,B)$ contained as a subbundle in the trivial bundle $\theta^n = (B\times F^n, \pi_1, B)$ is a ker-bundle if and only if over an open neighbourhood U_x of any point $x \in B$ with k-dimensional fibre $p^{-1}(x)$ one may choose a k-dimensional trivial subbundle $\theta^k|U_x \subset \theta^n|U_x$ such that $\xi|U_x \subset \theta^k|U_x \subset \theta^n|U_x$.

In the second part we give a condition for two ker-bundles given as kernels of two endomorphisms of θ^n, to be isomorphic. The existence of a cr-homotopy between ψ_0 and ψ_1 turns out to be necessary but not sufficient condition for $\ker\psi_0 \cong \ker\psi_1$.

We prove that any isomorphism $f : \ker\psi_0 \to \ker\psi_1$ can be extended to an isomorphism $F : \theta^{3n} \to \theta^{3n}$ so that the following diagram commutes:

$$\begin{tikzcd}
\ker\psi_0 \arrow[r, "i"] \arrow[d, "f"'] & \theta^n \oplus \theta^{2n} \arrow[d, "F"] \\
\ker\psi_1 \arrow[r, "i"] & \theta^n \oplus \theta^{2n}
\end{tikzcd}$$

Here i denotes embedding into the first factor.

As a corollary we get the implication:

$$\ker\psi_0 \cong \ker\psi_1 \Rightarrow \mathrm{im}(\psi_0 \oplus I_{2n}) \cong \mathrm{im}(\psi_1 \oplus I_{2n})$$

where I_{2n} stands for the identity map on θ^{2n}.

Theorem 2 applied to the theory of locally trivial vector bundles gives a necessary and sufficient condition for two bundles which are kernels of two endomorphisms of θ^n to be isomorphic. If $\psi_0, \psi_1 \in \mathrm{End}(\theta^n)$ and $\mathrm{rank}\psi_0 = \mathrm{rank}\psi_1 = \mathrm{const}$ then

$$\ker\psi_0 \cong \ker\psi_1 \Leftrightarrow \text{there exist } \alpha, \beta \in \mathrm{Aut}(\theta^{3n}) \text{ such that}$$
$$\beta\cdot(\psi_1 \oplus I_{2n})\cdot\alpha = \psi_0 \oplus I_{2n}.$$

We use the notation of [3] in this paper. We begin with some definitions and constructions.

A real (or complex) quasibundle, called shortly q-bundle, is a triple $\xi = (E,p,B)$ where $p : E \to B$ is a continuous map such that any fibre $p^{-1}(x) = E_x$, $x \in B$, is an n_x-dimensional vector space over the field F of real (or complex) numbers. A (mono-, epi-, izo-) morphism $\varphi : \eta \to \eta'$ of q-bundles is a pair of maps (u,f) such that the diagram

$$\begin{tikzcd} E \arrow[r, "u"] \arrow[d, "p"'] & E' \arrow[d, "p'"] \\ B \arrow[r, "f"] & B' \end{tikzcd}$$

commutes and the restriction of u to any fibre is a linear (mono-, epi-, izo-)morphism. A B-morphism is a morphism of the form (u, id_B). We consider only q-bundles with bounded dimension of fibres (i.e. $\dim_F E_x \leqslant m < \infty$) and such that the sets $B_{\overline{n}}(\xi) = \bigcup_{i \geqslant n} B_i(\xi)$ $(n = 0,1,\ldots,m)$, where $B_i(\xi) = \{x \in B : \dim_F E_x = i\}$, are subcomplexes of a finite CW-complex B. We use the notation $B_i, B_{\overline{n}}$, and $B_{\underline{n}} = \bigcup_{i \leqslant n} B_i$ when it is clear which q-bundle we mean. The category of q-bundles we denote by QVB.

Let $\xi = (E,p,B)$ be a q-bundle and let $x \in B_k$, $k \leqslant n \in N$. Then an n-dimensional hyper chart over an open neighbourhood U of x is a pair (U,φ) where φ is a U-monomorphism in the diagram

$$\begin{tikzcd} p^{-1}(U) = E_U \arrow[r, "\varphi"] \arrow[dr, "p"'] & U \times F^n \arrow[d, "\pi_1"] \\ & U \end{tikzcd}$$

We say that a family $\{(U_i,\varphi_i)\}_{i \in I}$ of hyper charts of a q-bundle $\xi = (E,p,B)$ is compatible if and only if for each pair of charts (U_i,φ_i) and (U_j,φ_j) of dimensions n_i and n_j respectively such that $U_{ij} = U_i \cap U_j \neq \emptyset$ and $n_i \geqslant n_j$ we have a U_{ij}-monomorphism f_{ij} and a commutative diagram

$$\begin{tikzcd} & U_{ij} \times F^{n_i} \\ E_{U_{ij}} \arrow[ur, "\varphi_i"] \arrow[dr, "\varphi_j"'] & \\ & U_{ij} \times F^{n_j} \arrow[uu, "f_{ij}"'] \end{tikzcd}$$

with the relation $f_{ki} \circ f_{ij} = f_{kj}$ which holds for $U_i \cap U_j \cap U_k \neq \emptyset$ and $n_k \geqslant n_i \geqslant n_j$.

A hyper atlas of a q-bundle $\xi = (E,p,B)$ is a family $J = \{(U_i,\varphi_i),f_{ij}\}_{i,j\in I}$ of compatible hyper charts with transition functions f_{ij} such that $\bigcup_{i\in I} U_i = B$.

Observe that any q-bundle ξ which is a subbundle of a locally trivial vector bundle η admits a hyper atlas determined by any atlas of η.

We say that a hyper atlas $J = \{(U_i,\varphi_i),f_{ij}\}_{i,j\in I}$ of a q-bundle $\xi = (E,p,B)$ is tight if and only if for any $n = 0,1,\dots,m$ and $x \in B_n$ there exists an n-dimensional hyper chart (a tight chart at x) (U_i,φ_i) in J with $x \in U_i$. Observe that ξ admits a tight hyper atlas only if the set $B_{\underline{n}}$ for any n are open in B.

Let $\xi = (E,p,B)$ be a q-bundle with a hyper atlas $J = \{(U_i,\varphi_i),f_{ij}\}_{i,j\in I}$. Consider the space $\bar{E} = \coprod_{i\in I} U_i \times F^{n_i}$. We define the following relation in $\bar{E}$: for $(x,v) \in U_i \times F^{n_i}$, $(y,w) \in U_j \times F^{n_j}$ $(x,v) \sim (y,w)$ if and only if $x = y$ and $f_{ki}(x,v) = f_{kj}(y,w)$ for some $k \in I$. This relation can be extended to an equivalence in an obvious way. Let $E(J) = \bar{E}/\sim$ and let $\pi : E(J) \to B$ be defined by $\pi([x,v]) = x$. Then $S(J) = (E(J),\pi,B)$ is an im-bundle (i.e. as we proved in [3] there exists an endomorphism of a trivial bundle such that S(J) is its image) called the span of J.

Recall that for a finite hyper atlas J on $\xi = (E,p,B)$ there exists a B-monomorphism of S(J) onto $\theta^n = (B\times F^n,\pi_1,B)$ and hence also ξ into θ^n.

Theorem 1. Let $\xi = (E,p,B)$ be a q-bundle with the bounded dimension of fibres and such that the sets $B_{\bar{n}}(\xi)$ $(n = 0,1,\dots,m)$ are subcomplexes of a finite CW-complex B. Then the following conditions are equivalent:

a) there exists an embedding of ξ into n-dimensional trivial bundle θ^n such that for any sequence $\{x_i\}_{i\in N} \in B_k$ with $\lim_{i\to\infty} x_i = x \in B_l$ $(l \geqslant k)$ and $\lim_{i\to\infty} E_{x_i} = V \in G_k(F^n)$ we have $V \subsetneq E_x$, (Here $G_k(F^n)$ denotes the Grassmann manifold of k-dimensional subspaces in F^n),

b) ξ admits a finite tight hyper atlas,

c) there exists $n \in N$ and B-endomorphism (given for any embedding of ξ into Θ^n) $\psi : \Theta^n \to \Theta^n$ of the n-dimensional trivial bundle Θ^n over B such that $\xi = \ker \psi = \ker \psi^2$,

d) there exists B-morphism $\varphi : \eta_1 \to \eta_2$ of locally trivial bundles of finite type such that ξ and $\ker \varphi$ are isomorphic.

Proof. The implications $c \Rightarrow d$ and $d \Rightarrow a$ are obvious. At first we prove $a \Rightarrow b$. Assume that $\xi \subset \Theta^n$ and the condition (a) is satisfied. Then, observe that for any point $x \in B$, $x \in B_k(\xi)$ one may construct an open neighbourhood $V \ni x$ and V-monomorphism of the k-dimensional trivial bundle over V into $\Theta^n{}_{|V}$ such that $\xi_{|V} \subset \Theta^k{}_{|V} \subset \Theta^n{}_{|V}$. Therefore the orthogonal (in the canonical riemannian metric) complement of $\Theta^k{}_V$ in $\Theta^n{}_V$ admits (n-k) linearly independent sections in a neighbourhood $U \subset V$ of the point x. Let $\xi^\perp = (E^\perp, \pi_1, B)$ denotes the q-bundle which is the orthogonal complement of $\xi \subset \Theta^n$ i.e. $E^\perp = \{(x,v) \in B \times F^n :\ v \in E^\perp_x\}$. It is obvious that for any point $x \in B$, $x \in B_k(\xi) = B_{n-k}(\xi^\perp)$, there exists an open neighbourhood U and (n-k) linearly independent sections of $\xi^\perp_{|U}$.

By [3] (Theorem 1) we know that $\xi^\perp$ is an im-bundle and hence admits the so called tight para atlas $J^\perp = \{U_i, \varphi^\perp_i), f^\perp_{ij}\}_{ij \in I}$. This means that there exist the family of U_i-monomorphisms $\varphi_i : U_i \times F^{n-n_i} \to E_{U_i}$ and the family $\{f_{ij}\}$ of U_{ij}-monomorphisms such that the following diagrams for $n-n_j \leqslant n-n_i$ commute over $U_{ij} \neq \emptyset$:

$$\begin{tikzcd}
U_{ij}\times F^{n-n_i} \arrow[dr, "\varphi_i^\perp"] & \\
& E_{U_{ij}} \\
U_{ij}\times F^{n-n_j} \arrow[uu, "f_{ij}^\perp"] \arrow[ur, "\varphi_j^\perp"'] &
\end{tikzcd}$$

Moreover the tightness of that atlas means that for any $x \in B$ with $x \in B_{n-k}(\xi^\perp) = B_k(\xi)$ there exists an (n-k)-dimensional the so called para chart $(U_i, \varphi_i^{\perp})$ such that $x \in U_i$ i.e. we have the commutative diagram:

$$\begin{tikzcd}
U_i \times F^{n-k} \arrow[rr, "\varphi_i^{\perp}"] \arrow[dr] & & E_{U_i} \subset U_i \times F^n \arrow[dl] \\
& U_i &
\end{tikzcd}$$

We may assume that the orthogonal complement of the image of any $\varphi_i^{\perp}$ is a trivial bundle over U_i. Define a hyper atlas J on ξ as follows: for $x \in B_k(\xi)$ let $\varphi_i : E_{U_i} \to F^k$ be the orthogonal projection over U_i into the trivial complement in $U_i \times F^n$ of the image $\varphi_i^{\perp}(U_i \times F^{n-k})$ of a tight chart. Observe that φ_i is monomorphic and tight in the neighbourhood U_i. Moreover the compatibility of the maps $\{(U_i, \varphi_i)\}_{i \in I}$ is preserved by the compatibility of the atlas J so we get a tight hyper atlas on ξ. This atlas can be made finite by the compactness of every $B_{\bar{k}}(\xi)$. This proves $a \Rightarrow b$. Proof of $b \Rightarrow c$: Let J be a tight, finite hyper atlas on ξ. There exists a natural number n such that $\xi \subset S(J) \subset \Theta^n$. Consider $\xi^{\perp}$ - the orthogonal complement of ξ in Θ^n. By [3] $\xi^{\perp}$ is an im-bundle so there exists a morphism $\psi : \Theta^n \to \Theta^n$ such that $\xi^{\perp} = \operatorname{im}\psi = \operatorname{im}\psi^2$ and therefore $\ker\psi = (\operatorname{im}\psi)^{\perp} = (\operatorname{im}\psi^2)^{\perp} = \ker\psi^2 = \xi$. This proves Theorem 1.

A q-bundle which is a kernel of a homomorphism of locally trivial bundles is called a ker-bundle. The following Lemma is useful to define morphisms of the category of ker-bundles.

Lemma 1. Let $\xi = (E, p, B)$ be a ker-bundle which is a subbundle of the n-dimensional trivial bundle over B and let $f : \xi \to \Theta^n$ be a B-morphism of q-bundles. Then there exists a B-morphism $F : \Theta^n \to \Theta^n$ such that $F_{|\xi} = f$.

Proof. We may assume that there exist:

i) $J = \{(U_p, \varphi_p), f_{pq}\}_{p,q \in A}$ a tight, finite hyper atlas on ξ where the set A is a finite set of points of B and for any $p \in A$ the hyper chart (U_p, φ_p) is tight at p, i.e. (U_p, φ_p) is k-dimensional when $p \in B_k$.

ii) a partition of unity $\{\eta_p\}_{p\in A}$ subordinated to $\{U_p\}_{p\in A}$.

iii) a family of retractions $\{r_p : U_p \to U_p \cap B_k(\xi)$ where $p \in A \cap B_k(\xi)\}_{p\in A}$,

iv) a B-morphism $\varphi : \theta^n \to \theta^n$ such that $\operatorname{im}\varphi = \operatorname{im}\varphi^2 = S(J) \subset \theta^n$ and $\varphi|_\xi = \mathrm{id}|_\xi$.

For any $p \in A$ we define a map (but not a morphism in the category QVB)

$$\varkappa_p : E(J)|_{U_p} \to E(J)|_{U_p},$$

$$\varkappa_p(x,v) = \begin{cases} (x,v) & \text{if } v \in E_x \\ (r_p(x),v) & \text{if } v \notin E_x. \end{cases}$$

We show that $\varkappa_p$ is continuous. Let $\lim\limits_{k\to\infty} (x_k,v_k) = (x_o,v_o) \in U_p \times F^n$. Then either (A) $v_k \in E_{x_k}$ and $v_o \in E_{x_o}$

or (B) $v_k \in E_{x_k}$ and $v_o \notin E_{x_o}$

or (C) $v_k \notin E_{x_k}$ and $v_o \in E_{x_o}$

or (D) $v_k \notin E_{x_k}$ and $v_o \notin E_{x_o}$.

By Theorem 1 (see condition (a)) we may eliminate the case (B). In the other cases (and its combinations) it is obvious that $\lim\limits_{k\to\infty} \varkappa_p(x_k,v_k) = \varkappa_p(x_o,v_o)$. We define a continuous map: $\bar{f} : S(J) \to \theta^n$, $\bar{f}(x,v) = (x, \sum\limits_{p\in A} \eta_p(x)(\pi_2 \circ f \circ \varkappa_p)(x)$ where $(x,v) \in E(J) \subset B \times F^n$ and π_2 denotes projection on the second factor. To prove that $\bar{f}$ is linear on each fibre observe that by definition $\varkappa_p(x,v) = (x',v)$, hence it is linear on the second factor. Because $\pi_2 \circ f(x',\cdot) : E_x \to \theta^n_x$, is linear we have the linearity of $\bar{f}(x,\cdot)$ for any $x \in B$. Therefore we ist a B-morphism $\bar{f} : S(J) \to \theta^n$. Observe that for $(x,v) \in E$

we have $\varkappa_p(x,v) = (x,v)$ and $\bar{f}(x,v) = (x,\pi_2 \cdot f(x,\sum_{p\in A} \eta_p(x)v) = f(x,v)$ and hence $\bar{f}_{|\xi} = f$. We define $F : \theta^n \to \theta^n$ by the formula $F = \bar{f} \circ \varphi$ (where $\mathrm{im}\varphi = S(J)$). The required property of F is obvious. This proves Lemma 1.

We define the category KerVB of ker-bundles as the full subcategory in the category QVB consisting of the q-bundles which are kernels of morphisms of locally trivial bundles.

Any ker-bundle we may represent by a map $\psi : B \to \mathrm{End}(F^n)$ (or equivalently by a $(n\times n)$-matrix with coefficients in the ring $\Lambda(B)$ of continuous functions on B with values in F). Then we denote such ker-bundle by $\ker\psi$. A ker-bundle $\ker\psi$ has $\ker\psi(x) \subset F^n$ as fibres. Any B-morphism of two ker-bundles over B may be represented by a B-homomorphism of trivial bundles (Lemma 1) and hence by an $(n\times n)$-matrix with coefficients in $\Lambda(B)$.

Observe that for any B-morphism $\varphi : \xi \to \xi'$ of q-bundles, where ξ is a ker-bundle, the kernel of φ is also a ker-bundle.

Remark. Let $\xi_i = \ker \varphi_i$ $(i = 1,2)$ be two ker-bundles given by $\varphi_1 \in \mathrm{End}(\theta^n)$ and φ_2 $\mathrm{End}(\theta^m)$ – two B-endomorphisms. Then one may define the generalisation of the classical operations:

the Whitney sum $\xi_1 \oplus \xi_2 \subset \theta^n \oplus \theta^m$ has as its fibres $(\xi_1 \oplus \xi_2)_x = \xi_{1|x} \oplus \xi_{2|x}$; it is a kernel of $\varphi_1 \oplus \varphi_2$;

the tensor product $\xi_1 \otimes \xi_2 \subset \theta^n \otimes \theta^m$ has as its fibres $(\xi_1 \otimes \xi_2)_x = \xi_{1|x} \otimes \xi_{2|x}$; it is a kernel of $(\varphi_1 \otimes I_m) \oplus (I_n \otimes \varphi_2) \circ D : \theta^n \otimes \theta^m \to (\theta^n \otimes \theta^m) \oplus (\theta^n \otimes \theta^m)$ where D denotes the diagonal embedding.

The aim of this paragraph is to prove the following:

Theorem 2. Let $\varphi_1, \varphi_2 \in \mathrm{End}(\theta^n)$ be two B-endomorphisms of an n-dimensional trivial bundle over a finite CW-complex B such that the sets $B_k^-(\ker \varphi_i)$ $(i = 1,2;\ k \in N)$ are subcomplexes of B. Then there exists an isomorphism $f : \ker \varphi_1 \to \ker \varphi_2$ if and only if there exists a B-automorphism $\alpha \in \mathrm{Aut}(\theta^{3n})$ such that $\ker(\varphi_1 \oplus I_{2n}) = \ker((\varphi_2 \oplus I_{2n})\alpha)$.

Proof. It is obvious that the existence of such $\alpha \in \mathrm{Aut}(\theta^{3n})$ implies that $\ker \varphi_1 \cong \ker \varphi_2$. For the converse let $f : \ker \varphi_1 \to \ker \varphi_2$ be an isomorphism of ker bundles. By Lemma 1 there exists $F \in \mathrm{End}(\theta^n)$ such that $F(\ker \varphi_1) = \ker \varphi_2$. Let us consider the B-homomorphism $\psi : \theta^n \to \theta^{2n}$ defined by $\psi = (F \oplus \varphi_1) \circ D$ where D denotes the diagonal embedding of θ^n into θ^{2n}. Observe that ψ is a monomorphism and so $\mathrm{im}\,\psi$ is an n-dimensional trivial subbundle of θ^{2n}. There exist two following projections:

$A : \theta^{2n} \to \mathrm{im}\,\psi \cong \theta^n$ and $H : \theta^{2n} \to (\mathrm{im}\,\psi)^{\perp}$ where $(\mathrm{im}\,\psi)^{\perp}$ denotes the orthogonal complement of $\mathrm{im}\,\psi$ in θ^{2n}.

We define $\alpha : \theta^{3n} \to \theta^{2n} \oplus \theta^n$ by the formula

$$\alpha(x, v_1, v_2, v_3) = (x, \psi_x(v_1) + H_x(v_2, v_3), A_x(v_2, v_3)).$$

Observe that the condition $\psi_x(v_1) + H_x(v_2, v_3) = 0$ implies $v_1 = 0$ and $(v_2, v_3) \in \mathrm{im}\,\psi(x)$ so for $(v_2, v_3) \neq (0,0)$ we have $A(v_2, v_3) \neq 0$. Since for v $\ker \varphi_1(x)$ we have $\alpha(x, v, 0, 0) = (x, F_x(v), 0, 0)$ hence α is the required isomorphism. This proves Theorem 2.

Corollary 1. The condition $\ker \varphi_1 \cong \ker \varphi_2$ for $\varphi_1, \varphi_2 \in \mathrm{End}(\theta^n)$ implies that $\mathrm{im}(\varphi_1 \oplus I_{2n}) \cong \mathrm{im}(\varphi_2 \oplus I_{2n})$ where I_{2n} denotes the identity automorphism of θ^{2n}.

Proof. Observe that for any $\varphi : \theta^m \to \theta^m$ we have the isomorphism (of im-bundles) $\varphi : \mathrm{coim}\,\varphi \to \mathrm{im}\,\varphi$ where $\mathrm{coim}\,\varphi$ is the orthogonal complement of $\ker \varphi$ in θ^m. By Theorem 2 we have an isomorphism $\mathrm{coim}(\varphi_1 \oplus I_{2n}) \cong \mathrm{coim}(\varphi_2 \oplus I_{2n})$ so there exists an isomorphism $\mathrm{im}(\varphi_1 \oplus I_{2n}) \cong \mathrm{im}(\varphi_2 \oplus I_{2n})$.

Remark. The above result can be improved. The isomorphism $\ker \varphi_1 \overset{f}{\cong} \ker \varphi_2$ implies in fact $\mathrm{im}(\varphi_1 \oplus I_n) \cong \mathrm{im}(\varphi_2 \oplus I_n)$. To prove it observe that the q-bundle $\xi = (E, \pi_1, B \times [0,1])$ where $E = \{((x,t), v(1-t), tf(x)v) \in (B \times [0,1]) \times \ker \varphi_1(x) \times \ker \varphi_2(x) \subset B \times [0,1] \times F^n \times F^n\}$ is a ker-bundle such that $\dim \xi_{(x,t)} = \dim \xi_{(x,0)}$. Therefore its orthogonal complement $\xi^{\perp} \subset \theta^n \oplus \theta^n$ is an im-bundle given by a image of a cr-homotopy between $\varphi_1 \oplus I_n$ and $I_n \oplus \varphi_2$. We proved in [3] (Theorem 2) that this implies $\operatorname{im}(\varphi_1 \oplus I_n) \cong \operatorname{im}(\varphi_2 \oplus I_n)$. Howsver it is not clear for us that Theorem 2 is true with 2n replaced by n.

Corollary 2. Let $\varphi \in \operatorname{End}(\theta^n)$ be an endomorphism of constant rank over B. Then there exists an automorphism $H \in \operatorname{Aut}(\theta^{3n})$ such that the following diagram commutes:

$$\begin{tikzcd}
\operatorname{coim}\varphi \arrow[r, hook] \arrow[d, "\varphi"] & \theta^n \oplus \theta^{2n} \arrow[d, "H"] \\
\operatorname{im}\varphi \arrow[r, hook] & \theta^n \oplus \theta^{2n}
\end{tikzcd}$$

Proof. Observe that if φ is of constant rank then $\operatorname{im}\varphi$ and $\operatorname{coim}\varphi$ are also ker-bundles and therefore Theorem 2 may be applied for $f = \varphi$.

Corollary 3. Let $\varphi \in \operatorname{End}(\theta^n)$ be an endomorphism of constant rank over B. Then there exists an automorphism $H \in \operatorname{Aut}(\theta^{3n})$ such that the image $H(\ker\varphi \oplus \theta^{2n}) = \operatorname{coker}\varphi \oplus \theta^{2n}$ where $\operatorname{coker}\varphi$ is the orthogonal complement of $\operatorname{im}\varphi$ in θ^n.

Proof. The automorphism $H \in \operatorname{Aut}(\theta^{3n})$ from Corollary 2 is the required isomorphism.

Corollary 4. Let $\varphi_1, \varphi_2 \in \operatorname{End}(\theta^n)$ be two B-endomorphisms of constant rank. Then $\ker\varphi_1 \cong \ker\varphi_2$ if and only if there exist two B-automorphisms $\alpha, \beta \in \operatorname{Aut}(\theta^{3n})$ such that $\beta \circ (\varphi_2 \oplus I_{2n}) \circ \alpha = \varphi_1 \oplus I_{2n}$.

Proof. Assume that $\beta \circ (\varphi_2 \oplus I_{2n}) \circ \alpha = \varphi_1 \oplus I_{2n}$ where $\alpha, \beta \in \operatorname{Aut}(\theta^{3n})$. Then α restricted to $\ker(\varphi_1 \oplus I_{2n})$ determines the required isomorphism. Conversely, let $f : \ker\varphi_1 \to \ker\varphi_2$ be an isomorphism. The isomorphism $\alpha \in \operatorname{Aut}(\theta^{3n})$ we define as in the proof of Theorem 2. The isomorphism $\beta \in \operatorname{Aut}(\theta^{3n})$ we define as the following composition:

$$\begin{tikzcd}
\theta^{3n} \tilde{\Rightarrow} \arrow[d, "\cong"', "\text{(the orthogonal projections)}"] \\
\operatorname{im}\varphi_2 \oplus \theta^{2n} \oplus \operatorname{coker}\varphi_2 \arrow[d, "\cong"', "\text{(the inverse of H given in Corollary 2)}"] \\
\operatorname{coim}\varphi_2 \oplus \theta^{2n} \oplus \ker\varphi_2 \arrow[d, "\cong"', "\text{(the isomorphism } \alpha^{-1}\text{)}"] \\
\operatorname{coim}\varphi_1 \oplus \theta^{2n} \oplus \ker\varphi_1 \arrow[d, "\cong"', "(\varphi_1 \oplus \text{ isomorphism given in Corollary 3)}"] \\
\operatorname{im}\varphi_1 \oplus \theta^{2n} \oplus \operatorname{coker}\varphi_1 \arrow[d, "\cong"', "\text{(the natural embedding)}"] \\
\theta^{3n}
\end{tikzcd}$$

This is the required automorphism of θ^{3n}.

REFERENCES

[1] D. Husemoller : Fibre bundles. New York 1966.
[2] M. Karoubi : K-theory, Springer 1977.
[3] J. Samsonowicz : Images of vector bundles morphisms, Bull. Acad. Polon. Sci., 34, No 9-10 (1986), 599-609.

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NOTE ON CONVEX INTEGRAL FUNCTIONALS

Some more refined results on convex integral functional are investigated. This note is also a supplement to paper [3]. The necessity of investigations of convex integral functionals is broadly explained by Rockafellar in [4], [5]. Problems which we shall study here arose by investigations of variational problems and differential equations in Hilbert spaces in [3]. However, some results obtained here are interesting by themselves and generalize several statements from [5].

1. In what follows, we assume X to be a separable reflexive Banach space with norm $|\cdot|$ and Y its dual, paired by $(\cdot,\cdot)$. X_w will denote the space X supplied with its weak topology. Let (E,M,dt) be a measure space such that M is a σ-algebra of subsets of E and dt is a positive σ-finite measure on M which is complete. We shall refer to a function $f : E \times X \to (-\infty,+\infty]$ as an integrand and, for each t in E, we denote by f_t the function $x \to f(t,x)$. f will be called a convex integrand if f_t is convex for each t in E. The conjugate of the integrand f is the integrand g on $E \times Y$ defined by

$$g(t,y) = \sup\{(x,y) - f(t,x) \mid x \in X\}.$$

The general theory of conjugate functions asserts that in this case g_t is convex and lower semicontinuous. Let B be the σ-algebra of Borel subsets of X. $M \otimes B$ will denote the σ-algebra of subsets of $E \times X$ generated by products of sets in M and B. If f is $M \otimes B$-measurable on $E \times X$ and, for each t in E, f_t is lower semicontinuous, then f is said to be normal. There exist other equivalent definitions of a normal integrand (see e.g. [4] p.221). It is known that the integrand g conjugate to the normal integrand f is a normal convex integrand [1].

Let us consider a convex normal integrand g on $E \times Y$ (which may be conjugated, for instance, to a normal integrand f) and let K be any compact set in Y_W. Denote by K_1 a subset of $L^p(E,Y)$, $1 \leq p \leq +\infty$, of measurable functions $y : E \to Y$ such that $y(t) \in K$ for t in E. We state

Proposition 1. If there exists at least one $y_0(\cdot) \in K_1$ such that $\int_E g(t,y_0(t))dt > -\infty$, then

$$(1) \qquad \sup\left\{\int_E g(t,y(t))dt \,\middle|\, y(\cdot) \in K_1\right\} = $$

$$= \int_E \sup\{g(t,y) \mid y \in K\}dt.$$

Proof. By Lemma III. 39 [1], the function $h(t) = \sup\{g(t,y) \mid y \in K\}$ is a measurable function. In view of the assumption and the inequality

$$(2) \qquad g(t,y_0(t)) \leq h(t),$$

the integral in both sides of (1) is well defined. Moreover, there is a summable function $s_0(t)$ such that $g(t,y_0(t)) \leq s_0(t)$ in E. As the inequality in (2) also holds for each $y(\cdot) \in K_1$, we have the inequality $\leq$ in (1). To show the converse inequality, let us take any real number b such that

$$\int_E h(t)dt > b.$$

Hence there is a summable function $s_1(t)$ such that $h(t) > s_1(t)$ in E and

$$\int_E s_1(t)\,dt > b. \tag{3}$$

Define the multifunction $F : E \to K$ by

$$F(t) = \{y \in K \mid g(t,y) > s_1(t)\}.$$

By the measurability of $(t,y) \to g(t,y) - s_1(t)$, the graph of F belongs to $M \otimes B$; it is non-empty. Therefore, by Theorem III.22 [1], there is a measurable function $y_1(t)$, $y_1(t) \in F(t)$ in E. As K is bounded, $y_1(\cdot)$ is bounded.

Further, in view of (3), we may choose a measurable set $S \subset E$ of finite measure such that

$$\int_S s_1(t)\,dt + \int_{E\setminus S} s_0(t)\,dt > b. \tag{4}$$

Put

$$y(t) = \begin{cases} y_1(t) & \text{for } t \text{ in } S, \\ y_0(t) & \text{for } t \text{ in } E \setminus S. \end{cases}$$

Of course, we have $y(\cdot) \in L^p(E,Y)$, $g(t,y(t)) \geq s_1(t)$ in S and $g(t,y(t)) \geq s_0(t)$ in $E \setminus S$. These, along with (4), yield

$$\int_E g(t,y(t))\,dt > b.$$

Hence we infer the needed inequality $\geq$ in (1) to end the proof.

Let now the set K be convex and compact in Y_w. If $dt(E) < +\infty$ and $p > 1$, then the set K_1 is also convex and compact with respect to the weak topology on $L^p(E,Y)$ (in the case $p = \infty$ we always consider the weak* topology on $L^p(E,Y)$). Let us consider the functional

$$I(y(\cdot)) = \int_E g(t,y(t))\,dt \tag{5}$$

defined on $L^p(E,Y)$ (we adopt the convention that $I(y(\cdot)) = +\infty$ if the function $t \to g(t,y(t))$ is not integrable). It is convex. Let further the set K_1 be endowed with the topology induced from the weak topology on $L^p(E,Y)$. Then K_1 happens to be a topological compact Hausdorff space (by the assumption that $p > 1$).

Theorem 1. Suppose the functional I to be lower semicontinuous and finite in K_1. Then the function $h: t \to \sup\{g(t,y) \mid y \in K\}$ is summable in E.

Proof. By Proposition 1, the function h is integrable. Thus, in view of (1), it suffices to justify that $\sup\{I(y(\cdot)) \mid y(\cdot) \in K_1\}$ is finite. But this is a well-known result in convex analysis.

2. All the following results are inspired by paper [6] and they are of fundamental importance in applying existence theorems from the calculus of variations to differential equations in [3]. We assume $E = [0,T]$ with the Lebesgue measure dt and we shall write f_t instead of f.

Let f_t and g_t be normal convex proper integrands, not necessarily conjugate to each other, such that, in $[0,T] \times X$, $f_t(x) \geq 0$ and $g_t(x) \geq h_t(x)$ where h_t is a normal integrand satisfying the growth condition:

$$\int_0^T h_t^*(y(t))dt \quad \text{is finite for all } y(\cdot) \text{ in } L^\infty(0,T,X)$$

(h_t^* denotes the conjugate of $h_t(\cdot)$). By the growth condition and the conjugance relation, $\int_0^T h_t(v(t))dt > -\infty$ for all $v(\cdot)$ in $L^1(0,T,X)$. Let $A(X)$ denote the space of absolutely continuous functions $x : [0,T] \to X$. We put, for a convex closed set $C \subset X$, $A_C(X) = \{x(\cdot) \in A(X) \mid x(0) = x_0,\ x_0$ any fixed point in X, $x(T) \in C\}$. Let us define on the space $L^\infty(0,T,X) \times L^1(0,T,X)$ the convex functional

$$J(x(\cdot),v(\cdot)) = \int_0^T (f_t(x(t)) + g_t(v(t)))dt. \qquad (6)$$

We assume further that, for at least one $x(\cdot)$ in $A_C(X)$, $J(x(\cdot)) = J(x(\cdot),\dot{x}(\cdot))$ is finite. Then, for this $x(\cdot)$, $\int_0^T h_t(x(t))dt < +\infty$ and $\int_0^T g_t(\dot{x}(t))dt < +\infty$. Since $\int_0^T g_t^*(y(t))dt \leqslant \int_0^T h_t^*(y(t))dt < +\infty$ in $L^\infty(0,T,X)$, thus, by Corollary in [4] p.227, the functional $\int_0^T g_t(v(t))dt$ is lower semicontinuous in $L^1(0,T,X)$. In turn, by Fatou s lemma, the functional $\int_0^T f_t(x(t))dt$ is lower semicontinuous in $L^\infty(0,T,X)$. Hence and from the convexity of functional (6) we conclude that it is lower semicontinuous in the weak topology of the space $L^\infty(0,T,X) \times L^1(0,T,X)$. In particular, we obtain that $J(x(\cdot))$ is weakly sequentially lower semicontinuous in $A(X)$. The last statement results from Theorem 3 in [3], as well. By Theorem 4 from [3], $J(x(\cdot))$ attains its minimum in $A_C(X)$.

Let $B^\infty = X \times L^\infty(0,T,X)$ be the linear space under the norm $\|[d,y(\cdot)]\|_\infty = \max\{|d|, \|y(\cdot)\|_{L^\infty}\}$, which is dual to $A(X)$ under the pairing

$$<x(\cdot),[d,y(\cdot)]> = (x(\cdot),d) + \int_0^T (x(t),y(t))dt. \qquad (7)$$

For any $[a,y(\cdot)] \in B^\infty$, we denote the functional J^y as

$$J^y(x(\cdot)) = \int_0^T (f_t(y(t) + x(t)) + g_t(\dot{x}(t)))dt, \quad x(\cdot) \in A(X). \qquad (8)$$

By the above considerations, J^y is well defined on $A(X)$. We define the functional i on B^∞ by

$$i(a,y(\cdot)) = i(y(\cdot)) = \inf\{J^y(x(\cdot)) \mid x(\cdot) \in A_C\}. \qquad (9)$$

Note that if, for each $y(\cdot) \in L^\infty(0,T,X)$, there is $\bar{x}(\cdot)$ in $A_C(X)$ such that $J^y(\bar{x}(\cdot))$ is finite, then, by Theorem 4 from [3], the infimum in (9) is attained for each $y(\cdot)$ in $L^\infty(0,T,X)$. Further, we shall consider only the functional J^y that has this property.

Proposition 2. With the assumptions made above, the functional $i(y(\cdot))$ is sequentially lower semicontinuous with respect to the weak* topology on $L^\infty(0,T,X)$.

Proof. For each $y(\cdot)$ in $L^\infty(0,T,X)$, we have $x_y(\cdot)$ in $A_C(X)$ such that

$$i(y(\cdot)) = J^y(x_y(\cdot)).$$

Functional (6) is lower semicontinuous in the weak topology of $L^\infty(0,T,X) \times L^1(0,T,X)$. Note further that if a sequence $\{x_n(\cdot)\} \subset A(X)$ is weakly convergent in $A(X)$, then it is also convergent in the weak* topology on $L^\infty(0,T,X)$ (it suffices to notice that then $\{x_n(t)\}$ is weakly convergent in X for each t in $[0,T]$ and $\sup_n \|x_n(\cdot)\|_{L^\infty} \leqslant \sup_n \|x_n(\cdot)\|_{A(X)} < +\infty$). The above means that if any $\{y_n(\cdot)\} \subset L^\infty(0,T,X)$ converges in $L^\infty(0,T,X)$ weakly to $y(\cdot)$ and any $\{x_n(\cdot)\} \subset A(X)$ is weakly convergent in $A(X)$ to $\bar{x}(\cdot)$, then

$$(10) \quad \liminf_{n\to\infty} J^{y_n}(x_n(\cdot)) \geqslant J^y(\bar{x}(\cdot)) \geqslant J^y(x_y(\cdot)) = i(y(\cdot)).$$

Therefore, to prove the proposition, it is enough to show that, for each weakly* convergent $\{y_n(\cdot)\}$ from $L^\infty(0,T,X)$, there is $x_{y_{n_k}}(\cdot)$ in $A_C(X)$ weakly convergent in $A(X)$. To this effect, let $\{y_n(\cdot)\} \subset L^\infty(0,T,X)$ be a sequence convergent weakly* to $y(\cdot)$ and $\{x_{y_n}(\cdot)\}$ a sequence from $A_C(X)$ corresponding to it. Then either

$$\liminf_{n\to\infty} J^{y_n}(x_{y_n}(\cdot)) = +\infty$$

or there is $c_0 \in R$ such that the set

$$A_{c_0} = \left\{ x_{y_n}(\cdot) \in A_C(X) \,\middle|\, J^{y_n}(x_{y_n}(\cdot)) \leqslant c_0 \right\}$$

has infinitely many elements. The first alternative implies that the needed inequality in (10) is satisfied. From the second one and Theorem 2 [3] we infer that there exists a subsequence $\left\{ x_{y_{n_k}}(\cdot) \right\}$ weakly convergent in $A(X)$, and this completes the proof.

Along with the functional $J(x(\cdot))$ defined on $A_C(X)$, we need its dual defined on $A(X)$ by

$$J_D(p(\cdot)) = (x_0, p(0)) + m(-p(T)) + \int_0^T (g_t^*(p(t)) + f_t^*(\dot{p}(t)))dt$$

where m is the support function of the set C. It is not difficult to see, by the assumptions made on functional (6), that J_D is well defined on $A(X)$ and convex.

The following theorem is analogous to a part of Theorem 3 from [6] p.196.

Theorem 2. The functional i is convex on B^∞ and its conjugate on $A(X)$ (with respect to pairing (7)) is given by

$$i^* = J_D.$$

The conjugate of J_D on B^∞ is given by

$$i(a, y(\cdot)) = J_D^*(a, y(\cdot)). \qquad (11)$$

Since in the proof of Theorem 3 from [6] it is not essential that the space R^n (in our case X) is of a finite dimension, therefore the first assertion of our theorem is justified quite analogously to the first part of Theorem 3 [6] (therein $\Phi_{l,L}$ corresponds to our i and $\Phi_{m,M}$ to our J_D). The second assertion of our theorem follows from the first one, Proposition 2 and the property of the biconjugate (in such a situation $i^{**} = i$).

Taking in (11) $(a, y(\cdot)) = (0,0)$, by the definition of the conjugate, we easily obtain

Corollary. With the same assumptions, we have

$$\inf\{J(x(\cdot)) \mid x(\cdot) \in A_C(X)\} = -\inf\{J_D(p(\cdot)) \mid p(\cdot) \in A(X)\}.$$

REFERENCES

[1] C. Castaing, M. Valadier: Convex analysis and measurable multifunctions, Lecture Notes Math. 580, Berlin-Heidelberg-New York: Springer (1977).

[2] I. Ekeland, R. Temam: Convex analysis and variational problems, North-Holland Publishing Company, Amsterdam - Oxford, American Elsevier Publishing Company, Inc. New York (1976).

[3] A. Nowakowski, D. Zakrzewska: Variational problems and differential equations in Hilbert spaces, Demonstratio Math. 22 (1989).

[4] R.T. Rockafellar: Convex integral functionals and duality, In: Contributions to Nonlinear Functional Analysis, Ed. Eduardo H. Zarantonello, Acad. Press, New York-London (1971).

[5] R.T. Rockafellar: Integral functionals, normal integrands and measurable selections, In: Nonlinear Operators and the Calculus of Variations, Bruxelles 1975.

[6] R.T. Rockafellar: Conjugate convex functional in optimal control and the calculus of variations, J. Math. Anal. Appl. 32, 174-222 (1970).

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ON UNCONDITIONAL CONVERGENCE OF LINEAR OPERATORS

Let $\{A_n\}$ be a series of linear operators mapping R^k into R^l. Each operator A_n is represented as a matrix $(a^n_{i,j})$, $i = 1,\ldots,l$, $j = 1,2,\ldots,k$. In the theory of stability of difference equation essential role is played (see for example [1], Chapter 4, Lemma 30,1, [2] Chapter 7, sec. 2, proof of Theorem 1, [4] Chapter 1, § 3, Theorem 2) by the following simple.

Theorem 1. If for each arbitrary bounded sequence $\{x_n\} \subset R^k$ the series $\sum A_n x_n$ is convergent then $\sum_{n=1} |a^n_{i,j}| < +\infty$ for $i = 1,2,\ldots,l$, $j = 1,2,\ldots,k$.

Thus it is a natural question to ask how is a situation in infinite dimensional Banach spaces, and it is investigated by the present paper.

Let X,Y be two Banach spaces, either over reals or over complex numbers. By $\|\ \|$ we shall denote it the norms in both spaces, since does not lead to a missunderstanding. Similarly the norms of linear continuous operators acting from X into Y we shall denote by the same symbol $\|\ \|$.

Let $\{A_n\}$ be a sequence of continuous linear operators acting from the space X into the space Y. We shall say that the series $\sum_{n=1}^{\infty} A_n$ is absolutely convergent, if

$$C = \sum_{n=1}^{\infty} \|A_n\| < +\infty. \tag{1}$$

We shall say that the series $\sum_{n=1}^{\infty} A_n$ is unconditionally convergent if for each sequence of number $\{\varepsilon_n\}$ equal either to 0 or to 1 the series $\sum_{n=1}^{\infty} \varepsilon_n A_n$ is convergent in the operator norm topology. We shall say that the series $\sum_{n=1}^{\infty} A_n$ is unconditionally bounded if there is a constant $C > 0$ such that for arbitrary sequence of numbers $\{\varepsilon_n\}$, ε_n equal either to 0 or 1, we have

$$(2) \qquad \left\| \sum_{n=1}^{N} \varepsilon_n A_n \right\| \leqslant C \quad \text{for } N = 1,2,\ldots .$$

We shall say that the series $\sum_{n=1}^{\infty} A_n$ is (b)-unconditionally convergent if for an arbitrary bounded sequence $\{x_n\}$ of elements of X the series $\sum_{n=1}^{\infty} A_n(x_n)$ is convergent. We shall say that the series $\sum_{n=1}^{\infty} A_n$ is (b)-unconditionnally bounded if there is a constant $C > 0$ such that for each sequence $\{x_n\}$ of elements of X with the norms non-greater than 1, $\|x_n\| < 1$, $n = 1,2,\ldots,$ we have

$$(3) \qquad \left\| \sum_{n=1}^{N} A_n(x_n) \right\| \leqslant C, \quad N = 1,2,\ldots .$$

We shall say that the series $\sum_{n=1}^{\infty} A_n$ is pointwise unconditionally convergent if for all $x \in X$ and all sequences $\{\varepsilon_n\}$, where ε_n is equal either 0 or 1, the series $\sum_{n=1}^{\infty} \varepsilon_n A_n(x)$ is convergent. We shall say that the series $\sum_{n=1}^{\infty} A_n$ is pointwise uniconditionally bounded if there is $C > 0$ such that for each $x \in X$ and each sequence of numbers $\{\varepsilon_n\}$, which are equal either 0 or 1, we have

$$(4) \qquad \left\| \sum_{n=1}^{N} \varepsilon_n A_n(x) \right\| < C, \quad N = 1,2,\ldots .$$

By the classical Riemann theorem in finite dimensional spaces unconditional and absolutely convergence are equivalent. If implies that for X and Y finite dimensional the all introduced above notions are equivalent.

The equivalence of (b)-unconditional boundness and absolute convergence play an essential role (see for example [4]) in the theory of stability of finite dimensional difference equations. In the theory of stability of differential equations in finite dimensional spaces an essential role play a corresponding integral equivalence (see for example [1], [2]).

It is interesting to investigate relations between introduced notions also in infinite dimensional spaces, having in mind, possibility to extend the methods used in stability theory in finite dimensional case on infinite dimensional case.

The paper is devoted to the subject. However, I ought to say, that the obtained results, from the point of view of stability theory, have rather negative character.

Of course each absolutely convergent series is unconditionally convergent, (b)-unconditionally convergent, pointwise unconditionally convergent. Each unconditionally convergent series is unconditionally bounded. Each (b)-unconditionally convergent series is (b)-unconditionally bounded. Each (b) unconditionally bounded series is pointwise unconditionally bounded.

Those implications follows almost immediately from the definitions. Observe that all introduced notions of unconditional convergence and unconditional boundness have the following property. The sum of two unconditional convergent (bounded) series in a sence defined above is an unconditional convergent (bounded) series in the same sense. If a series $\sum_{n=1}^{\infty} A_n$ is unconditionally convergent (bounded) in a sense defined above then for each continuous linear operator B mapping Y into Z (mapping Z into X) the series $\sum_{n=1}^{\infty} BA_n \left(\sum_{n=1}^{\infty} A_n B\right)$ is unconditionally convergent (bounded) in the same sense. As an obvious consequence of Banach-Steinhaus theorem we obtain

Proposition 1. Each pointwise unconditionally bounded series is unconditionally bounded.

The similar result for unconditional convergence does not hold as follows from

Example 1. Let X be a Banach space with an unconditional basis (e_n) (i.e. such a basis that the expansion with respect to this basis are unconditionally convergent). We assume that the basis is normal $\|e_n\| = 1$, $n = 1,2,\ldots$. Let $\{f_n\}$ denote the basis functionals.

Since the basis is unconditional the series $\sum\limits_{n=1}^{\infty} P_n$, where $P_n(x) = f_n(x)e_n$ is pointwise unconditionally convergent. On the other hand $\|P_n\| \geq 1$ hence the series $\sum\limits_{n=1}^{\infty} P_n$ is not unconditionally convergent.

Now we shall discuss relations between pointwise unconditionally convergence (boundness), (b)-unconditionally convergence (boundness) and absolute convergence.

Theorem 2. If X is finite dimensional, then pointwise unconditionally convergence (boundness) implies (b)-unconditionally convergence (boundness).

Proof. Suppose that the space X is k-dimensional. Then there is k one dimensional projections P_j, $j = 1,\ldots,k$ such that the identity on X, I is equal

$$I = P_1 + \ldots + P_k. \tag{5}$$

Observe that P_jX is one dimensional. Since for one dimensional X the (b)-unconditionally convergence (boundness) and pointwise unconditional convergence (boundness) coincide, we have that for each pointwise unconditionally convergent (bounded) series $\sum\limits_{n=1}^{\infty} A_n$ the series

$$\sum_{n=1}^{\infty} A_nP_j, \quad j = 1,2,\ldots,k$$

are (b)-unconditionally convergent. Therefore the series $\sum_{n=1}^{\infty} A_n = \sum_{n=1}^{\infty} \sum_{j=1}^{\infty} A_n P_j$ is (b)-unconditionally convergent (bounded).

Theorem 3. Let X be an infinite dimensional space. Then there is a sequence $\{A_n\}$ of linear continuous operator acting from X to Y such that the series $\sum_{n=1}^{\infty} A_n$ is unconditionally convergent but it is not (b)-unconditionally bounded.

Proof. By Dvoretzky-Rogers theorem [3] there is a sequence $\{f_n\}$ of elements of the conjugate space $f_n \in X^*$, $n = 1,2,\dots$, such that the series $\sum_{n=1}^{\infty} f_n$ is unconditionally convergent, but it is not absolutely convergent. It implies that the series of operators $A_n(x) = f_n(x)y$ where $y \neq 0$ is an arbitrary fixed element of Y is unconditionally convergent. On the other hand the series $\sum_{n=1}^{\infty} f_n$ is not absolutely convergent, i.e.

(6)
$$\sum_{n=1}^{\infty} \|f_n\| = +\infty .$$

Let ε be an arbitrary positive number. Let $\{x_n\}$ be an element of norm one such that

(7)
$$\|f_n\| - \frac{\varepsilon}{2^n} \leqslant f_n(x_n) \leqslant \|f_n\|, \quad n = 1,2,\dots .$$

Observe that (7) and (6) implies that

(8)
$$\sum_{n=1}^{\infty} f_n(x_n) = +\infty$$

Thus the series of operators $\sum_{n=1}^{\infty} A_n$ is not (b)-unconditionally bounded.

Theorem 4. Let Y be a finite dimensional space. Then each (b)-unconditionally bounded series is absolutely convergent.

Proof. At the beginning we shall show Proposition 4 for one dimensional space Y. In this case we can identify operator acting from X into Y with continuous linear functionals. Let $\{f_n\}$ be a sequence of linear continuous functionals. Suppose that the series $\sum_{n=1}^{\infty} f_n$ is (b)-unconditionally bounded. It means that for all bounded sequences of elements $\{x_n\}$ the series $\sum_{n=1}^{\infty} f_n(x_n)$ is unconditionally bounded.

Let ε be an arbitrary positive number. Let $\{x_n\}$ be a sequence of elements of norm one such that

$$(7) \qquad \left(1 - \frac{\varepsilon}{2^n}\right)\|f_n\| \leq f_n(x_n) \leq \|f_n\|.$$

Then

$$\sum_{n=1}^{\infty} f_n(x_n) \geq \sum_{n=1}^{\infty} \|f_n\| - \varepsilon$$

and the series $\sum_{n=1}^{\infty} \|f_n\|$ is absolutely convergent.

Now we shall prove the Theorem in the case when Y is k-dimensional. In this case the identity in Y can be represented as a sum of k, one dimensional projection operators P_j, $j = 1,2,\ldots,k$

$$I = P_1 + \ldots + P_k.$$

Observe that if a series $\sum_{n=1}^{\infty} A_n$ is (b)-unconditionally bounded then the series $\sum_{n=1}^{\infty} P_j A_n$, $j = 1,\ldots,k$ are also (b)-unconditionally bounded. Since $P_j A_n$ are one dimensional the series $\sum_{n=1}^{\infty} P_j A_n$, $j = 1,2,\ldots,k$ are absolutely convergent, and thus the series $\sum_{n=1}^{\infty} A_n = \sum_{n=1}^{\infty} \sum_{j=1}^{\infty} P A_n$ is also absolutely convergent.

Theorem 5. If the space Y is infinite dimensional, then there is a (b)-unconditionally convergent series $\sum_{n=1}^{\infty} y_n$ which is not absolutely convergent.

Proof. Since Y is infinite, then there is a sequence $\{y_n\}$, $y_n \in Y$ such that the series $\sum_{n=1}^{\infty} y_n$ is unconditionally convergent and it is not absolutely convergent. Let $\{f_n\}$ be an arbitrary sequence, of linear continuous functionals of norm one defined on X, $f_n \in X^*$.

Let $A_n(x) = f_n(x)y_n$. Since the series $\{y_n\}$ unconditionally convergent and for each bounded sequence of elements $\{x_n\}$ the sequence $\{f_n(x_n)\}$ is bounded, the series of operators $\sum_{n=1}^{\infty} A_n$ is (b)-unconditionally convergent. On the other hand, $\|A_n\| = \|y_n\|$, and since $\sum_{n=1}^{\infty} \|y_n\| = +\infty$, $\sum_{n=1}^{\infty} \|A_n\| = +\infty$.

Remark. Proposition 5 can be formulated in a little stronger way. Namely if X = Y is infinite dimensional, then there is a series of invertible operator B_n such that the series $\sum_{n=1}^{\infty} B_n$ is (b)-unconditionally convergent, but it is not absolutely convergent.

Indeed, we construct as before $A_n(x) = f_n(x)y_n$. Since A_n are one dimensional it is easy to find a sequence $\varepsilon_n > 0$ such that $\sum \varepsilon_n < 1$ and the operators $B_n = \varepsilon_n I - A_n$ are invertible. The constructed sequence B_n has the requerted properties.

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REFERENCES

[1] R. Brockett: Finite Dimensional Linear Systems. New York, 1970.

[2] C.A. Desoer: Notes for a Second Course on Linear Systems, New York 1970.

[3] A. Dvoretzky, C.A. Rogers: Absolute and unconditional convergence in normed linears spaces, Proc. Nat. Acad. Sci. USA 36 (1950) 37-57.

[4] A. Halanay, D. Wexler: Teoria Calitativu a Sistem cu Impulscuri, Bucureşti 1968.

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PROFESSOR EDWARD OTTO (1908-1986)

Professor Edward Otto was born in Lwów on October 13, 1908 in the family of a railroad employee. After obtaining the diploma of a secondary school, in September 1927 he was admitted to the General Faculty of Lwów Polytechnic. He entered the mathematical class taught by the known mathematicians belonging to the Polish Mathematical School: Professors Kazimierz Kuratowski, Włodzimierz Stożek and docents Stefan Kaczmarz, Władysław Nikliborc and Władysław Orlicz. He also attended a course in descriptive geometry conducted by Professor Antoni Plamitzer. After five years of study he graduated with M.Sc. in mathematics. He wrote a diploma work in descriptive geometry under the direction of Professor Plamitzer. A part of this work was published in 1933 under the title "Uber Punkten der Ordnung C" in Monatshefte für Mathematik u. Physik as the first publication of E. Otto. He started his didactic work in descriptive geometry in the chair of Professor Plamitzer. He also continued his study toward Ph.D. in mathematics with Professor Plamitzer as thesis adviser. In 1936 he obtained a Ph.D. on the basis of the thesis entitled "On n-linear correspondences in one-current structures". In 1937 he obtained a position of substituting professor at the Chair of Descriptive Geometry of the Faculty of Engineering at the Technical University of Warsaw (Warsaw Polytechnic). By this fact he started his many-years work at TU of Warsaw lasting till his retirement in 1978. On the position of substituting professor he worked till September 1939, when the Second World War started. During the war he took part in the secret teaching at the university level and prepared his habilitation thesis.

After the war Professor Edward Otto continued his work at the Technical University of Warsaw In 1945 he presented to Professor Plamitzer his habilitation thesis "On some properties of algebraic curves". Professor Plamitzer expressed his positive opinion to the Faculty of Engineering. On the basis of this opinion, Edward Otto obtained the degree of docent at the Technical University of Warsaw. He started lectures and exercises in mathematics and descriptive geometry. He raised fast in professional career: in 1946 he obtained the title of extraordinary professor in mathematics, in 1961 the title of ordinary professor.

Professor Edward Otto did not restrict himself to lecturing only. Together with his co-workers he took up the task of preparing necessary textbooks. As the result of his efforts, he published a textbook on descriptive geometry, a textbook on nomography and several textbooks on differential and integral calculus (as a co-author). All those textbooks have had several editions and are still in use. The textbook on nomography was published in English translation by Pergamon Press. Together descriptive geometry (in two parts), which also have had several edition.

There was a considerable increase in the number of universities and technical universities in Poland after the war. At the same time there was a lack of qualified faculty to teach at those universities. Many teaching positions were occupied by faculty without doctorate. Professor Edward Otto took up the task of developing the faculty in mathematics and descriptive geometry at technical universities. He was at that time the only professor in descriptive geometry. To educate the faculty, Professor Otto started an all-Poland Seminar on Descriptive and Projective Geometry. This seminar was connected with all-Poland conferences on descriptive geometry. The seminar was meeting once in a month, and these meetings allow to solve many problems in descriptive geometry. To help to publish new results, Professor Otto has originated a new scientific journal devoted to descriptive geometry - "Scientific Fascicles on Descriptive Geometry". This journal was published yearly, and Professor Otto has become its chief editor. The fascicles have published original search results in descriptive geometry. As the result of the effort of Professor Otto, the level of teaching and research in descriptive geometry has been significantly raised up. In his seminar, he educated 30 Ph.D.'s, 10 docents and 3 professors.

The activity of professor Edward Otto has not been restricted to teaching and research only, he was also a good organizer of scientific life. He was always ready to help others. For twenty years he took part in organizing the

Mathematical Olympic for high school students. He was a member of the editorial board of Demonstratio Mathematica since its origin. But undoubtedly his most important achievement was his share in creating the Institute of Mathematics at the Technical University of Warsaw. The idea to establish the institute was initiated in 1963 by professors of mathematics at Technical University of Warsaw: W. Pogorzelski, S.Straszewicz and E. Otto. The institute was founded in 1963 and Professor Otto was nominated as Director of the Institute. At the same time the scientific council was named. Professor Otto had been the director of the institute for 12 years. During his term, the institute obtained the law to promotion of Ph.D. in mathematics and applied mathematics, as well as to habilitation. The former chairs of mathematics were transformed into scientific-didactic departments covering basic fields of mathematics taught at the Technical University. The faculty of the institute amounted to more than 100 members, among them 20 professors and docents. This great scientific potential was directed by Professor Edward Otto. For his achievements as director, he obtained the award of the Ministry.

Professor Edward Otto was a distinguished scientific worker in mathematics and descriptive geometry. After his first paper mentioned previously, he published before the war two papers in topology, one with Professor K.Kuratowski, the other with S.Eilenberg. They concerned properties of dimension and were published in Fundamenta Mathematicae. After the war he published several papers in descriptive geometry and numerical methods. The papers concerned colinear transformations of second order surfaces, linear involutions on line, projective transformations of quadrics, oblique projections of n-dimensional spaces. He proved a generalization of Darboux's theorem, he also generalized Pappus and Pascal theorems and he found a new method of solving linear equations. He reported his results at many scientific conferences.

Professor Edward Otto was also very active at the Polish Mathematical Society. He was a Secretary General of the Society in the years 1948 - 1951, a member of the Main Board in 1963 - 1965, and the Director of Mathematical Olympic in the years 1957 - 1964. For his active scientific, didactic and social work he was awarded with the state distinctions: in 1957 with the Cavalier Cross and in 1978 with the Officer Cross.

Professor Edward Otto belongs to the long line of mathematicians working in descriptive geometry originating with Gaspard Monge. His contribution to descriptive geometry are important, his textbooks are still read by students - they are the only textbooks on descriptive geometry on academic level written by Polish authors.

In 1978 Professor Edward Otto has retired from Technical University of Warsaw. With the status of Professor Emeritus he remained active and took part in seminars on descriptive geometry as well as in works on applied mathematics. He was adviser and referee of doctoral thesis. Till the end of his life he was a member of the editorial board of Demonstratio Mathematica and contributed to the editorial work. He died after a long illness on May 10, 1986 leaving the sadness among his pupils and co-workers. To memorize his achievement, the

Scientific Council has decided to publish a special volume of Demonstratio Mathematica with papers dedicated to his memory. We hope that this volume will pay the best tribute to his memory. Professor Edward Otto was one of the last living members of the Polish Mathematical School from before the war. He will remain in our memory as a distinguished member of this School, whose contributions to mathematics in technical universities are of special importance.

We are enclosing a list of scientific and didactic publications of Professor Edward Otto.

BIBLIOGRAPHY OF PROFESSOR EDWARD OTTO

List of scientific publications

1. Über Punkten der Ordnung C. Monatshefte für Mathematik u. Physik 40 (1933)88-92.
2. O odpowiedniościach n-linjowych w utworach jednobieżnych (Correspondances n-linéaires dans les formes unicursales). Wiadomości Matematyczne 43 (1937) 87-130.
3. Quelques propriétés caractèristiques de la dimension. Fundamenta Mathematicae 31 (1938) 149-153 (joint with S. Eilenberg).
4. Sur les espaces à connexite n-dimensionnelle. Fundamenta Mathematicae 32 (1939) 259-264 (joint with K. Kuratowski).
5. O pochodnej wykładniczej (On exponential derivative). Roczniki Polskiego Towarzystwa Matematycznego Seria 2. Z. 1 (1955) 122-125.
6. Przekształcenia kolineacyjne powierzchni drugiego stopnia na

siebie (Colinear transformations of a second order surface into itself). Zeszyty Naukowe Geometria Wykreślna, Z. 1 (1959) 105-109.

7. Inwolucje liniowe zależne na prostej (Dependent linear involution on the line). Zeszyty Naukowe Geometria Wykreślna, Z. 2(1961) 11-14.
8. Przekształcenia rzutowe kwadryk prostokreślnych (Projective transformations of straight-line quadrics). Zeszyty Naukowe Geometria Wykreślna, Z. 2 (1961) 15-22 (joint with B. Grochowski).
9. Działalność prof. dra Witolda Pogorzelskiego w Politechnice Warszawskiej (Activity of Prof. Dr Witold Pogorzelski in Technical University of Warsaw). Zeszyty Naukowe Politechniki Warszawskiej, Matematyka Z. 2 (1964) 113-116.
10. O wielkości średniej rzutu figury (On the mean value of a projection of a figure). Zeszyty Naukowe Geometria Wykreślna, Z. 3 (1964) 9-11.
11. Rozwiązanie układu równań liniowych metodą ortogonalizacji (A solution of linear system by means of the orthogonalization method). Zeszyty Naukowe Politechniki Warszawskiej, Matematyka Z. 1 (1964) 103-106.
12. Uogólnienie twierdzenia Darboux na układy równań (A generalization of the Darboux theorem to the system of equations). Zeszyty Naukowe Politechniki Warszawskiej, Matematyka Z. 6 (1965) 117-120.
13. O przekształceniach rzutowych przestrzeni P^n na P^m dla $n>m$ (On projective transformations of the space P^n into P^m for $n>m$). Zeszyty Naukowe Geometria Wykreślna, Z. 4 (1966) 5-11.

14. O rzutowaniu skośnym przestrzeni n-wymiarowych (On a oblique projection of the n-dimensional spaces). Zeszyty Naukowe Geometria Wykreślna, Z. 4 (1966) 13-21 (joint with B. Grochowski).
15. Własności charakterystyczne krzywych normalnych (Characteristic properties of the normal curves). Zeszyty Naukowe Geometria Wykreślna Z. 5 (1967) 5-7.
16. Wspomnienie o profesorze Adamie Zawadzkim (Recollection about Professor Adam Zawadzki). Zeszyty Naukowe Geometria Wykreślna, Z. 8 (1973) 5-7.
17. The address delivered at Technical University of Warsaw on the 19th January 1974 during the ceremony of conferring the dignity of doctor honoris causa upon emeritus ordinary Professor dr Stefan Straszewicz. Demonstratio Mathematica 7 (1974) 279-283.
18. Papus'sche Eigenschaft des Linienraumes. Demonstratio Mathematica 11 (1978) 851-852.
19. Professor Stefan Straszewicz. Demonstratio Mathematica 11 (1978) 5-6.
20. An elementary proof of Fourier's formula. Demonstratio Mathematica 13 (1980) 1005-1008.
21. A theorem on axes of relative screw motions of three rigid bodies of a spatial kinematic chain. Demonstratio Mathematica 21 (1988) 853-860 (joint with J. Knapczyk).

List of textbooks (only 1st editions)

1. Matematyka wyższa. Łódź 1948, pp. 450.
2. Geometria wykreślna. Warszawa 1950 Czytelnik, pp. 272.

2. Matematyka dla chemików. Warszawa 1952 PZWS, pp. 416.
4. Zbiór zadań z geometrii wykreślnej. Part 1. Warszawa 1952 PWN, pp. 186.
5. Nomografia. Warszawa 1956 PWN, pp. 220.
6. Matematyka. Podręcznik dla wydziałów budowlanych politechniki. Warszawa 1959 PWN, pp. 500.
7. Zbiór zadań z geometrii wykreślnej. Part 2. Warszawa 1962 PWN, pp. 243. (joint with F. Otto).
8. Nomography. Warszawa 1963 Pergamon Press, pp. 313.
9. Matematyka. Podręcznik dla inżynierskich studiów zawodowych. Vol. 1, Warszawa 1969 PWN, pp. 399 (joint with A. Hać, A. Plucińska, W. Stankiewicz, B. Wieprzkowicz, W. Wojtowicz).
10. Matematyka. Podręcznik dla inżynierskich studiów zawodowych. Vol. 2, Warszawa 1970 PWN, pp. 256 (joint with A. Hać, T. Kornatowski, A. Plucińska, W. Stankiewicz, B. Wieprzkowicz, J. Wojtowicz).
11. Krzywe stożkowe. Warszawa 1971 PZWS, pp. 127.
12. Matematyka. Podręcznik dla inżynierskich studiów zawodowych. Vol. 3, Warszawa 1971 PWN, pp. 328 (joint with A. Hać, T. Kornatowski, A. Plucińska, W. Stankiewicz, B. Wieprzkowicz, J. Wojtowicz).
13. Matematyka. Dla wydziałów budowlanych i mechanicznych. Vol. 1, Warszawa 1973 PWN, pp. 354 (joint with A. Hać, T. Kornatowski, A. Plucińska, W. Stankiewicz, B. Wieprzkowicz, J. Wojtowicz).
14. Matematyka. Dla wydziałów budowlanych i mechanicznych. Vol. 2, Warszawa 1974 PWN, pp. 514 (joint with A. Hać,

T. Kornatowski, A. Plucińska, W. Stankiewicz, B. Wieprzkowicz, J. Wojtowicz).

15. Matematyka. Dla wydziałów budowlanych i mechanicznych. Vol. 3, Warszawa 1974 PWN, pp. 400 (joint with A. Hać, T. Kornatowski, A. Plucińska, W. Stankiewicz, B. Wieprzkowicz, J. Wojtowicz).

16. Podręcznik geometrii wykreślnej. Warszawa 1975 PWN, pp. 416 (joint with F. Otto).

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A THEOREM ON AXES OF RELATIVE SCREW MOTIONS OF THREE RIGID BODIES OF A SPATIAL KINEMATIC CHAIN

1. Introduction

Over a hundred years ago S. Aronhold [1] (1872) and A. Kennedy [6] (1886) formulated independently a theorem on colinearity of three instant centers of rotations of three rigid links of a plane kinematic chain. Next, this theorem has been generalized for covering spherical and spatial kinematic chains with revolute and cylindrical pairs, see e.g. works by O. Bottema, B. Roth [3], K. H. Hunt [5], J. S. Beggs [2], F. M. Dimentberg [4] and F. L. Litvin [7].

Kinematics of a plane kinematic chain can be regarded as a special case of kinematics of a spherical chain with point of intersection of axes tending to infinity. Interrelations of angular displacements for a spatial chain with cylindrical pairs are identic with those for an equivalent spherical chain, what results from the principle of transference [4].

Analogies existing between kinematics of plane, spherical and spatial kinematic chains make it possible to generalize the Aronhold-Kennedy theorem for an arbitrary kinematic chain which has been noticed by E. Otto [8]. In the present paper a trial is being made to give an analytical proof of the theorem on the common perpendicular line to the three axes of a spatial kinematic chain composed of arbitrary kinematic pairs.

2. A theorem on the common perpendicular line to the axes of relative screw motions of three links of a spatial kinematic chain

Consider velocities of relative motions of three links: "i-1", "i", and "i+1" of a spatial kinematic chain (Fig. 1) which is composed of two cylindrical pairs (possessing two degrees of freedom each) and one contact pair with five degrees of freedom. Let us fix to each of these links a rectangular coordinate system, defined as follows. The axis z_{i-1} lies on the axis of the cylindrical pair "i-1,i"; the axis z_i lies on the axis of the cylindrical pair "i,i+1" and both axes x_{i-1} and x_i lie on the common perpendicular to z_{i-1} and z_i.

Velocity of the point P_{i-1} belonging to the link "i-1" in relation to the system "i" may be found as follows:

(1) $$\vec{v}_{P,i-1} = \dot{\lambda}_i \vec{z}_{i-1} + \dot{\theta}_i (\vec{z}_{i-1} \times \vec{r}_{P,i-1})$$

where denote: $\vec{r}_{P,i-1}$- vector of the position of point P_{i-1} in the system "i-1", $\dot{\lambda}_i$, $\dot{\theta}_i$ - projections on z_{i-1} axis of the linear and angular velocities of the link "i-1" in relation to the link "i", and $\vec{z}_{i-1}$- the unit vector of z_{i-1} axis.

Velocity of point P_{i+1} belonging to the link "i+1" in relation to system "i" can be defined in a similar manner:

$$\vec{v}_{P,i+1} = \dot{\lambda}_{i+1}\vec{z}_i + \dot{\theta}_{i+1}(\vec{z}_i \times \vec{r}_{P,i+1}). \tag{2}$$

Evaluating the cross-products in the formulas (1), (2) one gets

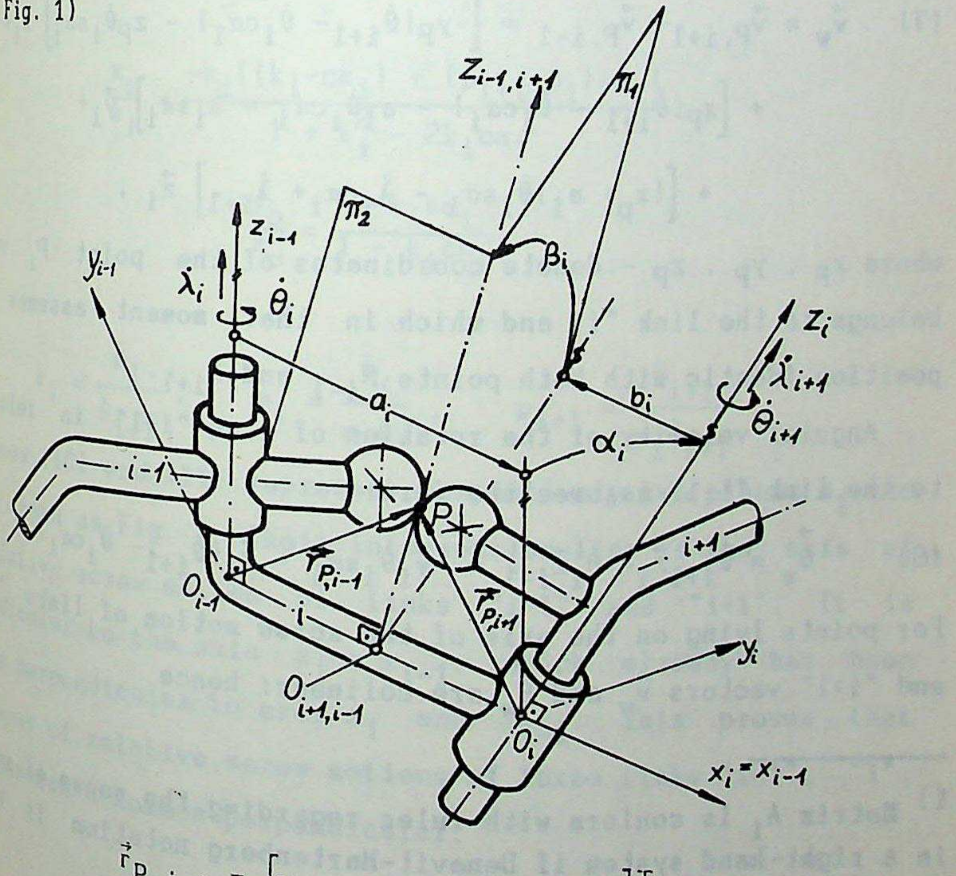
$$\vec{v}_{P,i-1} = \dot{\lambda}_i\vec{z}_{i-1} + \dot{\theta}_i(-\vec{x}_{i-1}y_{P,i-1} + \vec{y}_{i-1}x_{P,i-1}), \tag{3}$$

$$\vec{v}_{P,i+1} = \dot{\lambda}_{i+1}\vec{z}_i + \dot{\theta}_{i+1}(-\vec{x}_i y_{P,i+1} + \vec{y}_i y_{P,i+1}). \tag{4}$$

Assuming that the instant positions of points P_{i-1} and P_{i+1} are identic with the point P_i belonging to the link "i" one can transform vectors involved in (3) and (4) into the system "i"

$$\vec{r}_{P,i-1} = A_i\vec{r}_{P,i} + \begin{bmatrix} a_i & 0 & 0 \end{bmatrix}^T, \tag{5}$$

where symbol a_i denotes the distance between axes z_i and z_{i-1} (see Fig. 1)


$Z_{i-1,i+1}$
π_1
π_2
β_i
z_{i-1}
y_{i-1}
$\dot{\lambda}_i$
$\dot{\theta}_i$
z_i
$\dot{\lambda}_{i+1}$
$\dot{\theta}_{i+1}$
a_i
b_i
α_i
i-1
P
i+1
$\vec{r}_{P,i-1}$
$\vec{r}_{P,i+1}$
O_{i-1}
i
$O_{i+1,i-1}$
y_i
O_i
$x_i = x_{i-1}$

$$\vec{r}_{P,i-1} = \begin{bmatrix} x_{P,i-1} & y_{P,i-1} & z_{P,i-1} \end{bmatrix}^T$$

$$\vec{r}_{P,i+1} = \vec{r}_{P,i} = \left[x_{P,i} \quad y_{P,i} \quad z_{P,i}\right]^T$$

$$A_i = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\alpha_i & -s\alpha_i \\ 0 & s\alpha_i & c\alpha_i \end{bmatrix}, \quad s\alpha_i = \sin\alpha_i \ , \quad c\alpha_i = \cos\alpha_i .^{1)}$$

Hence,

$$(6) \quad \begin{cases} x_{P,i-1} = a_i + x_{P,i} \ , & x_{i-1} = x_i \ , \\ y_{P,i-1} = y_{P,i}c\alpha_i - z_{P,i}s\alpha_i \ , & y_{i-1} = y_i c\alpha_i - z_i s\alpha_i \ , \\ z_{P,i-1} = y_{P,i}s\alpha_i + z_{P,i}c\alpha_i \ , & z_{i-1} = y_i s\alpha_i + z_i c\alpha_i \ , \end{cases}$$

Substituting (5) to (3) and (4) one can present the relative velocity of points P_{i-1} and P_{i+1} in the following form:

$$(7) \quad \vec{v}_w = \vec{v}_{P,i+1} - \vec{v}_{P,i-1} = \left[-y_P(\dot{\theta}_{i+1} - \dot{\theta}_i c\alpha_i) - z_P\dot{\theta}_i s\alpha_i\right]\vec{x}_i + \left[x_P(\dot{\theta}_{i+1} - \dot{\theta}_i c\alpha_i) - a_i\dot{\theta}_i c\alpha_i - \dot{\lambda}_i s\alpha_i\right]\vec{y}_i + \left[(x_P + a_i)\dot{\theta}_i s\alpha_i - \dot{\lambda}_i c\alpha_i + \dot{\lambda}_{i+1}\right]\vec{z}_i \ ,$$

where x_P , y_P , z_P - denote coordinates of the point P_i which belongs to the link "i" and which in that moment assumes the position identic with both points P_{i-1} and P_{i+1}.

Angular velocity of the rotation of link "i+1" in relation to the link "i-1" assumes the form:

$$(8) \quad \vec{\dot{\theta}}_w = \dot{\theta}_{i+1}\vec{z}_i - \dot{\theta}_i\vec{z}_{i-1} = -\vec{y}_i\dot{\theta}_i s\alpha_i + \vec{z}_i(\dot{\theta}_{i+1} - \dot{\theta}_i c\alpha_i).$$

For points lying on the axis of the screw motion of links "i-1" and "i+1" vectors $\vec{v}_w$ and $\vec{\dot{\theta}}_w$ are colinear: hence

---

1) Matrix A_i is conform with rules regarding the sense of angles in a right-hand system if Denavit-Hartenberg notation is being used.

(9) $$\vec{\dot{\theta}}_w \times \vec{v}_w = \begin{vmatrix} \vec{x}_i & \vec{y}_i & \vec{z}_i \\ \dot{\theta}_{wx,i} & \dot{\theta}_{wy,i} & \dot{\theta}_{wz,i} \\ v_{wx,i} & v_{wy,i} & v_{wz,i} \end{vmatrix} = 0.$$

Substituting expressions for projections of vectors $\vec{\dot{\theta}}_w$ and $\vec{v}_w$ as defined by (7) and (8) into equation (9) we get the following set of two independent equations:

$$-\dot{\theta}_i s\alpha_i\left[\dot{\theta}_i(x_p + a_i)s\alpha_i - \dot{\lambda}_i c\alpha_i + \dot{\lambda}_{i+1}\right] +$$

$$-(\dot{\theta}_{i+1} - \dot{\theta}_i c\alpha_i)\left[x_p(\dot{\theta}_{i+1} - \dot{\theta}_i c\alpha_i) - a_i\dot{\theta}_i c\alpha_i + \dot{\lambda}_i s\alpha_i\right] = 0,$$

$$(\dot{\theta}_{i+1} - \dot{\theta}_i c\alpha_i)\left[-y_p(\dot{\theta}_{i+1} - \dot{\theta}_i c\alpha_i) - z_p\dot{\theta}_i s\alpha_i\right] = 0.$$

Hence

(10) $$\frac{x_p}{a_i} = \frac{-k_i[(k_i - c\alpha_i) + (p_{i+1} - p_i)s\alpha_i]}{1 + k_i^2 - 2k_i c\alpha_i},$$

(11) $$\frac{y_p}{z_p} = \frac{-k_i s\alpha_i}{1 - k_i c\alpha_i},$$

where

$$k_i = \frac{\dot{\theta}_i}{\dot{\theta}_{i+1}}, \quad p_i = \frac{\dot{\lambda}_i}{a_i\dot{\theta}_i}, \quad p_{i+1} = \frac{\dot{\lambda}_{i+1}}{a_i\dot{\theta}_{i+1}}.$$

Equations (10) and (11) represent respectively two planes π_1 and π_2, as shown as Fig. 1. Their intersection line is the axis of the relative screw motion of links "i-1" and "i+1". It is perpendicular to the axis $x_i = x_{i-1}$ which already has been assumed perpendicular to axes z_i and z_{i-1}. This proves that three axes of relative screw motions of three links "i-1", "i", "i+1" possess a common perpendicular.

3. Applications

This theorem may be applied for instance to finding the relative velocity of links "i-1" and "i+1". Consider point P on the axis $z_{i-1,i+1}$. Its velocity $\vec{v}_w$ is perpendicular to axis x_i, and thus the projection $v_{wx,i} = 0$. The angle β_i between axes $z_{i-1,i+1}$ and z_i (see Fig. 1) may be found from (11), it follows that

$$\text{(12)} \qquad \mathrm{tg}\,\beta_i = k_i s\alpha_i/(-1 + k_i c\alpha_i).$$

Summing projections of the components of the vector $\vec{v}_w$ onto axis $z_{i-1,i+1}$ one can find projection of the velocity of translation of link "i+1" in relation to "i-1"

$$\text{(13)} \qquad \dot{\lambda}_w = v_{wy,i} s\beta_i + v_{wz,i} c\beta_i.$$

From (13), (7) and (12) there follows:

$$\text{(14)} \qquad \dot{\lambda}_w = \left[x_p(\dot{\theta}_{i+1} - \dot{\theta}_i c\alpha_i) - a_i\dot{\theta}_i c\alpha_i - \dot{\lambda}_i s\alpha_i\right] s\beta_i + \left[(x_p + a_i)\dot{\theta}_i s\alpha_i - \dot{\lambda}_i c\alpha_i + \dot{\lambda}_{i+1}\right] c\beta_i ,$$

where

$$\text{(15)} \qquad \begin{cases} s\beta_i = \dfrac{-k_i s\alpha_i}{\sqrt{1+k_i^2 - 2k_i c\alpha_i}} , \\[2ex] c\beta_i = \dfrac{1 - k_i c\alpha_i}{\sqrt{1+k_i^2 - 2k_i c\alpha_i}} . \end{cases}$$

The absolute value of the angular velocity of rotation of "i+1" in relation to "i-1" is equal to the lenght of vector $\vec{\dot{\theta}}_w$.

$$\text{(16)} \qquad \dot{\theta}_w = \dot{\theta}_{i+1}\sqrt{1+k_i^2 - 2k_i c\alpha_i} .$$

Parameter of the screw motion with axis $z_{i-1,i+1}$ is usually defined as ratio of absolute values of translation and rotation. Taking into account (14)-(16) one gets

$$w = \frac{v_w}{\dot{\theta}_w} = a_i \frac{|k_i s\alpha_i + (k_i - c\alpha_i)p_i k_i + (1-k_i c\alpha_i)p_{i+1}|}{\sqrt{1+k_i^2 - 2k_i c\alpha_i}}$$

where $k_i = \dot{\theta}_i / \dot{\theta}_{i+1}$, $p_i = \dot{\lambda}_i / a_i\dot{\theta}_i$, $p_{i+1} = \dot{\lambda}_{i+1} / a_i\dot{\theta}_{i+1}$.

If axes z_{i-1} and z_i are parallel, and if $\dot{\lambda}_i = \dot{\lambda}_{i+1} = 0$, then

$$p_i = p_{i+1} = 0, \quad \beta_i = 0,$$

$$x_p/a_i = k_i/(1 - k_i), \quad y_p = 0, \quad w = 0.$$

This means that the relative screw motion between links "i+1" and "i-1" reduces to pure rotation about an axis parallel to the axes z_{i-1} , and z_i .

If, on the other hand, axes z_{i-1} and z_i intersect, then $a_i = 0$, $x_p = 0$, $y_p/z_p = -k_i s\alpha_i/(1-k_i c\alpha_i)$, $w=0$. This means that the relative screw motion reduces to pure rotation about an axis on which the point of intersection of z_{i-1} and z_i lies.

Note, that if values of parameters k_i, p_i, p_{i+1} and of dimensions a_i and α_i are given, then the relative motion of links "i-1" and "i+1" is fully defined. If out of these five values the parameter k_i is allowed to vary in a continuous way, then the ruled surface can be found on which the screw axis lies.

If the kinematic chain comprizes "n" links (n>3) then the theorem proven in section 2 for n=3 may be applied consecutively n-2 times. Among many applications of this procedure let us quote seeking for velocities of all links of any open-chain manipulator in relation to the base.

REFERENCES

[1] S. Aronhold : Grundzüge der kinematischen Geometrie. Verh. d. Ver. z. Beförderung des Gewerbefleißes in Preussen 51, 1872.

[2] J. Beggs : Advanced Mechanism. MacMillan Comp., New York 1966.

[3] O. Bottema, B. Roth : Theoretical Kinematics. North Holland Publ. comp., Amsterdam, New York, Oxford 1979.

[4] F. M. Dimentberg : Screw Theory and its Applications. Moskva 1978. (in Russian).

[5] K. H. Hunt : Kinematic Geometry of Mechanisms. Clarendon Press, Oxford 1978.

[6] A. B. W. Kennedy : The Mechanics of Machinery. Macmillan, London 1886.

[7] F. L. Litvin : Theory of Gears, Moskva 1986. (in Russian).

[8] E. Otto : Generalized Aronhold-Kennedy Theorem. Manuscript, Warsaw 1985.

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LES INTÉGRALES PARAMÉTRIQUES MULTIPLES DÉPENDANT DES DÉRIVÉES DU SECOND ORDRE

Dedicated to the memory of Professor Edward Otto

Introduction

Dans le Calcul des Variations on pouvait depuis la première moitié du XIX siècle trouver l'étude sporadique des intégrales dites paramétriques (voir [1], [2], [4], [6], [8]). Le présent article a pour but de donner une généralisation des résultats de C. Kobb [4] (et partiellement aussi de E. Zermelo [8]) concernant des conditions à vérifier par la fonction formante d'une intégrale k-tuple (dépendant de n≥k fonctions et de leurs dérivées du premier et second ordre), pour qu'elle soit une intégrale paramétrique. Ces conditions dépendent de la définition acceptée de l'hypercourbe ou de l'hypersurface (voir [7]). Nous suivrons dans notre définition les idées présentées par K. Tatarkiewicz ([5] vol. II, p. 141).

De même la notion "l'intégrale paramétrique" sera remplacée (voir [5]) par l'intégrale dépendant d'une (hyper)surface et l'intégrale invariantative (intégrale localement invariantative).

1. Les hypersurfaces et les hypercourbes

Soient deux entiers k,n tels que $1 \leq k < n$ et soit la sphère fermée

$$K^k(a,r) := \{x: |x-a| \leq r,\ x \in R_k\},$$

où $a = (a^1,\dots,a^k)$, $x = (x^1,\dots,x^k)$ et $|x|$ est la distance euclidienne. Nous allons écrire en bref $K^k := K^k(\theta,r)$, où $\theta := (0,\dots,0)$. Soit enfin $S^k := \partial' K^k$ la surface de la sphère K^k.

Soit une transformation $\hbar = (h^1,\dots,h^k) : R_k \longleftrightarrow R_k$ de classe C^r, $r \geq 1$, et avec le jacobien vérifiant la condition

$$D_{1,\dots,k}\hbar(y) := \frac{\partial(h^1,\dots,h^k)}{\partial(y^1,\dots,y^k)}(y^1,\dots,y^k) > 0$$

pour tout $(y^1,\dots,y^k) =: y \in R_k$. Désignons par H l'ensemble des toutes telles transformations $\hbar$.

En plus, définissons un élargissement

$\overset{*}{\hbar} = (\overset{*}{h}{}^1,\dots,\overset{*}{h}{}^n) : R_n \longleftrightarrow R_n$ donné par la formule

$\overset{*}{\hbar} := (\overset{*}{h}{}^1,\dots,\overset{*}{h}{}^k, n_{j_{k+1}},\dots,n_{j_n})$, où

$$\overset{*}{h}{}^i(y^1,\dots,y^k,y^{k+1},\dots,y^n) := h^i(y^1,\dots,y^k)$$

pour i = 1,...,k et pour $(y^1,\dots,y^k,y^{k+1},\dots,y^n) \in R_n$, la fonction n_{j_i} étant définie par la formule $n_{j_i}(x^1,\dots,x^n) = x^i$ pour tout $(x^1,\dots,x^n) \in R_n$.

Nous avons

$$D_{1,\dots,n}\overset{*}{\hbar}(y^1,\dots,y^k,y^{k+1},\dots,y^n) = D_{1,\dots,k}\hbar(y^1,\dots,y^k) > 0.$$

Désignons par $\overset{*}{H}$ l'ensemble de toutes les transformations $\overset{*}{\hbar} \in C^r$.

Posons $Z^{k,n} := \{(y^{k+1},\dots,y^n) : y^i = 0,\ i = k+1,\dots,n\}$

et admettons $\overset{*}{K}{}^{k,n} := K^k \times Z^{k,n}$, $\overset{*}{S}{}^k := S^k \times Z^{k,n}$.

Désignons par D^r et G^r les familles de tous les ensembles

(1) $$D := \hbar(K^k) \text{ et } G := \hbar(S^k),$$

respectivement, telles que $\hbar \in H$, et par $\overset{*}{D}{}^r$ et $\overset{*}{G}{}^r$ les familles de tous les ensembles

(1') $$\overset{*}{D} := \overset{*}{\hbar}\,(K^k) \text{ et } \overset{*}{G} := \overset{*}{\hbar}\,(\overset{*}{S}{}^k),$$

respectivement, telles que $\overset{*}{\hbar} \in \overset{*}{H}$. Soit W l'ensemble de toutes les transformations $C^r \ni æ = (x^1,\dots,x^n) : R_n \longleftrightarrow R_n$, $r \geq 1$, avec $D_{1,\dots,n}\, æ(t^1,\dots,t^n) > 0$.

Considérons aussi l'ensemble U^r, $r \geq 1$, des transformations

(2) $$C^r \ni u = (u^1,\dots,u^k) : R_k \longleftrightarrow R_k$$

avec

(3) $$D_{1,\dots,k}\, u(y^1,\dots y^k) > 0$$

et telles que $D_1 \in D^r$ implique $u(D_1) := D_2 \in D^r$ et réciproquement.

Soit $\overset{*}{U}{}^r$ l'ensemble des transformations

$$\overset{*}{u} := (u^1,\dots,u^k, n_{j_{k+1}},\dots,n_{j_n}) : R_n \longleftrightarrow R_n$$

avec $u = (u^1,\dots,u^k) \in U^r$.

Définition 1. Si P est un ensemble de couples $[æ,D]$ tels que $æ \in W$, $D \in D^r$, alors aux éléments $[æ,D] \in P := W \times D^r$ correspondent biunivoquement les éléments $[æ,D] \in \overset{*}{P} := W \times \overset{*}{D}{}^r$.

Définition 2. Les éléments $[æ_1,D_1],[æ_2,D_2] \in P$

sont en relation $\mathcal{R}$ à savoir

$$[\varkappa_1, D_1]\ \mathcal{R}\ [\varkappa_2, D_2], \tag{4}$$

s'il existe une transformation $\overset{*}{u} \in \overset{*}{U}^r$ telle que

$$(\varkappa_1 \circ \overset{*}{u})|_{D_2} = \varkappa_2|_{D_2} \tag{5}$$

($\varkappa_1 \circ \overset{*}{u}$ étant une composition des transformations $\varkappa_1$, $\overset{*}{u}$ et $z|_A$ désignant une transformation z restreinte à l'ensemble A).

On peut démontrer le théorème suivant.

Théorème 1. La relation $\mathcal{R}$ forme une équivalence.

La relation $\mathcal{R}$ étant une équivalence dans P elle divise P en classes d'abstraction.

Définition 3. Les éléments $\vec{\mathcal{P}} \in P/\mathcal{R}$ seront appelés hypersurfaces orientées.[1]

Soit $Q := W \times G^r$ un ensemble de couples $[\varkappa, G]$ tels que $\varkappa \in W$, $G \in G^r$. Aux éléments $[\varkappa, G]$ correspondent biunivoquement les éléments $[\varkappa, \overset{*}{G}] \in \overset{*}{Q} := W \times \overset{*}{G}^r$ à l'aide de la définition suivante.

Définition 4. Les éléments $[\varkappa_1, G_1]$, $[\varkappa_2, G_2]$ sont en relation $\mathcal{R}'$ à savoir

$$[\varkappa_1, G_1]\ \mathcal{R}'\ [\varkappa_2, G_2],$$

s'il existe une transformation $\overset{*}{u} \in \overset{*}{U}^r$ telle que

$$[\varkappa_1 \circ \overset{*}{u}]|_{\overset{*}{G}_2} = \varkappa_2|_{\overset{*}{G}_2}.$$

On peut démontrer le théorème suivant.

Théorème 2. La relation $\mathcal{R}'$ forme une équivalence.

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[1] Soient un ensemble A et une équivalence $\mathcal{R}$ définie dans A. Par $A/\mathcal{R}$ nous allons désigner l'ensemble des classes d'abstraction de A par rapport à $\mathcal{R}$.

La relation $\mathcal{R}'$ est donc une équivalence définie dans l'ensemble Q et elle le divise en classes d'abstraction.

Définition 5. Les éléments $\vec{\Gamma} \in Q/\mathcal{R}'$ seront appelés hypercourbes orientées.

Selon nos définitions les représentants des classes d'abstraction qui forment les hypersurfaces (hypercourbes) orientées sont les couples $[\ae, D]$ avec $D \in D^{I}$, ($[\ae, G]$ avec $G \in G^{I}$).

Définition 6. L'hypersurface orientée $\vec{\mathcal{P}}$ est tendue avec la même orientation sur la courbe orientée $\vec{\Gamma}$ s'il existe des représentants $[\ae, D] \in \vec{\mathcal{P}}$ et $[y, G] \in \vec{\Gamma}$ tels que $\ae = y$, $G = \partial' D$.

Théorème 3. Si $[\ae, D]$ est un représentant de l'hypersurface orientée $\vec{\mathcal{P}}$, alors $[\ae, \partial' D]$ est un représentant de l'hypercourbe orientée sur laquelle $\vec{\mathcal{P}}$ est tendue avec la même orientation.

La réciproque est évidemment faux - si $[\ae_1, \partial' D]$ et $[\ae_2, \partial' D]$ sont des représentants de la même hypersurface orientée, alors il peut arriver que le couples $[\ae_1, D]$, $[\ae_2, D]$ soient des représentants de deux hypersurfaces orientées différentes $\vec{\mathcal{P}}_1$ et $\vec{\mathcal{P}}_2$ (il suffit alors qu'il soit $\ae_1|_{\overset{*}{D}} \neq \ae_2|_{\overset{*}{D}}$). Mais alors, vu la définition 6, ces deux hypersurfaces orientées $\vec{\mathcal{P}}_1$, $\vec{\mathcal{P}}_2$ sont tendues sur la même hypercourbe orientée avec la même orientation. Cette remarque nous permet d'introduire la définition suivante.

Définition 7. $\vec{\Gamma} \in Q/\mathcal{R}'$ étant une hypercourbe orientée, désignons par $P(\vec{\Gamma})$ l'ensemble de tous les couples $[\ae, D]$ tels que $[\ae, \partial' D] \in \vec{\Gamma}$ et appelons-le l'ensemble des hypersurfaces orientées tendues (avec la même orientation) sur l'hypercourbe orientée $\vec{\Gamma}$.

Nous avons $P(\vec{\Gamma}) \subset P$ et, ce qu'on peut facilement vérifier, $P(\vec{\Gamma})/\mathcal{R} \subset P/\mathcal{R}$.

Définition 8. Si $D_1, D_2 \in D^{\Gamma}$ et $D_1 \subset D_2$, alors $[\ae, D_1]$ sera dite sous-hypersurface orientée de l'hypersurface orientée $[\ae, D_2]$.

2. Une condition du type de Zermelo

2.1. Lemme.

Soit un domaine fermé $D \subset R_n$ et une fonction $C^0 \ni g : D \longrightarrow R$. La fonction $g|_Q$ est intégrable dans un domaine fermé $Q \subset D$. Posons donc

$$\mathfrak{G}(Q) := \int_Q g, \tag{6}$$

$\mathfrak{G}$ étant une fonction additive de l'ensemble Q.

Lemme 1. Soient un point $S \in D$ et un nombre $\varepsilon_0 > 0$ tels que $K^n(S, \varepsilon_0) \subset D$ et soit une famille des domaines fermés Q_ε, définis pour chaque $\varepsilon \in (0, \varepsilon_0)$ tels que $Q_\varepsilon \subset K^n(S, \varepsilon)$ et $\mu(Q_\varepsilon) > 0$ pour chaque $\varepsilon > 0$. Alors on a

$$\lim_{\varepsilon \to 0} \frac{\mathfrak{G}(Q_\varepsilon)}{\mu(Q_\varepsilon)} = g(S). \tag{7}$$

La formule (7) définit la dérivée de la fonctionelle $\mathfrak{G}$ donnée par (6).

2.2. Les intégrales dépendant d'une hypersurface

Soit une fonction

$$C^2 \ni f : R_p \longrightarrow R \tag{8}$$

de p variables, où

$$p = n + \frac{n}{2}(k^2 + 3k)$$

(9)

et $1<k<n$, (voir sec. 1) et désignons ses arguments $x^1,\dots,x^n,x^1_1,\dots,x^n_1,\dots,x^1_k,\dots,x^n_k,\dots,x^1_{11},\dots,x^n_{11},x^1_{12},\dots,x^n_{12},\dots,x^1_{kk},\dots,x^n_{kk}$.

En supposant $x^i_{\alpha\beta} = x^i_{\beta\alpha}$ pour $i = 1,\dots,n$ et $\alpha,\beta = 1,\dots,k$ nous pouvons écrire en bref $f = f(x^i, x^i_\alpha, x^i_{\alpha\beta})$.

Introduisons une fonctionelle $\mathcal{I} : P \longrightarrow R$ (voir déf. 1), appelée ultérieurement "intégrale", donnée par la formule

(10) $$\mathcal{I}[(x^1,\dots,x^n),D] = \int_D f(x^i, x^i_\alpha, x^i_{\alpha\beta}),$$

où x^i_α et $x^i_{\alpha\beta}$ désignent les dérivées $\partial x^i/\partial t^\alpha$ et $\partial^2 x^i/\partial t^\alpha \partial t^\beta$, respectivement, et $\int_D$ désigne une intégrale k-tuple (f étant dite alors une fonction formante de l'intégrale $\mathcal{I}$).

Définition 9. Si

(11) $$\varkappa_1 \mathcal{R} \varkappa_2 \text{ implique } \mathcal{I}[\varkappa_1,D_1] = \mathcal{I}[\varkappa_2,D_2],$$

pour chaque deux couples $[\varkappa_i,D_i] \in P(\vec{\Gamma})$, $i = 1,2$, alors nous dirons que l'intégrale $\mathcal{I}$ est invariantative dans $P(\vec{\Gamma})$ par rapport à la relation $\mathcal{R}$.

Définition 10. Si pour tous les couples $[\varkappa,D_1]$, $[\psi,D_2]$ tels que

$$[\psi,D_2]\,\mathcal{R}\,[\varkappa,D_1],$$

où $D_1 \subset D$ et $[\varkappa,D] \in P(\vec{\Gamma})$ nous avons

(12) $$\mathcal{I}[\psi,D_2] = \mathcal{I}[\varkappa,D_1],$$

alors l'intégrale $\mathcal{I}$ est dite localement invariantative.

Si pour chaque représentant d'une hypersurface orientée appartenant à $P(\vec{\Gamma})$ la valeur de l'intégrale $\mathcal{I}$ est la même, alors est invariantative dans $P(\vec{\Gamma})$. Si pour chaque représentant d'une sous-hypersurface orientée de chaque hypersurface orientée appartenant à $P(\vec{\Gamma})$ la valeur de l'intégrale $\mathcal{I}$ est la même, alors $\mathcal{I}$ est localement invariantative.

On a donc le résultat suivant.

Théorème 4. Chaque intégrale $\mathcal{I}$ localement invariantative est invariantative.

Si l'intégrale $\mathcal{I}$ est invariantative dans $P(\vec{\Gamma})$ par rapport à $\mathcal{R}$, alors on peut s'en servir en introduisant une nouvelle fonctionelle $\mathcal{J} : P(\vec{\Gamma})/\mathcal{R} \longrightarrow R$ par la formule

$$\mathcal{J}\,[\mathfrak{P}] = \mathcal{I}\,[\mathfrak{x}, D] \tag{13}$$

pour tous les $[\mathfrak{x}, D] \in \mathfrak{P} \in P(\vec{\Gamma})/\mathcal{R}$.

Définition 11. Si l'intégrale $\mathcal{I}$ est localement invariantative, alors la fonctionelle correspondante $\mathcal{J}$ est dite une fonctionelle locale des surfaces.

2.3. Une condition fonctionelle

Supposons que l'intégrale $\mathcal{I}$ donnée par la formule (10) est localement invariantative. Vu la définition 2, il existe donc une transformation $\mathfrak{u} = (u^1, \dots, u^k) : D_1 \longleftrightarrow D_2$ telle que

$$\mathfrak{u} \in C^2,\quad D_{1,\dots,k}\,\mathfrak{u}(t^1, \dots, t^k) > 0, \tag{14}$$

$$(\mathfrak{x} \circ \mathfrak{u})|_{D_2} = \mathfrak{y}|_{D_2}. \tag{15}$$

L'intégrale $\mathcal{I}$ étant localement invariantative, on a donc

$$\int_{D_1} f\left(y^i, y^i_\alpha, y^i_{\alpha\beta}\right) = \mathcal{I}\,[\mathfrak{y}, D_1] = \mathcal{I}\,[\mathfrak{x}, D_2] = \int_{D_2} f\left(x^i, x^i_\alpha, x^i_{\alpha\beta}\right). \tag{16}$$

En posant (15) sous l'intégrale $\mathcal{I}\,[\mathfrak{y}, D_1]$ nous obtenons

$$(17)\quad \int_{D_1} f\left(y^i, y^i_\alpha, y^i_{\alpha\beta}\right) = \int_{D_2} f\left(x^i\circ u, x^i_\alpha \circ u, x^i_{\alpha\beta} \circ u\right) D_{1,\dots,k}u.$$

L'intégrale $\mathcal{J}[\mathfrak{x}, D_2]$ peut par contre s'écrire

$$(18)\quad \int_{D_2} f\left(x^i, x^i_\alpha, x^i_{\alpha\beta}\right) =$$

$$= \int_{D_2} f\left(x^i\circ u, [x^i_\alpha\circ u]u^\alpha_\beta\ , [x^i_{\gamma\delta}\circ u]u^\gamma_\alpha u^\delta_\beta + [x^i_\gamma\circ u]u^\gamma_{\alpha\beta}\right).$$

La formule (16) prend donc la forme

$$(19)\quad \int_{D_2} f\left(x^i\circ u, x^i_\alpha\circ u, x^i_{\alpha\beta}\circ u\right) D_{1,\dots,k}u =$$

$$= \int_{D_2} f\left(x^i\circ u, [x^i_\alpha\circ u]u^\alpha_\beta\ , [x^i_{\gamma\delta}\circ u]u^\gamma_\alpha u^\delta_\beta + [x^i_\gamma\circ u]u^\gamma_{\alpha\beta}\right).$$

En vertu de Lemme 1, nous obtenons pour chaque point $S = (t^1,\dots,t^k) \in D_2 \in u^{-1}(D_1)$ (et pour chaque u correspondant) l'égalité

$$(20)\quad f\left(x^i(u(S)), x^i_\beta(u(S))u^\beta_\alpha(S), x^i_{\gamma\delta}(u(S))u^\gamma_\alpha(S)u^\delta_\beta(S) + x^i_\gamma(u(S))u^\gamma_{\alpha\beta}(S)\right) =$$

$$= f\left(x^i(u(S)), x^i_\alpha(u(S)), x^i_{\alpha\beta}(u(S))\right) D_{1,\dots,k}u(S).$$

Soient les suites de nombres quelconques ${}_0x^i$, ${}_0x^i_\alpha$, ${}_0x^i_{\alpha\beta}$ tels que

$$(21)\quad \begin{cases} {}_0x^i_{\alpha\beta} = {}_0x^i_{\beta\alpha}, \quad i = 1,\dots,n,\ \alpha,\beta = 1,\dots,k, \text{ et } a^j_\alpha,\ a^j_{\alpha\beta} \\ \text{tels que } \det(a^j_\alpha) > 0,\ a^j_{\alpha\beta} = a^j_{\beta\alpha},\quad \alpha,\beta,j = 1,\dots,k. \end{cases}$$

En posant $x^i(u^1,\dots,u^k) := {}_0x^i + {}_0x^i_\alpha u^\alpha + {}_0x^i_{\alpha\beta}u^\alpha u^\beta$ nous obtenons donc $u^1=\ldots=u^k=0$, $x^i(0,\ldots,0)={}_0x^i$, $x^i_\alpha(0,\ldots,0)={}_0x^i_\alpha$, $x^i_{\alpha\beta}(0,\ldots,0)={}_0x^i_{\alpha\beta}$ pour tous i,α,β. Soient des fonctions $u^j(t^1,\ldots,t^k)=a^j_\alpha t^\alpha+\frac{1}{2}a^j_{\alpha\beta}t^\alpha t^\beta$, satisfaisant à (14) localement au voisinage du point $S=(t^1,\ldots,t^k)$. Ayant $x^i(u(S)) = x^i(0,\ldots,0)={}_0x^i$, $x^i_\alpha(u(S))={}_0x^i_\alpha$, $x^i_{\alpha\beta}(u(S))={}_0x^i_{\alpha\beta}$ et $u^\beta_\alpha(S)=a^\beta_\alpha$, $u^\gamma_{\alpha\beta}(S)=a^\gamma_{\alpha\beta}$ pour $S=(0,\ldots,0)$ de formule (20) on peut écrire

$$f\left({}_0x^i,\ {}_0x^i_\beta a^\beta_\alpha,\ {}_0x^i_{\gamma\delta}a^\gamma_\alpha a^\delta_\beta+{}_0x^i_\gamma a^\gamma_{\alpha\beta}\right)=f\left({}_0x^i,{}_0x^i_\alpha,\ {}_0x^i_{\alpha\beta}\right)\det(a^j_\alpha)$$

pour un arrangement quelconque des nombres (21). Posons

$$(22)\qquad f\left(x^i,x^i_\beta a^\beta_\alpha\ ,x^i_{\gamma\delta}a^\gamma_\alpha a^\delta_\beta+x^i_\gamma a^\gamma_{\alpha\beta}\right)=f\left(x^i,x^i_\alpha\ ,x^i_{\alpha\beta}\right)\det(a^j_\alpha).$$

Nous avons donc démontré le théorème suivant.

Théorème 5. Pour que l'intégrale (10) soit localement invariantative, il faut et il suffit que pour tous systèmes de nombres x^i, x^i_α, $x^i_{\alpha\beta}$, avec $x^i_{\alpha\beta}=x^i_{\beta\alpha}$, et a^γ_α, $a^\gamma_{\alpha\beta}$, avec $a^\gamma_{\alpha\beta}=a^\gamma_{\beta\alpha}$ et $\det(a^\gamma_\alpha)>0$, la condition (22) soit vérifiée.

2.4. Les conditions différentielles nécessaires

Nous allons maintenant introduire un système de deux équations aux dérivées partielles du type de Zermelo à vérifier par la fonction formante f.

Posons

$$h\left(x^i, x^i_\alpha, x^i_{\alpha\beta}; a^\beta_\alpha, a^\gamma_{\alpha\beta}\right) := f\left(x^i, x^i_\nu a^\nu_\alpha, x^i_{\nu\mu} a^\nu_\alpha a^\mu_\beta + x^i_\nu a^\nu_{\alpha\beta}\right).$$

Nous avons supposé que $f \in C^2$, donc aussi $h \in C^2$ et nous pouvons calculer

(23) $$\frac{\partial h(x^i, \dots, a^\gamma_{\alpha\beta})}{\partial a^\lambda_\delta} =$$

$$= \sum_{i=1}^{n} \frac{\partial f(\dots)}{\partial x^i_\delta} x^i_\lambda + 2 \sum_{i=1}^{n} \sum_{=1}^{k} \sum_{=1}^{k} \frac{\partial f(\dots)}{\partial x^i_{\delta\nu}} x^i_{\lambda\mu} a^\mu_\nu .$$

Posons $A = \det(a^\alpha_\beta)$, $\beta = 1, \dots, k$, et désignons par A^β_α des mineurs algébriques de A. Nous avons donc

(24) $$\frac{\partial A}{\partial a^\alpha_\beta} = A^\beta_\alpha .$$

La condition (22) étant une identité par rapport à tous ses arguments, nous pouvons différentier par rapport à a^λ_δ. Vu (23) et (24) nous obtenons ainsi la formule

(25) $$\frac{\partial h(x^i, \dots, a^\gamma_{\alpha\beta})}{\partial a^\alpha_\beta} = A^\beta_\alpha \, f(x^i, x^i_\alpha, x^i_{\alpha\beta}),$$

étant valable pour tous $a^\gamma_{\alpha\beta}$ tels que $A > 0$ et pour tous $a^\gamma_{\alpha\beta}$ tels que $a^\gamma_{\alpha\beta} = a^\gamma_{\beta\alpha}$. En particulier nous pouvons admettre

(26) $$a^\alpha_\beta = \delta^\alpha_\beta , \quad a^\gamma_{\alpha\beta} = 0 \quad (\delta^\alpha_\beta \text{ - le symbole de Kronecker}).$$

Nous avons donc $A = 1$, $A^\beta_\alpha = \delta^\beta_\alpha$ et nous obtenons le système

(27) $$\sum_{i=1}^{n} x^i_\alpha \cdot \frac{\partial f(\dots)}{\partial x^i_\beta} + 2 \sum_{i=1}^{n} \sum_{\mu=1}^{k} x^i_{\mu\beta} \cdot \frac{\partial f(\dots)}{\partial x^i_{\mu\alpha}} = \delta^\beta_\alpha \, f(\dots)$$

pour $\alpha,\beta = 1,\ldots,k$. Vu $x^i_{\alpha\beta} = x^i_{\beta\alpha}$. le système (27) contient au plus $k(k+1):2$ équations différentes.

Les dérivées du second membre de (22) par rapport à $a^\gamma_{\alpha\beta}$ s'annulent. En différentiant le premier membre de (22) par rapport à $a^\gamma_{\alpha\beta}$, nous obtenons les formules.

$$\frac{\partial h}{\partial a^\lambda_{\alpha\beta}} = \sum_{i=1}^{n} \frac{\partial f(\ldots)}{\partial x^i_{\alpha\beta}}\, x^i_\lambda$$

et, en y substituant (26), nous arrivons aux équations

$$\sum_{i=1}^{n} x^i_\gamma \cdot \frac{\partial f(\ldots)}{\partial x^i_{\alpha\beta}} = 0 \tag{28}$$

pour α, β, $\gamma = 1,\ldots,k$. Vu $x^i_{\alpha\beta} = x^i_{\beta\alpha}$, le système (28) contient au plus $k^2(k+1):2$ équations différentes. On peut donc énoncer le théorème suivant.

Théorème 6. Si l'intégrale $\mathcal{I}$ donnée par la formule (10) est localement invariantative, alors sa fonction formante f doit - pour toutes les suites de nombres x^i, x^i_α, $x^i_{\alpha\beta}$, $i=1,\ldots,n$, $\alpha,\beta = 1,\ldots,k$ - vérifier le système d'équations différentielles (27), (28) parmi lesquelles au plus $k(k+1)^2 : 2$ sont différentes.

OUVRAGES CITÉS

[1] N. Berutti Onesti : Sopra una classa di problemi variazionali di ordine n, Ann. Mat. Pura ed applicata 4 (91)

(1972) 129-161.

[2] C. Cinquini : Sopra i fundamenti di una classa di problemi variazionali dello spazio, Rend. Circ. Mat. Palermo 2 (6) (1957) 271-288.

[3] A. Kneser : Lehrbuch der Variationsrechnung. Braunschweig 1900.

[4] G. Kobb : Sur les maxima et les minima des intégrales doubles, Acta Math. 16 (1892) 65-140.

[5] K. Tatarkiewicz : Rachunek wariacyjny, Warszawa 1969, t. II, Warszawa 1970.

[6] K. Weierstrass : Vorlesungen über Variationsrechnung. Leipzig 1927.

[7] M. A. Wolanowski : A note on parametrical integrals, Comment. Math. 17 (1973) 265-267.

[8] E. Zermelo : Untersuchungen zur Variationsrechnung. Dissertation. Berlin 1894.

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CONVERGENCE SCALES FOR THE SUN POINT PROJECTIONS WITH DOUBLE CONVERGING SCALE

Dedicated to the memory
of Professor Edward Otto

Introduction

The double converging scale presented in this paper enables the drawing of shadows in such situations when the sun point projections are beyond of the limits of the drawing sheets. Useing the double converging scale we can avoid the unprofitable abbreviations of shadows arising in the cases when the sun point projections are not far enough.

1. The ranges of points on parallel base lines

We consider two parallel lines $l_1 \| l_2$ as the bases of two ranges of points: $l_1(A_1,B_1,C_1,\ldots)$, $l_2(A_2,B_2,C_2,\ldots)$. We assume that these ranges are in a correspondance in which the points of the pairs A_1A_2, B_1B_2, $C_1C_2,\ldots$ correspondent reciprocly to each other.

The ranges of points with different base lines are perspective iff all the straigth lines joining the pairs of their corresponding points pass through a common point which is called the center of the perspectivity of the ranges.

The ranges of points are similar iff the ratio:

$$\frac{|A_1B_1|}{|A_2B_2|} = \text{constant}$$

for every two pairs A_1B_1, A_2B_2 of their corresponding points.

Theorem 1. Two similar ranges of points on two parallel base lines are perspective.

Theorem 2. When one of two given perspective ranges of points with parallel base lines is translated on its base line then their center of perspectivity moves on a line which is parallel to the base lines of the ranges.

The proofs of these two theorems follow immediatly from the theorems inverse to the well known Thales' and Cavallerie's theorems.

2. The convergence scales with simple and double converging scale

Let be given two scales on two parallel lines $m \parallel c$ (Fig. 1) with constant moduli j_m, j_c, respectively. Setting in correspondence the points of these scales provided with the same numbers we obtain similar ranges of points (m), (c).

But since $m \parallel c$, then in virtue of Theorem 1 the ranges (m), (c) are perspective. We denote by S the center of their perspectivity. One may state that each straight line joining the points of the scales (m), (c) provided with the same numbers passes throught the fixed point S. In such a situation we say that the scales (m), (c) are the convergence scales for the point S. The scale (m) we call the main and the scale (c) the converging scale.

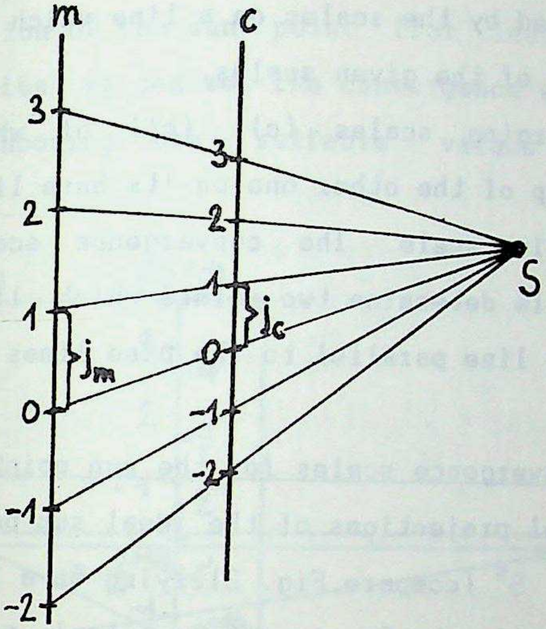

Fig. 1

Having the convergence scales for a point lying beyond of the limits of the paper sheet we draw lines through this point as the lines joining the pairs of corresponding points of these scales.

Now let us deliberatly translate the converging scale (c) on its base line c (Fig. 2). We denote by (c') the scale (c) in its new position.

The scales (m), (c') are convergence scales for a point S' which lies in virtue of Theorem 2 on a line through S parallel to the lines m, c.

Thus we have the following theorem:

Theorem 3. The translation of one of two given convergence scales on its base line causes a translation of the point determined by the scales on a line which is parallel to the base lines of the given scales.

The converging scales (c), (c') of which one is a translation map of the other one on its base line we call the double converging scale. The convergence scales with double converging scale determine two points which lie in virtue of Theorem 3 on a line parallel to the base lines of the scales.

3. The convergence scales for the sun point projections

The central projections of the ideal sun point S^∞ consist of two points S', S^s (compare Fig. 2) lying on a line perpendicular to the horizon line h: S' is the central projection of the horizontal projection of S^∞ (when no ambiguity can arise we say short "the horizontal projection"), S^s is the central projection of S^∞. The horizontal projection S' of the ideal sun point S^∞ must lie on the horizon line h.

The convergence scales for the sun point projections S', S^s, lying beyond of the limits of the drawing sheet have to have the double converging scale and (because of $S'S^s \perp h$) their base lines must be perpendicular to the horizon line h. The o-point of the converging scale (c) is the intersection point of its base line c with the horizon line h , but the distance of the base lines m, c, the ratio of the moduli of the scales (m), (c) and the position of the o-point of the translated converging scale (c') are free parameters. Picking any specific values for these parameters we determine the sun point projections and thereby the position of the ideal sun point. On the other hand to the given position of the sun point (for instance by its spherical coordinates) we can set the convergence scales for its projections by choosing the suitable values for their parameters.

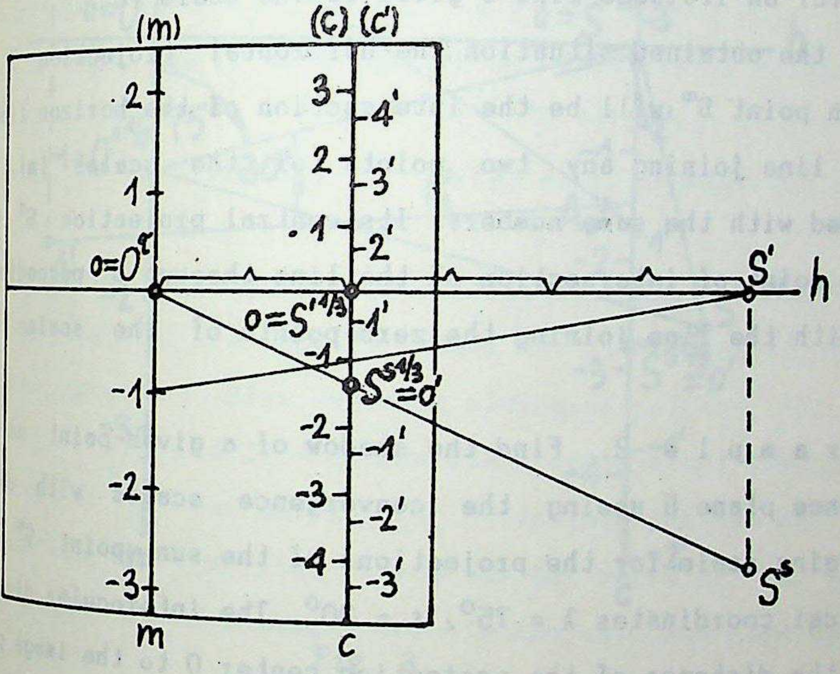

Fig. 2

Example 1. Determine the projections of the sun point by the convergence scales with double converging scale. The base lines m, c of the scales have to be perpendicular to the horizon line h on arbitrary choosen distance each from other. The zero-points of the scales (m), (c) must be the intersection points of the lines m, c with the horizon line h. The moduli of these scales may be at free choise and so may be the position of the o'-point of the scale (c') on the line c too.

We draw the lines $m, c \perp h$ (compare Fig. 2) and set on them the scales (m), (c) with o-points on the line h and with arbitrary but different moduli. A deliberate translation of the scale (c) on its base line c gives us the scale (c').

In the obtained situation the horizontal projection S' of the sun point S^{∞} will be the intersection of the horizon line h with a line joining any two points of the scales (m), (c) provided with the same numbers. Its central projection S^{s} will be the point of intersection of the line through S' perpedicular to h with the line joining the zero-points of the scales (m), (c').

Example 2. Find the shadow of a given point on the reference plane Π useing the convergence scales with double converging scale for the projections of the sun point S^{∞} with spherical coordinates $\lambda = 75^{o}$, $\varepsilon = 30^{o}$. The interocular distance (i.e. the distance of the projection center 0 to the image plane τ) $\delta = 10j$.

Taking the reduction factor $k = 1/5$ we determine at first the reduced projections $S^{,1/5}$, $S^{s1/5}$ of the sun point S^{∞} as follows:

1° The reduced horizontal projection $S^{,1/5}$ we find (Fig. 3) on the horizon line h by measuring the angle $\sphericalangle 0^{\tau}0^{x1/5}S^{,1/5} = 75^{o}$, where $0^{\tau}0^{x1/5} \perp h$ and $|0^{\tau}0^{x1/5}| =$

= 1/5.10j = 2j.

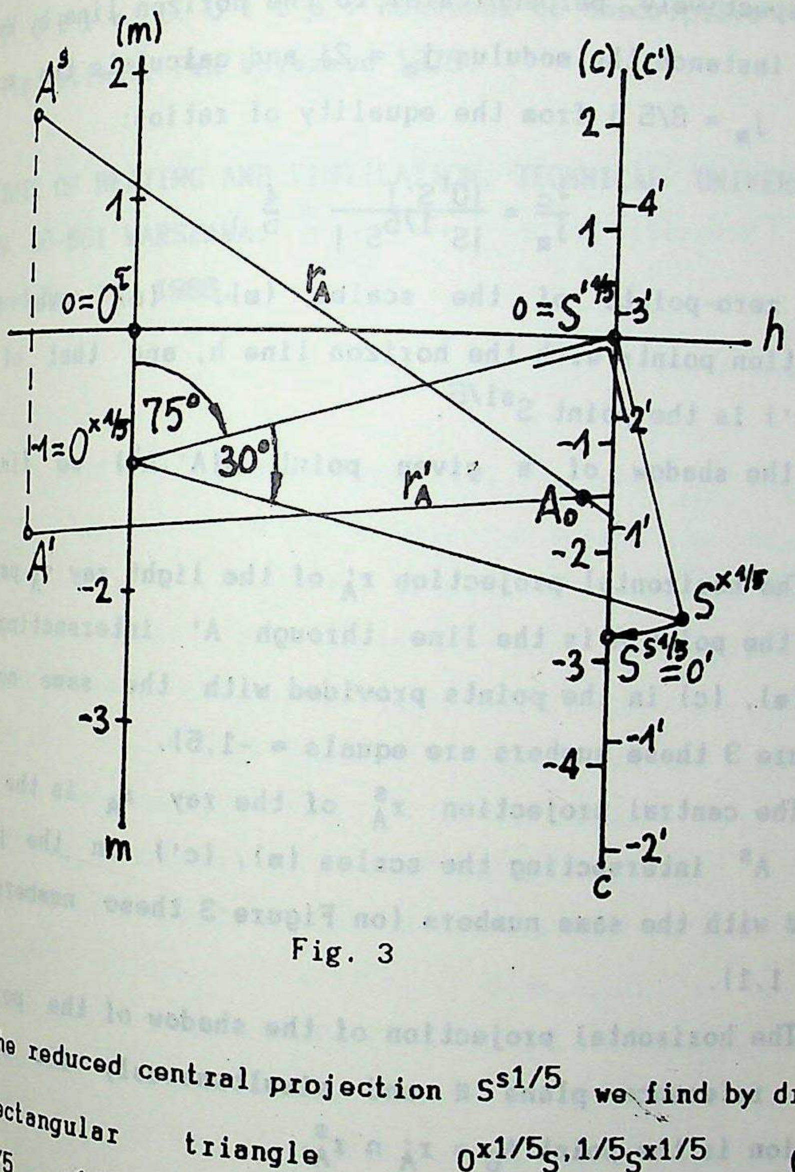
(m)
A^s
2
1
$0=0^{\xi}$
$-1=0^{x1/5}$
75°
30°
r_A
r'_A
A'
-2
-3
m
(c), (c')
2
4'
1
$0=S'^{1/5}$
3'
h
2'
-1
A_0
1'
-2
$S^{x1/5}$
-3
$S^{s1/5}=0'$
-1'
-4
-2'
c

Fig. 3

2° The reduced central projection $S^{s1/5}$ we find by drawing the rectangular triangle $0^{x1/5}S^{,1/5}S^{x1/5}$ (where $S^{,1/5}S^{x1/5} \perp 0^{x1/5}S^{,1/5}$ and $\sphericalangle S^{,1/5}0^{x1/5}S^{x1/5} = 30^o$) and by setting up the segment $S^{,1/5}S^{s1/5} \perp h$ and having the distance $|S^{,1/5}S^{s1/5}| = |S^{,1/5}S^{x1/5}|$ (see F.Otto, E.Otto [1], page 374).

Having the reduced projections of the sun point we draw the base lines m, c of the convergence scales through the points O^{τ}, $S'^{1/5}$ respectively, perpendicular to the horizon line h. We take then for instance the modulus $j_m = 2j$ and calculate the modulus $j_c = 4/5\ j_m = 8/5\ j$ from the equality of ratios:

$$\frac{j_c}{j_m} = \frac{|O^{\tau}S'|}{|S'^{1/5}S'|} = \frac{4}{5}\ .$$

The zero-points of the scales (m), (c) are their intersection points with the horizon line h, and that of the scale (c') is the point $S^{s1/5}$.

Now the shadow of a given point $A(A', A^s)$ we find as follows:

1° The horizontal projection r'_A of the light ray r_A passing through the point A is the line through A' intersecting the scales (m), (c) in the points provided with the same numbers (on Figure 3 these numbers are equals ≈ -1,5).

2° The central projection r^s_A of the ray r_A is the line through A^s intersecting the scales (m), (c') in the points provided with the same numbers (on Figure 3 these numbers are equal ≈ 1,1).

3° The horisontal projection of the shadow of the point A on the reference plane π and simultaneously its central projection is the point $A_0 = r'_A \cap r^s_A$.

REFERENCES

[1] F. Otto, E. Ottoo : Handbook of descriptive geometry (in Polish), PWN Warszawa 1975.

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ON THE GEOMETRY OF ORIENTED HYPERPLANES IN MULTIDIMENSIONAL METRIC-PROJECTIVE SPACES

Dedicated to the memory of Professor Edward Otto

Introduction

In the paper we are concerned with the geometry of oriented hyperplanes in a given metric-projective space. An analytical representation of analogous structures constructed from metric-affine spaces was given in [5]. For elliptic and hyperbolic planes a characterization was given in [3] and [7]. The case of 3-dimensional hyperbolic spaces can be considered as the Lie' construction (see[4]). It was shown in [7] how to construct the Beck Model in a more geometrical way with the help of Minkowskian spaces. This procedure can be generalized to hyperbolic spaces of arbitrary dimension, and even to arbitrary regular metric-projective spaces. It turns out that the structures in question are isomorphic to the generalized Benz-spaces i.e. to the geometries of appropriate quadrics.

Basic notions

Let V be an (n+1)-dimensional linear space over an ordered euclidean field F. Let us consider in V a form θ:

$$\theta(x,y) = \sum_{i=1}^{k} x_i y_i - \sum_{i=k+1}^{n+1} x_i y_i \quad \text{where} \quad n \le 2k,\ k \le n.$$

The proportionality relation $\approx$ in $V\setminus\{0\}$ is defined as usual. For shortness we write $V/_{\approx}$ instead of $(V\setminus\{0\})/_{\approx}$. In the horizon of V, i.e. in the n-dimensional projective space $P_n(F) = V/_{\approx}$ the form θ induces in a standard way the orthogonality $\perp$ (c.f.[1,8]). Moreover θ determines the polarity π and the quadric Q by the conditions:

$Q = \{ [u]_{\approx} : u \in V\setminus\{0\},\ \theta(u,u) = 0 \}$

$\pi([x_1,\dots,x_{n+1}]_{\approx}) = \{ [y]_{\approx} : y \in V\setminus\{0\},\ \theta(x,y) = 0\}$.

Let $\mathcal{H}$ be the set of all $(n-1)$ dimensional hyperplanes in $P_n(F)$. We define

$P^- = \{ [x]_{\approx} \in V/_{\approx} : \theta(x,x) < 0 \}$, $P^+ = \{ [x]_{\approx} \in V/_{\approx} : \theta(x,x) > 0 \}$.

$\mathcal{H}^+ = \{ U \in \pi^{-1}(U) \in P^+ \}$, $\mathcal{H}^- = \{ U \in \mathcal{H} : \pi^{-1}(U) \in P^- \}$.

Let $K(\theta) = \langle P^-, H^+; \in \rangle$. Clearly $K(\theta)$ are substructures of $P_n(F)$. Note that for $k=n$ we have: $K(\theta)$ is the n-dimensional Klein model over F, i.e. $K(\theta)$ is a n-dimensional hyperbolic space.

Note also that the metric-projective space $(P_n(F),\perp)$ is definable in $K(\theta)$-one must only define all pencils within $K(\theta)$ - with the help of Desargues configuration, for example (cf.[2]).

The form θ determines the metric geometry in V, since θ can be considered as a scalar product in V (cf.[6]). Let $\mathcal{P}$ be the set of all n-dimensional hyperplanes in V. The set $\mathcal{P}$ will be divided into three classes in accordance with the behaviour of its element of the horizon.

In a standard way we imbed V into the projective space $P_{n+1}(F)$. Then the horizon H_∞ of V is described by the equation $x_0=0$. For any hyperplane H in V we frequently identify H with its projective closure denoted by $\hat{H}$.

Let H have the equation $\sum_{i=1}^{n+1} \alpha_i x_i + \alpha_o = 0$. Then we define

$$H \in \mathcal{P}^0 \Leftrightarrow \theta(\alpha,\alpha) = 0 \quad \text{(degenerate hyperplane)},$$

$$H \in \mathcal{P}^+ \Leftrightarrow \theta(\alpha,\alpha) < 0 \quad \text{(exterior hyperplane)},$$

$$H \in \mathcal{P}^- \Leftrightarrow \theta(\alpha,\alpha) > 0 \quad \text{(interior hyperplane)}.$$

Clearly then $H \in \mathcal{P}^0$ iff $\hat{H} \cap H_\infty$ is tangent to Q,

$$H \in \mathcal{P}^+(P^-) \text{ iff } \hat{H} \cap H_\infty \in \mathcal{X}^+(\mathcal{X}^-).$$

The equation of H can be given in the form $\theta(\alpha,x) = \alpha_o$; if $H \notin \mathcal{P}^0$ then the vector $\frac{1}{\theta(\alpha,\alpha)}\alpha$ will be called a normal vector of H. Note that if $H \in \mathcal{P}^+$ then $\theta\left(\frac{1}{\theta(\alpha,\alpha)}\alpha, \frac{1}{\theta(\alpha,\alpha)}\alpha\right) = 1$. Analogously we classify the family $\mathcal{L}$ of all lines in V into three classes $\mathcal{L}^0, \mathcal{L}^+, \mathcal{L}^-$, in accordance with the type of their directions.

Let G(Q) be the group of projective automorphisms of Q, i.e. of projective collineations of $P_n(F)$, preserving $\mathcal{P}^-$. In this group we distinguish the subset $\Sigma(Q)$ consisting of symmetries in hyperspaces from $\mathcal{X}$.

For a hyperplane U in $P_n(F)$ let $\Sigma(Q)_{\langle U\rangle}$ be the subset of $\Sigma(Q)$ consisting of symmetries in hyperplanes containing U. Let $G^+(Q)_{\langle U\rangle}$ be the subgroup of G(Q) consisting of products of even number of elements of $\Sigma(Q)_{\langle U\rangle}$.

Note that if dim U = n-2 then the reduction principle holds in $\Sigma(Q)_{\langle U\rangle}$, in this case we have simply $G^{+}(Q)_{\langle U\rangle} = \Sigma(Q)_{\langle U\rangle}\cdot\Sigma(Q)_{\langle U\rangle}$.

With every $U \in \mathfrak{H}^{+}$ we correlate two orientations, which form "oriented hyperplanes $\tilde{U}'$, $\tilde{U}''$". A oriented hyperplane will be also called a proper axis; the set of proper axes will be denoted by A^{o}.

Let $A = A^{o} \cup \{\pi(p) : p \in Q\}$. Thus A consists of all proper axes and all the hyperplanes tangent to Q.

The meaning of "orientation of U" is not clear yet. Usually this is correlated with "halfspace with boundary U" but this makes sense not for all structures $K(\theta)$. We just consider axis as a formal object, its geometrical interpretation (or rather representation) will be given in the next section.

Now we pass to the definition of a cycle. First we note that the correlation π can be extended upon the family of all hyperplanes in $P_n(F)$.

Let $X \in A$ and U be a hyperplane in $P_n(F)$. We define

$$c(U,X) := G^{+}(Q)_{\langle U\rangle}[X] \cup \{\pi(p) ; p \in Q \cap \pi(U)\}$$

for U, X such that $\sim\pi(X) \in U$. Thus c(U,X) is the union of the orbit of X under $G^{+}(Q)_{\langle U\rangle}$ and the set of hyperplanes tangent to Q, containing U.

Finally we put

$$\mathfrak{C}_m := \{c(U,X) : U \text{ is a } (n-m)\text{-dimensional hyperplane in } P_n(F)\},$$

$$T_m(\theta) := \langle A, \mathfrak{C}_m ; \epsilon\rangle.$$

The structure $T_m(\theta)$ will be referred to as the structure (or geometry) of oriented hyperplanes over $K(\Theta)$ (i.e. over the metric-projective space $(P_n(F).\perp>)$).

Analogously we consider symmetries in (V,Θ). We denote by Iz the group of isometries of (V,Θ). For H-hyperplane in V let $Iz^+_{\langle H\rangle}$ be the subgroup of Iz consisting of even isometries fixing H point-wise. Clearly for every $H \in \mathcal{P}\setminus\mathcal{P}^0$ there is a symmetry in H; $Iz^+_{\langle H\rangle}$ consists of products of even numbers of reflections in hyperplanes $H \in \mathcal{P}\setminus\mathcal{P}^0$ such that $H' \subseteq H$.

Analytical representation of the geometry of oriented hyperplanes

Let us consider in V the family of all the hyperplanes and all the lines passing through o. For $q \in H_\infty$ let $\lambda(q)$ be the line in V, with direction q, passing through o. We have obviously:

Lemma 1. The function λ is an isomorphism between $P_n(F)(H_\infty)$ and the structure $\langle\{ L\in\mathcal{L} : o\in L \},\{ H\in\mathcal{P} : o\in H \},\subseteq\rangle$. Under this isomorphism $K(\Theta)$ is transformed onto the structure $\langle\{ L\in\mathcal{L}^- : o\in L \}, \{H\in \mathcal{P}^+ : o\in H \},\subseteq\rangle$. Every isometry of (V,Θ) induces a projective automorphism of $(P_n(F),\Theta)$, i.e. a projective automorphism of Q.

Every $H \in \mathcal{P}^+$ has exacily two normal vectors say u^1,u^2; $u^1=-u^2$. If $H \cap H_\infty=U$, $o \in H$ then one can identify oriented hyperplane $\tilde{U}$ with one of the vectors u^1, u^2. Thus we have constructed a function $\mu^o:A^o\longrightarrow V$.

Lemma 2. The function μ^o is a bijection between A^o and the set $C^o=\{ u \in V : \Theta(u,u) = 1 \}$.

Proof. Clearly if $\Theta(u,u) = 1$ then u can be considered as one of the normal vectors of the hyperplane $H: \Theta(x,u) = 0$. We identify u with some oriented hyperplane $\widehat{H\cap H_\infty}$, i.e. $\mu^o(\widehat{H\cap H_\infty}) = u$. Conversely, if $H \in \mathcal{P}^+$, v is a normal vector of H then $v \in C^o$.■

The equation $\Theta(x,x) = 1$ can be rewritten in the form

$$\sum_{i=1}^{k} x_i^2 - \sum_{i=k+1}^{n+1} x_i^2 = x_o^2, \quad x_o = 1.$$

Note that the set described by the conditions $\sum_{i=1}^{k} x_i^2 - \sum_{i=k+1}^{n+1} x_i^2 = x_o^2$, $x_o = 1$ coincides with Q. Its elements can be identified with hyperplanes tangent to Q, i.e. with elements of $A\setminus A^o$. On the other hand the standard isomorphism λ from Lemma 1 maps Q into $\mathcal{L}^o$ - lines tangent to C^o, passing through o. We extended μ upon A setting $\mu(U) = q$ for U tangent to Q in q. This together yields

Lemma 3. The function μ is a bijection between A and the quadric $C \subseteq P_{n+1}$ described by the equation

$$C:\ \sum_{i=1}^{k} x_i^2 - \left(\sum_{i=k+1}^{n+1} x_i^2 + x_o^2\right) = 0.$$

Now we are going to prove that this bijection maps cycles onto intersections of C and hyperplanes of P_{n+1}. The proof is based on some standard properties of metric-affine spaces.

Lemma 4. Let H be a k-dimensional nondegenerate hyperplane in V, $o \in H$. Let $u \in V$, $u \notin H$. Then there is a $(n-k+1)$-dimensional hyperplane H' in V such that $Iz^+_{\langle H\rangle}[u] \subseteq H'$.

Proof. Let $u \in H'$, $H' \perp H$, $\dim H' + \dim H = n+1$ (i.e. for arbitrary $a,b \in H$, $a',b' \in H'$: $\Theta(a-b,a'-b') = 0$).

In particular if U is a line then $Iz^{+}_{\langle U\rangle}[u]$ is contained in a n-dimensional hyperplane; if U is (n-1) dimensional, then $Iz^{+}_{\langle U\rangle}[u]$ is contained in a plane in V.

As a corollary of Lemma 4, and because F is Euclidean we obtain

L e m m a 5. If H is a k-dimensional hyperplane in V, $o \in H$, and $u \in C^{o}$ then there is a (n-k+1)-dimensional hyperplane in V such that $Iz^{+}_{\langle H\rangle}[u] = C^{o} \cap H'$. For every H′ intersecting C^{o} there exists H such that $H' \cap C^{o} = Iz^{+}_{\langle H\rangle}[u]$ for some $u \in C^{o}$.

Note that for arbitrary U-hyperplane in H_{∞}, nontangent to Q, the isomorphism λ induces an isomorphism of $G^{+}_{\langle U\rangle}$ and $Iz^{+}_{\langle \lambda U\rangle}$. Analogously we prove

L e m m a 6. If $L \in \mathfrak{L}^{o}$, $o \in L$; H is a k-dimensional hyperplane in V passing through o then there is a (n-k+1) dimensional hyperplane $U \subseteq H_{\infty}$ such that $Iz^{+}_{\langle H\rangle}[L \cap H_{\infty}] \subseteq U$. Therefore $Iz^{+}_{\langle H\rangle}[L \cap H_{\infty}] = Q \cap U$.

For H - arbitrary hyperspace in V containing o we denote by I(H) the family $\{L \in \mathfrak{L}^{o} : o \in L \subseteq H\}$. Clearly $I(H) = \lambda(\lambda^{-1}(H) \cap Q)$. Let $\dim H = k$; $H', H'' \perp H$, $v \in C^{o} \cap H'$; H, H', H''-hyperplanes in V.

L e m m a 7. If $o \in L \in \mathfrak{L}$ then $L \in I(H'') \Leftrightarrow \lambda^{-1}(L) \in \hat{H}' \cap C$, where $\hat{H}'$ is a projective closure of H′ in $P_{n+1}(F) \supseteq V$.

P r o o f. Obvious. From the assumptions $H' \parallel H''$, $\dim H' = \dim H''$.■

As a consequence of Lemmas 3, 5, 6, 7 and the definition of cycle we obtain

Corollary 8. The function μ maps the set of cycles of $T_m(\theta)$ onto the family of all intersections of C with m-dimensional hyperplanes in $P_{n+1}(F)$.

Thus we have obtained our final result in this section

Theorem 9. The geometry of oriented hyperspaces in a metric-projective space, i.e. the structure $T_k(\theta)$ is isomorphic to the incidence structure on the quadric C with blocks interpreted as intersections of C with k-dimensional hyperplanes in $P_{n+1}(F)$.

The form θ will be called of the type (k,l) if $l=n+1-k$. The quadric C is called to have type (k,l) if it has the equation $\sum_{i=1}^{k} x_i^2 - \sum_{i=k+1}^{n+1} x_i^2 = 0$. Just from the above construction we see that if θ has the type $(k,l+1)$ then the quadric isomorphic to $T_k(\theta)$ has the type $(k,l+2)$. As we have noted the form θ which describes n-dimensional hyperbolic geometry has the type $(n,1)$. Thus the quadric C isomorphic to the geometry of hyperbolic axes and cycles has the type $(n,2)$. In particular if $n=2$ then we obtain C as a Beck'Model (cf.[3]). If $n=3$ then C has the type $(3,2)$, i.e. it is a Lie'quadric (cf.[4]).

Comments

Any metric-projective geometry (P_n,θ) with self-conjugate points is uniquely determined by the quadric Q. For $U \in \mathfrak{Z}$ we define $\gamma(U) := U \cap Q$. Then $\gamma|_{\mathfrak{Z}^+ \cup \mathfrak{Z}^o}$ is a bijection between $\mathfrak{Z}^+ \cup \mathfrak{Z}^o$ and $\{U \cap Q : U \in \mathfrak{Z}^+ \cup \mathfrak{Z}^o\}$. Thus the family of axes A can be considered as a family of "oriented blocks" ((n-1)-dimensional "spheres") contained in Q. Let θ has the type (k, n+1-k); let θ' has the type (k-1, n-k). We consider a metric-affine space (V', θ') with $\dim V' = n-1$. Then Q can be treated as the inversive closure of (V', θ') (cf. [8]). In particular elements of $\gamma(\mathfrak{X}^+)$ correspond in V' to (n-2)-dimensional hyperspheres with positive radius and (n-2)-dimensional hyperplanes which normal vector has positive "length"; while elements of $\gamma(\mathfrak{X}^0)$ correspond to hyperspheres with radius zero and hyperplanes with degenerate normal vector. One can consider in (V', θ') the geometry of oriented hyperspheres and hyperplanes. This was done in [5] for example, and the analytical representation for such structures was given there.

On the other hand there is a well-known construction of the Lie' plane from the geometry of axes in ordered Euclidean planes; the construction with the idea to treat in the same way oriented lines and oriented circles. It is seen that, in fact, this construction gives the geometry of axes on the inversive closure of (V', θ'). Thus we see that the Lie' procedure (inversive closure) applied to the structures investigated in [5] gives the geometries $T_n(\theta)$.

If $U \in \mathfrak{X}^0$ i.e. if U is tangent to Q in q then $Q \cap U$ consists of points which conjugate with q, w.r. to Q; $Q \cap U$ is a projective cone. If $U \in \mathfrak{X}^+$ then $U \cap Q$ is a quadric in U; clearly then $\overline{U} \cap \overline{Q} > 1$. But it may happen that $U \cap Q \neq \emptyset$ for $U \in \mathfrak{X}^-$. In such a case we can consider the structure of all "oriented spheres" of Q. It is seen that this structure is an inversive closure of (V', θ') with all oriented $(n-2)$ dimensional hyperspheres and hyperplanes considered. On the other hand the technique used in the previous section shows that in this case we obtain two quadrics, C and $\bar{C}$; with equations $\theta(x,x) = x_0^2$. This observation indicates the way in which one can represent the geometry of oriented sections in the geometry on quadrics.

In the geometry of (ruled) quadrics an essential role is played by the relation of conjugacy. It is even quite frequently used as a primitive notion (c.f.[4]). Thus it is worthwhile to establish conditions for $\breve{U}_1$, $\breve{U}_2$ under which $\mu(\breve{U}_1)$, $\mu(\breve{U}_2)$ conjugate w.r. to C. First we note

Lemma 10. If $H \| u$, $o \in H \in \mathcal{P}$ then $Iz^+_{\langle H \rangle}[u]$ is contained in the hyperplane tangent to $\dot{C}^o$ in u.

Proof. We just observe that if $H' \perp H$, $u \in H' \cap C^o$ then $u \perp H'$; thus H' is tangent to C^o in u. Next we apply Lemma 4;

Since we know that $u = \mu(\breve{U})$, $U = \lambda^{-1}(H_1)$, $u \perp H_1$ we see that these degenerate sections of C^o correspond to orbits $G^+_{\langle U_1 \rangle}[\breve{U}]$ where $U \perp U_1$, $\dim U = n-1$, $\dim U_1 = 0$.

Let $o \in L \in \mathfrak{L}^o$, $p = \lambda^{-1}(L)$, U-hyperplane tangent to C in p. It is seen that o is the pole of H_∞ w.r. to C, thus $o \in U$ and $\lambda^{-1}(U \cap H_\infty) = H = U \setminus H_\infty$. Therefore $U \setminus H_\infty \in \mathfrak{X}^o$. Consider arbitrary $u \in H \cap C^o$; let $H' \subseteq H$, $o, p \in \hat{H}'$. Then $H' \perp H$. We have $Iz^+_{\langle H' \rangle}[u] = H$.

This reasoning yields that (degenerate) sections of C^o with hyperplanes tangent "at infinity" correspond to orbits $G^{+}_{\langle U_1 \rangle}[\tilde{U}]$ where U_1 is tangent to Q. This relation can be also interpreted in a more analytical way. For two $u,v \in C^{o}$: u and v conjugate

$$(u\text{-}v) \text{ iff } \sum_{i=1}^{k} u_i v_i - \sum_{i=k+1}^{n+1} u_i v_i = 1, \text{ i.e. if } \theta(u,v) = 1.$$

Let us calculate $\theta(u-v,u-v) = \theta(u,u) + \theta(v,v) - 2\theta(u,v) = 2-2\theta(u,v)$. Therefore for $u,v \in C^{o}$: $u\text{-}v \Leftrightarrow \theta(u-v) = 0$. If u-v, $u \neq \pm v$ then there is a unique plane M containing o, u, v; clearly M is tangent to Q at the horizon. Let H_u, H_v be hyperplanes with normal vectors u,v respectively. Then $\lambda^{-1}(H_u), \lambda^{-1}(H_v)$ are orthogonal to $\lambda^{-1}(M)$ with respect to Q. Therefore $\lambda^{-1}(H_u), \lambda^{-1}(H_v)$ have a common point with Q. In standard classical hyperbolic geometry this means that $\lambda^{-1}(H_u)$ and $\lambda^{-1}(H_v)$ are parallel.

REFERENCES

[1] E. Artin : Geometric algebra . New York 1957.

[2] F. Bachmann : Aufbau der Geometrie aus dem Spiegelungsbegriff, Springer Verlag 1973.

[3] H. Beck : Zur Lieschen Kugelgeometrie im Nichteuklidischen Raum, Jahresbericht d.DMV.

[4] W. Benz : Vorlesungen über Geometrie der Algebren, Springer Verlag 1973.

[5] L. Dubikajtis, H. Guściora : On the geometry

of oriented equiaxial hyperquadrics, Bull. Acad. Polon. Sci. Sér. Math. 17 (1969), 29-35.

[6] E. Kusak : Linear spaces with generalized orthogonality, [submitted to] Bull. Acad. Polon. Sci. Sér. Math.

[7] K. Prażmowski : A note on cycle in elliptic geometry, [submitted to] J. of Geometry.

[8] Rozenfeld : Noneuclidean Geometries, Moscow 1956, (in Russian).

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ON LOCALLY COUNTABLY GENERATED DIFFERENTIAL SPACES

*Dedicated to the memory
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In this paper we give a characterization of Sikorski's differential spaces [6] which are locally countably generated by a family of real-valued functions. In section 1 we study the properties of the differential space $(R^N, \mathcal{E}_N)$ of all real sequences with the natural differential structure $\mathcal{E}_N$ generated by projection [2]. In Section 2 we describe some properties of differential spaces which are locally diffeomorphic to open subsets of R^N.

1. R^N as a differential space

Let R^N be the set of all real sequences. Every element $x \in R^N$ is a map $x\colon N \longrightarrow R$ from the set of natural numbers into the set of real numbers. We will denote it by $x = (x_i)_{i \in N}$.

Denote by π_i for $i \in N$ the projection of R^N onto the i-th coordinate given by

(1)
$$\pi_i(x) = x_i \quad \text{for } x = (x_i) \in R^N.$$

Let $P = \{\pi_i\colon i \in N\}$ be the set of all such projections. By τ_P denote the weakest topology in R^N in which all projections from P are continuous, i.e. the product topology.

Let $\mathcal{E}_N$ be the differential structure on R^N generated by the set P. By definition $\alpha\in\mathcal{E}_N$ iff for any $x\in R^N$ there exists an open neighbourhood $V \in \tau_P$ of x, natural numbers $i_1,\dots,i_k$ and a C^∞ function $\omega : R^k \longrightarrow R$ such that

$$(2) \qquad \alpha(x) = \omega(x_{i_1},\dots,x_{i_k}) \quad \text{for } x \in V.$$

Thus $\alpha\in \mathcal{E}_N$ iff locally it is a C^∞ function of a finite number of variables.

For any $\alpha \in \mathcal{E}_N$ and $i \in N$ we set

$$(3) \qquad \partial_{ix}(\alpha) := \omega'_{|i}(x_{i_1},\dots,x_{i_k}) ,$$

where ω comes from (2); here $\omega'_{|i} = \frac{\partial\omega}{\partial x_i}$ is understood to be zero if i is not one of the i_j. Clearly, the definition is independent of the choice of ω and $\partial_{ix}\colon \mathcal{E}_N \longrightarrow R$ is tangent vector at $x \in R^N$ to the differential space $(R^N,\mathcal{E}_N)$. The vectors ∂_{ix}, $i \in N$ are linearly independent since obviously

$$(4) \qquad \partial_{ix}(\pi_j) = \delta_{ij} \quad \text{for } i,j \in N.$$

For any $\alpha \in \mathcal{E}_N$ the set $I_{\alpha,x} := \{i\in N \; \partial_{ix}(\alpha) \neq 0\}$ is finite. For every $x\in R^N$ and every $r = (r_i)_{i\in N}\in R^N$, the vector $w \in T_x(R^N)$ given by

$$(5) \qquad w(\alpha) = \sum_{i\in I_{\alpha,x}} r_i\partial_{ix}(\alpha) \quad \text{for } \alpha \in \mathcal{E}_N$$

clearly satisfies $w(\pi_j) = r_i$ for $i \in N$. It is easy to see that w given by (5) is the unique vector with this property. In the sequel, it will be written as a formal sum $w = \sum_{i\in N} r_i\partial_{ix}$.

Thus we have proved

Lemma 1. The mapping $K\colon T_x(R^N) \longrightarrow R^N$ given by

$$K(v) = (v(\pi_1), v(\pi_2), \dots) \quad \text{for } v \in T_x(R^N) \tag{6}$$

is an isomorphism of vector spaces.

Now, let $\mathfrak{X}(R^N)$ be the $\mathcal{E}_N$-module of all smooth vector fields tangent to $(R^N, \mathcal{E}_N)$. It is easy to see that the mappings $\partial_i\colon R^N \longrightarrow TR^N$, $i \in N$, given by

$$\partial_i(x) = \partial_{ix} \quad \text{for } x \in R^N, \tag{7}$$

belong to $\mathfrak{X}(R^N)$ and are $\mathcal{E}_N$ - linearly independent.

For every sequence $(\varphi_i)_{i\in N}$ in $\mathcal{E}_N$, denote by $\sum_{i\in N} \varphi_i\partial_i$ the vector field W tangent to $(R^N, \mathcal{E}_N)$ given by

$$W(x) := \sum_{i\in N} \varphi_i(x)\partial_{ix} \quad \text{for } x \in R^N. \tag{8}$$

It is easy to see that $W(\pi_i) = \varphi_i$ for $i \in N$. Thus $W = \sum_{i\in N} \varphi_i\partial_i \in \mathfrak{X}(R^N)$. So we may uniquely represent an arbitrary vector field $X \in \mathfrak{X}(R^N)$ as a formal sum

$$X = \sum_{i\in N} X(\pi_i)\partial_i.$$

We have proved

Lemma 2. The mapping $L\colon \mathfrak{X}(R^N) \longrightarrow \mathop{\times}_{i\in N} \mathcal{E}_N$ given by

$$L(X) = (X(\pi_1), X(\pi_2), \dots) \quad \text{for } X \in \mathfrak{X}(R^N) \tag{9}$$

is an isomorphism of $\mathcal{E}_N$-modules.

Lemma 3. For any vector $v \in T_x(R^N)$ there exists a smooth vector field $X \in \mathfrak{X}(R^N)$ such that $v = X(x)$.

Proof. If $v = \sum_{i\in N} \lambda_i\partial_{ix}$, we put $X = \sum_{i\in N} \lambda_i\partial_i$.

Now, let (M,C) be a differential space. Let $h_M\colon A^k(M) \longrightarrow \Omega^k(M)$, $k \in N$, be the homomorphism of the C-module of smooth pointwise k-forms on M [1] into the C-module of global k-forms [8] given by

$$(10) \qquad h_M(\omega)(X_1), \ldots, X_k)(p) = \omega(X_1(p), \ldots, X_k(p))$$

for any $\omega \in A^k(M)$, $X_1, \ldots, X_k \in \mathfrak{X}(M)$ and $p \in M$.

In general h_M is not an isomorphism [4]. Now we prove

Proposition 1. If for every $p \in M$ and any $v \in T_pM$ there exists a vector field $X \in \mathfrak{X}(M)$ such that $v = X(p)$ then h_M is a monomorphism.

Proof. Let $\omega \in A^k(M)$ be such that $h_M(\omega) = 0$. Then for any $v_1, \ldots, v_k \in T_pM$ there exist vector fields $X_1, \ldots, X_k$ such that $X_1(p) = v_1, \ldots, x_k(p) = v_k$, and so $\omega(v_1, \ldots, v_k) = (h_M\omega)(X_1, \ldots$ $\ldots, X_k)(p) = 0$. Therefore $\omega = 0$.

Corollary 1. For the differential space $(R^N, \mathcal{E}_N)$ the homomorphism h_{R^N} is a monomorphism.

Corollary 2. In the graded algebra $A(R^N)$ there exists a unique operator $\partial\colon A^k(R^N) \longrightarrow A^{k+1}(R^N)$, $k = 0,1,2,\ldots$ with the well-known properties of exterior derivative.

Proof. In [4] we introduced, for any differential space (M,C), the homogeneous ideal $\mathfrak{M}(M)$ in the graded algebra $A(M)$, consisting of all pointwise differential forms which are formal differential of 0, and we proved the inclusion $\mathfrak{M}(M) \subset \ker h_M$. In our case, by Corollary 1, $\ker h_{R^N} = \{0\}$. Now the corollary follows from Theorem 3.1 in [4].

For an arbitrary $\omega \in A^k(R^N)$ and for each $p \in R^N$ there exists an open neighbourhood $V \in \tau_p$ such that

$$\omega|V = \sum_I \alpha^{i_1 \dots i_k} d\pi_{i_1} \wedge \dots \wedge d\pi_{i_k} |V,$$

where $\alpha^{i_1 \dots i_k} \in (\mathcal{E}_N)_V$, $(i_1, \dots, i_k) \in I \subset N^k$ and I is a finite subset of indices [4]

$$\partial\omega = \sum_I d\alpha^{i_1 \dots i_k} \wedge (d\pi_{i_1} \wedge \dots \wedge d\pi_{i_k})|V.$$

In the algebra, $\Omega(M)$ we have the exterior derivative $d: \Omega^k(M) \longrightarrow \Omega^{k+1}(M)$, $k = 0,1,\dots,$ given by the well-known formula [6]. The following diagram commutes:

$$\begin{tikzcd}
A^k(R^N) \arrow[r, "h_{R^N}"] \arrow[d, "\partial"'] & \Omega^k(R^N) \arrow[d, "d"] \\
A^{k+1}(R^N) \arrow[r, "h_{R^N}"] & \Omega^{k+1}(R^N)
\end{tikzcd}.$$

Lemma 4. Let $\alpha \in \mathcal{E}_N$ be a smooth function on R^N. If $d\alpha = 0$ then $\alpha = \text{const}$.

Proof. Since R^N is connected, it is enough to show that α is locally constant. For ever $x \in R^N$ there exists an open connected neighbourhood V of x and a smooth function ω on R^k such that

$$\alpha|V = \omega \circ (\pi_{i_1}, \dots, \pi_{i_k})|V.$$

nce $d\alpha|V = \left(\sum_{j=1}^{k} \omega'_{|j} \circ (\pi_{i_1}, \dots, \pi_{i_k}) \, d\pi_{i_k} \right)|V = 0.$

ce $d\pi_{i_1}|V, \dots, d\pi_{i_k}|V$ are linearly independent,

$$\omega'_{|j} \circ (\pi_{i_1}, \dots, \pi_{i_k})|V = 0 \quad \text{for } j = 1, \dots, k,$$

or equivalently

$$\omega'_{|j}|(\pi_{i_1},\dots,\pi_{i_k})(V) = 0 \quad \text{for } j = 1,\dots,k.$$

The set $(\pi_{i_1},\dots,\pi_{i_k})(V)\subset R^k$ is open and connected. Thus $\omega|(\pi_{i_1},\dots,\pi_{i_k})(V)$ is constant, and so α is locally constant, which completes the proof.

Now, we will introduce a topology in the $\mathcal{E}_N$-module $\mathfrak{X}(R^N)$ which leads to another interpretation of formal sums $\sum_{i\in N}\varphi_i\partial_i$ with $\varphi_i\in\mathcal{E}_N$. Recall that $d\pi_i\in\Omega^1(R^N)$, $i\in N$, is the global 1-form given by

$$(d\pi_i)(X) = X(\pi_i) \quad \text{for } X\in\mathfrak{X}(R^N). \tag{11}$$

For any $\omega\in\Omega^1(R^N)$ and $x\in R^N$ let $\omega_x:\mathfrak{X}(R^N)\longrightarrow R$ be given by

$$\omega_x(X) = \omega(X)(x) \quad \text{for } X\in\mathfrak{X}(R^N). \tag{12}$$

Set $\mathcal{F} := \{(d\pi_i)_x : i\in N,\ x\in R^N\}$. Then $\mathcal{F}$ is a set of real-valued functions on $\mathfrak{X}(R^N)$. Let $\tau_{\mathcal{F}}$ be the weakest topology on $\mathfrak{X}(R^N)$ in which all functions in $\mathcal{F}$ are continuous. It is easy to see that $\tau_{\mathcal{F}}$ is a Hausdorff topology.

Now we may prove

Proposition 2. For any sequence $(\varphi_i)_{i\in N}$ in $\mathcal{E}_N$,

$$\sum_{i\in N}\varphi_i\partial_i = \lim_{n\to\infty}\sum_{i=1}^{n}\varphi_i\partial_i,$$

where the limit is taken in $(\mathfrak{X}(R^N),\tau_{\mathcal{F}})$.

Proof. Put $X = \sum_{i\in N}\varphi_i\partial_i$ and $S_n = \sum_{i=1}^{n}\varphi_i\partial_i$ for $n\in N$.

Let $V\in\tau_{\mathcal{F}}$ be an arbitrary neighbourhood of X. There exist sets $\{i_1,\dots,i_k\}\subset N$, $\{p_1,\dots,p_k\}\subset R^N$ and an open interval $Q\subset R^N$

such that

$$X \in ((d\pi_{i_1})_{p_1}, \dots, (d\pi_{i_k})_{p_k})^{-1}(Q) \subset V.$$

Hence $((d\pi_{i_1})_{p_1}, \dots, (d\pi_{i_k})_{p_k})(X) \in Q$, i.e. $(\varphi_{i_1}(p_1), \dots, \varphi_{i_k}(p_k)) \in Q$

Putting $n_o = \max(i_1, \dots, i_k)$ we may observe that

$$S_n(\pi_{i_j}) = X(\pi_{i_j}) \quad \text{for } j = 1, \dots, k \text{ and } n \geq n_o.$$

Thus $S_n \in ((d\pi_{i_1})_{p_1}, \dots, (d\pi_{i_k})_{p_k})^{-1}(Q) \subset V$ for $n \geq n_o$. We have shown that for any open neighbourhood V of X there exists $n_o \in N$ such that $S_n \in V$ for $n \geq n_o$. This means that $X = \lim_{n \to \infty} S_n$.

Now let $\hat{\mathfrak{F}}$ be the differential structure on $\mathfrak{X}(R^N)$ generated by $\mathfrak{F}$. If $\omega \in h_{R^N}(A^1(R^N))$ then $\omega_p \in \hat{\mathfrak{F}}$ for all $p \in R^N$. Indeed, every $p \in R^N$ has an open neighbourhood V such that $\omega|V = \sum_{j=1}^{k} \omega^{i_j} d\pi_{i_j}$, where $\omega^{i_j} \in \mathcal{E}$ for j=1,...,k. Hence $\omega = \sum_{j=1}^{k} \omega^{i_j}(p)(d\pi_{i_j})$. So $\omega_p \in \hat{\mathfrak{F}}$.

Proposition 3. For every $\omega \in h_{R^N}(A^1(R^N))$ and any sequence $(\varphi_i)_{i \in N}$ in $\mathcal{E}_N$,

$$\omega\Big(\sum_{i \in N} \varphi_i \partial_i\Big) = \sum_{i \in N} \varphi_i \omega(\partial_i).$$

Proof. The sum $\sum_{i \in N} \varphi_i \omega(\partial_i)$ is locally finite. Indeed, for every $p \in R^N$ there exists an open neighbourhood $V \ni p$ such that

$$\omega|V = \sum_{j=1}^{k} \omega^{i_j} d(\pi_{i_j}|V).$$

Hence and from (4) it follows that the set $\{i \in N : \omega(\partial_i)|V \neq 0\}$ is finite. Now we have,

$$\omega\Big(\sum_{i\in N}\varphi_i\partial_i\Big)|V = \Big(\sum_{j=1}^{k}\omega^{i_j}\varphi_{i_j}\Big)|V =$$

$$= \Big(\sum_{j=1}^{k}\varphi_{i_j}\omega(\partial_{i_j})\Big)|V = \Big(\sum_{i\in N}\varphi_i\omega(\partial_i)\Big)|V,$$

which finishes the proof.

Remark. It is an open question whether the monomorphism h_{R^N} from Corollary 1 is an isomorphism.

2. Countably generated and locally countably generated differential spaces

A differential space (M,C) is said to be countably generated by a countable set $C_o = \{\alpha_i : i\in N\}$ of real functions on M if $C = (scC_o)_M$.

Now, let $\phi : M \longrightarrow R^N$ be mapping defined by

(13) $$\phi(p) = (\alpha_i(p))_{i\in N} \quad \text{for } p \in M.$$

Evidently ϕ is a smooth mapping of (M,C) into $(R^N,\mathcal{E}_N)$. Indeed, this follows from the equality $\pi_i\circ\phi = \alpha_i$ for $i \in N$.

Let $\gamma : (M,C) \longrightarrow (\phi(M),\mathcal{E}_{N\phi(M)})$ be the mapping ϕ onto the image $\phi(M)$ with the natural differential structure induced from $(R^N,\mathcal{E}_N)$ on $\phi(M)$ [7]. IF $\ell_{\phi(M)}$ is the natural imbedding $\phi(M) \longrightarrow R^N$ then $\ell_{\phi(M)}\circ\gamma = \phi$. Of course the differential structure $\mathcal{E}_{N\phi(M)}$ is countably generated by the set $\{\pi_i|\phi(M) : i\in N\}$. The function $\pi_i|\phi(M)$ is the i-th component of the imbedding map $\ell_{\phi(M)}$.

Similarly to Lemma 2.1 in [5] on can prove

Lemma 5. Let (M,C) be a differential space countably generated by the set $C_o = \{\alpha_i : i \in N\}$. Then

(i) the sets of the form $\phi^{-1}(Q)$ form a base of weakest topology τ_C on M in which all functions in C are continuous, where Q is an open set from the base of the Tikhonov topology of R^N. The topologies τ_C and $\tau_{\mathcal{E}_{N\phi(M)}}$ have the same cardinality,

(ii) the mapping $\gamma : (M,C) \longrightarrow (\phi(M), \mathcal{E}_{N\phi(M)})$ is open,

(iii) τ_C is the Hausdorff topology iff $\gamma : M \longrightarrow \phi(M)$ is a homeomorphism,

(iv) if $\phi(p_1) = \phi(p_2)$ for some p_1, p_2 in M then $\alpha(p_1) = \alpha(p_2)$ for each $\alpha \in C$.

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Proposition 4. A differential space (M,C) is countably generated if and only if there exists a smooth mapping $\phi : (M,C) \longrightarrow (R^N, \mathcal{E}_N)$ such that $\gamma^* : \mathcal{E}_{N\phi(M)} \longrightarrow C$ is an isomorphism of linear rings. Moreover, the set $\{\gamma^*(\pi_i | \phi(M)) : i \in N\}$ is a set of generators for the differential structure C.

Proof. ⇒ Let (M,C) be countably generated by $C_o = \{\alpha_i : i \in N\}$, and let $\phi : (M,C) \longrightarrow (R^N, \mathcal{E}_N)$ and $\gamma : (M,C) \longrightarrow (\phi(M), \mathcal{E}_{N\phi(M)})$ be defined before the formulation of Lemma 5. We will prove that $\gamma^* : \mathcal{E}_{N\phi(M)} \longrightarrow C$ is an isomorphism of linear rings. Since γ is a surjection, γ^* is a monomorphism. Now we will prove that γ^* is "onto". For any $\alpha \in C$, let $\omega_\alpha : \phi(M) \longrightarrow R$ be defined by

(14)
$$\omega_\alpha(q) = \alpha(p) \quad \text{for } q \in \phi(M),$$

where $p \in M$ is such that $q = \phi(p)$. The correctness of this definition follows from Lemma 5 (iv). Moreover,

$$\omega_\alpha \circ \tilde{\phi} = \alpha \,. \tag{15}$$

It remains to show that ω_α is a smooth function on $(\phi(M), \mathcal{E}_{N\phi(M)})$. Fix $q \in \phi(M)$ and choose $p \in M$ such that $\phi(p) = q$. In some open neighbourhood $V \in \tau_C$ of p the function α has a representation

$$\alpha | V = \omega \circ (\alpha_{i_1}, \ldots, \alpha_{i_k}) | V, \tag{16}$$

where $\omega : R^k \longrightarrow R$ is a C^∞ function.

Put $\omega_{i_1 \ldots i_k} = \omega \circ (\pi_{i_1}, \ldots, \pi_{i_k})$. Of course $\omega_{i_1 \ldots i_k} \in \mathcal{E}_N$.

Evidently from Lemma 5 it follows that $\tilde{\phi}(V)$ is open set containing q. Moreover, from (15) and (16) we have

$$\omega_{i_1 \ldots i_k} \circ \tilde{\phi} | V = \omega_\alpha \circ \tilde{\phi} | V .$$

Hence $\omega_\alpha | \tilde{\phi}(V) = \omega_{i_1 \ldots i_k} | \tilde{\phi}(V)$. Thus $\omega_\alpha \in \mathcal{E}_{N\phi(M)}$.

$\Leftarrow$ Conversely, let $\phi : (M,C) \longrightarrow (R^N, \mathcal{E}_N)$ be a smooth mapping such that $\tilde{\phi}^* : \mathcal{E}_{N\phi(M)} \longrightarrow C$ is a ring isomorphism; evidently the set $\{\pi_i | \phi(M) : i \in N\}$ generates the differential structure $\mathcal{E}_{N\phi(M)}$. Since $\tilde{\phi}^*$ is an isomorphism the countable set $\{\phi^*(\pi_i | \phi(M)) : i \in N\}$ generates the differential structure C, which completes the proof.

A differential space (M,C) is said to be locally countably generated if for every $p \in M$ there exists an open neighbourhood $V \ni p$ such that the induced differential space (V, C_V) is countably generated.

One can prove

Corollary 3. A differential space (M,C) is locally countably generated iff for each $p \in M$ there exists an open neighbourhood V of p and a smooth map $f\colon (V,C_V) \longrightarrow (f(V),\mathcal{E}_{Nf(V)})$ such that $f^*\colon \mathcal{E}_{Nf(V)} \longrightarrow C_V$ is an isomorphism of linear rings. If this condition is satisfied and (M,C) is a Hausdorff differential space then f is a diffeomorphism, so (M,C) is locally diffeomorphic to a subset $(R^N,\mathcal{E}_N)$ with the natural differential structure.

Among the differential spaces which are locally countably generated we can consider the class of differential spaces locally diffeomorphic to an open subset of $(R^N,\mathcal{E}_N)$.

A differential space (M,C) is said to be an R^N-manifold if for every $p \in M$ there exist an open neighbourhood V of p and a diffeomorphism $F\colon (V,C_V) \longrightarrow (F(V),\mathcal{E}_{NF(V)})$ onto an open subset $F(V) \subset R^N$. Every such diffeomorphism F is said to be a chart of the R^N-manifold (M,C).

Now, let $p \in M$ and let $F\colon V \longrightarrow R^N$ be a chart of (M,C) such that $V \ni p$. It is easy to observe that $F_{*p} \circ (\ell_V)_*^{-1}\colon T_pM \longrightarrow T_{F(p)}R^N$ is an isomorphism of tangent spaces, where $\ell_V\colon V \longrightarrow M$ is the inclusion map. Every chart $F\colon V \longrightarrow R^N$ has the form $F = (F_i)_{i \in N}$ where $F_i \in C_V$ for $i \in N$.

Put

(17)
$$\frac{\partial}{\partial F_i}\Big|_p = F_{*p}^{-1}(\partial_{iF(p)}) \quad \text{for } i \in N.$$

For any sequence $(r_i)_{i \in N} \in R^N$ there exists a unique vector

$w \in T_pM$ such that $w(F_i) = r_i$ for $i \in N$. The vector w is given by

$$(18) \qquad w = F^{-1}_{*p}\left(\sum_{i\in N} r_i \partial_{iF(p)}\right).$$

In the sequel it will be written as a formal sum $\sum_{i\in N} \lambda_i \frac{\partial}{\partial F_i}\Big|_p$, where $\lambda_i = w(F_i)$ for $i \in N$.

For any sequence $(\varphi_i)_{i\in N}$ in $\mathcal{E}_N$ denote by $\sum_{i\in N} \varphi_i \frac{\partial}{\partial F_i}$ the vector field from $\mathfrak{X}(V)$ given by

$$(19) \qquad \left(\sum_{i\in N} \varphi_i \frac{\partial}{\partial F_i}\right)(q) := \sum_{i\in N} \varphi_i(q)\frac{\partial}{\partial F_i}\Big|_q \quad \text{for } q \in V.$$

If $X \in \mathfrak{X}(M)$ then

$$X|V = \sum_{i\in N} \varphi_i \frac{\partial}{\partial F_i},$$

where $\varphi_i = X(F_i)$ for $i \in N$.

Proposition 5. Let (M,C) be an R^N-manifold. Then for any $v \in T_pM$ there exists a vector field $X \in \mathfrak{X}(M)$ such that $v = X(p)$.

Proof. Let $F = (F_i)_{i\in N}$ be a chart defined in a neighbourhood V of p, and suppose $v = \sum_{i\in N} \lambda_i \frac{\partial}{\partial F_i}\Big|_p$. Consider the vector field $X_1 = \sum_{i\in N} \lambda_i \frac{\partial}{\partial F_i} \in \mathfrak{X}(V)$. Let $\beta \in C$ be a function which separates p in V [7]. Let $X\colon \longrightarrow TM$ be the vector field defined by

$$X(q) = \begin{cases} \beta(q)X_1(q) & \text{for } q \in V, \\ 0 & \text{for } q \in M-V. \end{cases}$$

It is easy to see that $X \in \mathfrak{X}(M)$ and $X(p) = v$.

Lemma 3 and Proposition 5 yield

Corollary 4. If (M,C) is an R^N-manifold then the homomorphism $h_M\colon A^k(M) \longrightarrow \Omega^k(M)$, $k \in N$, is a monomorphism.

Corollary 5. In the graded algebra $A(M)$ there exists a unique exterior derivative $\eth: A^k(M) \longrightarrow A^{k+1}(M)$, $k = 0,1,2,\ldots$, satisfying the following conditions:

(i) $\eth$ is an R-linear mapping,

(ii) $\eth\alpha = d\alpha$,

(iii) $\eth \circ \eth = 0$,

(iv) $\eth(\omega\wedge\eta) = \eth\omega\wedge\eta + (-1)^{\deg\omega}\,\omega\wedge\eth\eta$ for any $\omega, \eta \in A(M)$.

Proof. The proof is the same as that of Corollary 2, with Corollary 1 replaced by Corollary 4.

The following diagram commutes:

$$\begin{tikzcd}
A^k(M) \arrow[r, "h_M"] \arrow[d, "\eth"'] & \Omega^{k}{}'(M) \arrow[d, "d"] \\
A^{k+1}(M) \arrow[r, "h_M"] & \Omega^{k+1}(M)
\end{tikzcd}$$

Lemma 4 implies

Proposition 6. Let (M,C) be a connected R^N-manifold. Then for any $\alpha \in C$, if $d\alpha = 0$ then $\alpha = \mathrm{const}$.

Proof. If $d\alpha = 0$ then for any chart $F: V \longrightarrow R^N$, $d(\alpha\circ F^{-1}) = (F^{-1})^*(d\alpha) = 0$. Suppose V is connected. Then by Lemma 4 the composition $\alpha\circ F^{-1}$ is constant. Thus α is locally constant. Since M is connected α is constant.

Proposition 7. If (M,C) and (N,D) are R^N-manifolds then the Cartesian product $(M \times N, C \times D)$ is an R^N-manifold.

Proof. One can easily see that the mapping $J: R^N \times R^N \longrightarrow R^N$ given by

(20) $$J(t,s) = (t_1, s_1, t_2, s_2, \ldots) \quad \text{for } t,s \in R^N$$

where $t = (t_i)_{i\in N}$, $s = (s_i)_{i\in N}$, is a diffeomorphism.

If $F: U \longrightarrow R^N$ and $G: V \longrightarrow R^N$ are charts of (M,C) and (N,D) respectively then $J\circ(F \times G): U \times V \longrightarrow R^N$ is a chart of $(M \times N, C \times D)$.

Now, let TM be the disjoint union of all tangent spaces T_pM, $p\in M$, and let $\pi: TM \longrightarrow M$ be the natural projection. Let TC be the differential structure on TM generated by the set $\{\alpha\circ\pi: \alpha\in C\} \cup \{d\alpha: \alpha\in C\}$ [1].

Proposition 8. If (M,C) is an R^N-manifold then (TM,TC) is an R^N-manifold.

Proof. If $F: V \longrightarrow R^N$ is a chart of (M,C) then $\bar{F}: \pi^{-1}(V) \longrightarrow R^N$ given by

(21) $\bar{F}(v) = J(F(\pi(V)),(v(F_i)))$ for $v \in \pi^{-1}(V)$

is a chart of (TM,TC).

Proposition 9. If (M,C) is an n-dimensional C^∞ manifold and (N,D) is an R^N-manifold then the Cartesian product $(M \times N, C \times D)$ is an R^N-manifold.

Proof. Let $F: U \longrightarrow R^n$ and $G: V \longrightarrow R^N$ be charts of (M,C) and (N,D) respectively. It is easy to prove that the map $L: R^n \times R^N \longrightarrow R^N$ given by

(22) $L(p,t) = (p_1,\dots,p_n,t_1,t_2,\dots)$ for $p \in R^n$ and $t \in R^N$

is a diffeomorphism. So the mapping $L\circ(F \times G): U \times V \longrightarrow R^N$ is a chart of $(M \times N, C \times D)$.

Example 1. Of course $(R^N, \mathcal{E}_N)$ is an R^N-manifold. Every open subset of R^N is an R^N-manifold.

Example 2. Consider the set $M = \{x\in R^N: x_1^2+x_2^2+x_3^2 = 1\}$ with the differential structure $(\mathcal{E}_N)_M$ induced from $(R^N,\mathcal{E}_N)$. One can prove that the mapping $\psi : M \longrightarrow S^2 \times R^N$ given by

$$\psi(x) = ((x_1,x_2,x_3),x_4,x_5,x_6,\ldots) \quad \text{for } x = (x_i) \in M \tag{23}$$

is a diffeomorphism. So $(M,(\mathcal{E}_N)_M)$ is an R^N-manifold diffeomorphic to $(S^2 \times R^N, C^\infty(S^2) \times \mathcal{E}_N)$.

REFERENCES

[1] A. Kowalczyk: Tangent differential spaces and smooth forms, Demonstratio Math. 13 (1980), 893-905.

[2] H. Matuszczyk, W. Waliszewski: An immersion of a differential space in a Cartesian space, Ann. Polon. Math. 37(1980), 131-134.

[3] J.W. Milnor, J.D. Stasheff: Characteristic Classes, Princeton University Press, Princeton, New Jersey 1974.

[4] W. Sasin: On some exterior algebra of the differential forms over a differential space, Demonstratio Math. 19 (1986), 1063-1075.

[5] W. Sasin, Z. Żekanowski: On locally finitely generated differential spaces, Demonstratio Math. 20 (1978), 477-487.

[6] R. Sikorski: Abstract covariant derivative, Colloq. Math. 18 (1967) 251-272.

[7] R. Sikorski: Differential modules, Colloq. Math. 24 (1974) 45-72.

[8] R. Sikorski: Introduction to Differential Geometry, Warszawa 1972 (in Polish).

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AN AXIOM SYSTEM DESCRIBING DEGENERATE HYPERBOLIC PLANES IN TERMS OF DIRECTED PARALLELITY

Dedicated to the memory
of Professor Edward Otto

Introduction

In this paper we give an axiom system for the class of degenerate hyperbolic planes and discuss some features of these structures. Usual, classical hyperbolic plane is a structure whose universe consists of points inside some fixed conic in a projective plane (c.f. [1]). In analogy we call a degenerate hyperbolic plane a structure in which point-universe is a interior of degenerate conic and lines are appropriate secants. Such a structure can be visualized as an open affine half-plane or (which is essentially the same) as an affine open stripe with parallel boundary lines. The structure of collineation group of degenerate hyperbolic planes was investigated in [6], [7] and [5]; several sufficient systems of primitive notions were shown in [5].

The language we choose contains as a primitive notion the relation of directed parallelity; in the affine model such a relation means that appropriate half lines meet at the horizon, i.e. on the boundary of the stripe. A general axiom system for directed parallelity was given in [2]; models of this axiom system in this paper are called quasihyperbolic structures. The class of quasihyperbolic planes includes the class of ordered affine planes (c.f.[3]), the class of hyperbolic planes, the class of Strambach planes (c.f.[4]) and the class of degenerate hyperbolic planes. Thus the aim of this paper is to distinguish degenerate hyperbolic planes from among all the quasihyperbolic planes. To this aim we define and investigate symmetries, which enable us to construct a field in our structure and to use this field in description of models of the axiom system we propose. Various configurational axioms are considered, which imply required properties of the family of reflections; in particular we try to formulate these axioms as statements on perspective position of certain figures. This is the technique which was already used in the case of Strambach planes (c.f.[4]).

Other axiom systems describing the class of degenerate hyperbolic planes can be found in [8] (where our structures are described as open convex subsets of pappian affine plane, whose boundary is formed by two lines) and in [7] (where the planes in question are considered as group-planes over suitable Hjelmslev'Groups).

Preliminaries, intended models and basic axioms

Let $\mathbf{F} = \langle F;+,\cdot,0,1,<\rangle$ be an ordered field, let $\mathbf{A}(F) = \langle F^2,B\rangle$ be the ordered affine plane over F, where B is the betweenness relation (c.f.[9]). For a≠b we denote by $\overrightarrow{ab}$ the halfline with origin a passing through b. Let $W = \{\langle x,y\rangle \in F^2 : -1<x<1\}$ and hyperplanes tangent "at infinity" correspond to orbits $G^{+}_{\langle U_1\rangle}[\tilde{U}]$ where U_1 is tangent to Q. This relation can be also interpreted in a more analytical way. For two $u, v \in C^{o}$: u and v conjugate (u-v) iff $\sum_{i=1}^{k} u_i v_i - \sum_{i=k+1}^{n+1} u_i v_i = 1$, i.e. if $\Theta(u,v) = 1$.

Let us calculate $\Theta(u-v, u-v) = \Theta(u,u) + \Theta(v,v) - 2\Theta(u,v) = 2 - 2\Theta(u,v)$. Therefore for $u, v \in C^{o}$: $u\text{-}v \Leftrightarrow \Theta(u-v) = 0$. If u-v, $u \neq \pm v$ then there is a unique plane M containing o, u, v; clearly M is tangent to Q at the horizon. Let H_u, H_v be hyperplanes with normal vectors u, v respectively. Then $\lambda^{-1}(H_u), \lambda^{-1}(H_v)$ are orthogonal to $\lambda^{-1}(M)$ with respect to Q. Therefore $\lambda^{-1}(H_u), \lambda^{-1}(H_v)$ have a common point with Q. In standard classical hyperbolic geometry this means that $\lambda^{-1}(H_u)$ and $\lambda^{-1}(H_v)$ are parallel.

REFERENCES

[1] E. Artin : Geometric algebra . New York 1957.

[2] F. Bachmann : Aufbau der Geometrie aus dem Spiegelungsbegriff, Springer Verlag 1973.

[3] H. Beck : Zur Lieschen Kugelgeometrie im Nichteuklidischen Raum, Jahresbericht d.DMV.

[4] W. Benz : Vorlesungen über Geometrie der Algebren, Springer Verlag 1973.

[5] L. Dubikajtis, H. Guściora : On the geometry

of oriented equiaxial hyperquadrics, Bull. Acad. Polon. Sci. Sér. Math. 17 (1969), 29-35.

[6] E. Kusak : Linear spaces with generalized orthogonality, [submitted to] Bull. Acad. Polon. Sci. Sér. Math.

[7] K. Prażmowski : A note on cycle in elliptic geometry, [submitted to] J. of Geometry.

[8] Rozenfeld : Noneuclidean Geometries, Moscow 1956, (in Russian).

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Wiesław Sasin

ON LOCALLY COUNTABLY GENERATED DIFFERENTIAL SPACES

Dedicated to the memory
of Professor Edward Otto

In this paper we give a characterization of Sikorski's differential spaces [6] which are locally countably generated by a family of real-valued functions. In section 1 we study the properties of the differential space $(R^N, \mathcal{E}_N)$ of all real sequences with the natural differential structure $\mathcal{E}_N$ generated by projection [2]. In Section 2 we describe some properties of differential spaces which are locally diffeomorphic to open subsets of R^N.

1. R^N as a differential space

Let R^N be the set of all real sequences. Every element $x \in R^N$ is a map $x: N \longrightarrow R$ from the set of natural numbers into the set of real numbers. We will denote it by $x = (x_i)_{i \in N}$.

Denote by π_i for $i \in N$ the projection of R^N onto the i-th coordinate given by

(1)
$$\pi_i(x) = x_i \quad \text{for } x = (x_i) \in R^N.$$

Let $P = \{\pi_i : i \in N\}$ be the set of all such projections. By τ_P we denote the weakest topology in R^N in which all projections from P are continuous, i.e. the product topology.

Let $\mathcal{E}_N$ be the differential structure on R^N generated by the set P. By definition $\alpha\in\mathcal{E}_N$ iff for any $x\in R^N$ there exists an open neighbourhood $V \in \tau_P$ of x, natural numbers $i_1,\dots,i_k$ and a C^∞ function $\omega : R^k \longrightarrow R$ such that

$$(2) \qquad \alpha(x) = \omega(x_{i_1},\dots,x_{i_k}) \quad \text{for } x \in V.$$

Thus $\alpha\in\mathcal{E}_N$ iff locally it is a C^∞ function of a finite number of variables.

For any $\alpha \in \mathcal{E}_N$ and $i \in N$ we set

$$(3) \qquad \partial_{ix}(\alpha) := \omega'_{|i}(x_{i_1},\dots,x_{i_k})\,,$$

where ω comes from (2); here $\omega'_{|i} = \frac{\partial\omega}{\partial x_i}$ is understood to be zero if i is not one of the i_j. Clearly, the definition is independent of the choice of ω and $\partial_{ix}: \mathcal{E}_N \longrightarrow R$ is tangent vector at $x \in R^N$ to the differential space $(R^N,\mathcal{E}_N)$. The vectors ∂_{ix}, $i \in N$ are linearly independent since obviously

$$(4) \qquad \partial_{ix}(\pi_j) = \delta_{ij} \quad \text{for } i,j \in N.$$

For any $\alpha \in \mathcal{E}_N$ the set $I_{\alpha,x} := \{i\in N\ \partial_{ix}(\alpha) \neq 0\}$ is finite. For every $x\in R^N$ and every $r = (r_i)_{i\in N}\in R^N$, the vector $w \in T_x(R^N)$ given by

$$(5) \qquad w(\alpha) = \sum_{i\in I_{\alpha,x}} r_i\partial_{ix}(\alpha) \quad \text{for } \alpha \in \mathcal{E}_N$$

clearly satisfies $w(\pi_j) = r_i$ for $i \in N$. It is easy to see that w given by (5) is the unique vector with this property. In the sequel, it will be written as a formal sum $w = \sum_{i\in N} r_i\partial_{ix}$.

Thus we have proved

Lemma 1. The mapping $K\colon T_x(R^N) \longrightarrow R^N$ given by

$$K(v) = (v(\pi_1), v(\pi_2), \ldots) \quad \text{for } v \in T_x(R^N) \tag{6}$$

is an isomorphism of vector spaces.

Now, let $\mathfrak{X}(R^N)$ be the $\mathcal{E}_N$-module of all smooth vector fields tangent to $(R^N, \mathcal{E}_N)$. It is easy to see that the mappings $\partial_i\colon R^N \longrightarrow TR^N$, $i \in N$, given by

$$\partial_i(x) = \partial_{ix} \quad \text{for } x \in R^N, \tag{7}$$

belong to $\mathfrak{X}(R^N)$ and are $\mathcal{E}_N$ - linearly independent.

For every sequence $(\varphi_i)_{i \in N}$ in $\mathcal{E}_N$, denote by $\sum_{i \in N} \varphi_i \partial_i$ the vector field W tangent to $(R^N, \mathcal{E}_N)$ given by

$$W(x) := \sum_{i \in N} \varphi_i(x) \partial_{ix} \quad \text{for } x \in R^N. \tag{8}$$

It is easy to see that $W(\pi_i) = \varphi_i$ for $i \in N$. Thus $W = \sum_{i \in N} \varphi_i \partial_i \in \mathfrak{X}(R^N)$. So we may uniquely represent an arbitrary vector field $X \in \mathfrak{X}(R^N)$ as a formal sum

$$X = \sum_{i \in N} X(\pi_i) \partial_i .$$

We have proved

Lemma 2. The mapping $L\colon \mathfrak{X}(R^N) \longrightarrow \mathop{\times}_{i \in N} \mathcal{E}_N$ given by

$$L(X) = (X(\pi_1), X(\pi_2), \ldots) \quad \text{for } X \in \mathfrak{X}(R^N) \tag{9}$$

is an isomorphism of $\mathcal{E}_N$-modules.

Lemma 3. For any vector $v \in T_x(R^N)$ there exists a smooth vector field $X \in \mathfrak{X}(R^N)$ such that $v = X(x)$.

Proof. If $v = \sum_{i \in N} \lambda_i \partial_{ix}$, we put $X = \sum_{i \in N} \lambda_i \partial_i$.

Now, let (M,C) be a differential space. Let $h_M\colon A^k(M) \longrightarrow \Omega^k(M)$, $k\in N$, be the homomorphism of the C-module of smooth pointwise k-forms on M [1] into the C-module of global k-forms [8] given by

$$(10)\qquad h_M(\omega)(X_1),\dots,X_k)(p) = \omega(X_1(p),\dots,X_k(p))$$

for any $\omega \in A^k(M)$, $X_1,\dots,X_k \in \mathfrak{X}(M)$ and $p \in M$.

In general h_M is not an isomorphism [4]. Now we prove

Proposition 1. If for every $p\in M$ and any $v\in T_pM$ there exists a vector field $X\in\mathfrak{X}(M)$ such that $v = X(p)$ then h_M is a monomorphism.

Proof. Let $\omega\in A^k(M)$ be such that $h_M(\omega) = 0$. Then for any $v_1,\dots,v_k\in T_pM$ there exist vector fields $X_1,\dots,X_k$ such that $X_1(p) = v_1,\dots,x_k(p) = v_k$, and so $\omega(v_1,\dots,v_k) = (h_M\omega)(X_1,\dots$ $\dots,X_k)(p) = 0$. Therefore $\omega = 0$.

Corollary 1. For the differential space $(R^N,\mathcal{E}_N)$ the homomorphism h_{R^N} is a monomorphism.

Corollary 2. In the graded algebra $A(R^N)$ there exists a unique operator $\mathfrak{d}\colon A^k(R^N) \longrightarrow A^{k+1}(R^N)$, $k = 0,1,2,\dots$ with the well-known properties of exterior derivative.

Proof. In [4] we introduced, for any differential space (M,C), the homogeneous ideal $\mathfrak{M}(M)$ in the graded algebra $A(M)$, consisting of all pointwise differential forms which are formal differential of 0, and we proved the inclusion $\mathfrak{M}(M)\subset\ker h_M$. In our case, by Corollary 1, $\ker h_{R^N} = \{0\}$. Now the corollary follows from Theorem 3.1 in [4].

For an arbitrary $\omega\in A^k(R^N)$ and for each $p\in R^N$ there exists an open neighbourhood $V \in \tau_p$ such that

$$\omega|V = \sum_I \alpha^{i_1\ldots i_k}\, d\pi_{i_1}\wedge\ldots\wedge d\pi_{i_k}|V,$$

where $\alpha^{i_1\ldots i_k}\in(\mathcal{E}_N)_V$, $(i_1,\ldots,i_k)\in I\subset N^k$ and I is a finite subset of indices [4]

$$\partial\omega = \sum_I d\alpha^{i_1\ldots i_k}\wedge(d\pi_{i_1}\wedge\ldots\wedge d\pi_{i_k})|V.$$

In the algebra, $\Omega(M)$ we have the exterior derivative $d:\Omega^k(M)\longrightarrow\Omega^{k+1}(M)$, $k = 0,1,\ldots,$ given by the well-known formula [6]. The following diagram commutes:

$$\begin{tikzcd}
A^k(R^N) \arrow[r, "h_{R^N}"] \arrow[d, "\partial"'] & \Omega^k(R^N) \arrow[d, "d"] \\
A^{k+1}(R^N) \arrow[r, "h_{R^N}"] & \Omega^{k+1}(R^N)
\end{tikzcd}$$

Lemma 4. Let $\alpha\in\mathcal{E}_N$ be a smooth function on R^N. If $\partial\alpha = 0$ then $\alpha = \text{const}$.

Proof. Since R^N is connected, it is enough to show that α is locally constant. For ever $x\in R^N$ there exists an open connected neighbourhood V of x and a smooth function ω on R^k such that

$$\alpha|V = \omega\circ(\pi_{i_1},\ldots,\pi_{i_k})|V.$$

Hence $d\alpha|V = \left(\sum_{j=1}^{k}\omega'_{|j}\circ(\pi_{i_1},\ldots,\pi_{i_k})\,d\pi_{i_k}\right)|V = 0.$

Since $d\pi_{i_1}|V,\ldots,d\pi_{i_k}|V$ are linearly independent,

$$\omega'_{|j}\circ(\pi_{i_1},\ldots,\pi_{i_k})|V = 0 \quad \text{for } j = 1,\ldots,k,$$

or equivalently

$$\omega'_{|j}|(\pi_{i_1},\dots,\pi_{i_k})(V) = 0 \quad \text{for } j = 1,\dots,k.$$

The set $(\pi_{i_1},\dots,\pi_{i_k})(V) \subset R^k$ is open and connected. Thus $\omega|(\pi_{i_1},\dots,\pi_{i_k})(V)$ is constant, and so α is locally constant, which completes the proof.

Now, we will introduce a topology in the $\mathscr{E}_N$-module $\mathfrak{X}(R^N)$ which leads to another interpretation of formal sums $\sum_{i\in N} \varphi_i \partial_i$ with $\varphi_i \in \mathscr{E}_N$. Recall that $d\pi_i \in \Omega^1(R^N)$, $i\in N$, is the global 1-form given by

(11) $$(d\pi_i)(X) = X(\pi_i) \quad \text{for } X \in \mathfrak{X}(R^N).$$

For any $\omega \in \Omega^1(R^N)$ and $x \in R^N$ let $\omega_x : \mathfrak{X}(R^N) \longrightarrow R$ be given by

(12) $$\omega_x(X) = \omega(X)(x) \quad \text{for } X \in \mathfrak{X}(R^N).$$

Set $\mathscr{F} := \{(d\pi_i)_x : i\in N,\ x\in R^N\}$. Then $\mathscr{F}$ is a set of real-valued functions on $\mathfrak{X}(R^N)$. Let $\tau_{\mathscr{F}}$ be the weakest topology on $\mathfrak{X}(R^N)$ in which all functions in $\mathscr{F}$ are continuous. It is easy to see that $\tau_{\mathscr{F}}$ is a Hausdorff topology.

Now we may prove

Proposition 2. For any sequence $(\varphi_i)_{i\in N}$ in $\mathscr{E}_N$,

$$\sum_{i\in N} \varphi_i \partial_i = \lim_{n\to\infty} \sum_{i=1}^{n} \varphi_i \partial_i,$$

where the limit is taken in $(\mathfrak{X}(R^N), \tau_{\mathscr{F}})$.

Proof. Put $X = \sum_{i\in N} \varphi_i \partial_i$ and $S_n = \sum_{i=1}^{n} \varphi_i \partial_i$ for $n \in N$.

Let $V \in \tau_{\mathscr{F}}$ be an arbitrary neighbourhood of X. There exist sets $\{i_1,\dots,i_k\} \subset N$, $\{p_1,\dots,p_k\} \subset R^N$ and an open interval $Q \subset R^N$

$\partial W = \{-1,1\}\times F$. For points a,b,c,d $\in$ W we define

$ab\Uparrow cd :\Leftrightarrow a=b \vee c=d \vee (a\neq b \wedge c\neq d \wedge [(a_1=b_1 \wedge c_1=d_1 \wedge \mathrm{sgn}(a_2-b_2) = \mathrm{sgn}(c_2-d_2)) \vee \overrightarrow{ab} \cap \overrightarrow{cd} \cap \partial W \neq \emptyset])$.

The relation $\Uparrow$ will be called the directed parallelity. Note, that $B|W$ is definable in terms of $\Uparrow$:

$$B(a,b,c) \Leftrightarrow ab\Uparrow bc, \text{ for } a,b,c \in W.$$

Let L be the collinearity relation (defined by $\Uparrow$ or B) and let $\mathcal{L}$ be the class of lines in A(F). We set

$\mathcal{L}_W = \{k\cap W : k \in \mathcal{L}\}\setminus\{\emptyset\}$ and

$\Omega(F) = \langle W;\ \Uparrow,\ \mathcal{L}_W\rangle$.

The structure $\Omega(F)$ will be called the degenerate hyperbolic plane over F. Clearly $\mathcal{L}\subseteq\mathcal{P}(W)$ is definable in $\langle W;\Uparrow\rangle$, however it is more convenient to consider lines as a primitive notion. Moreover $\Uparrow$ is definable in $\langle W;\mathcal{L}_W\rangle$, like in the case of classical Klein'Model, but the defining formulas are long and complicated.

Let DH_2 be the class of all degenerate hyperbolic planes coordinatized by ordered fields. We are going to characterize DH_2 as a subclass of the class of quasihyperbolic planes. A quasihyperbolic plane is a structure $M = \langle P;\ \Uparrow,\ \mathcal{L}\rangle$, where $\mathcal{L}\subseteq\mathcal{P}(P)$ is the set of lines and $\Uparrow\subseteq(P^2)^2$, with the following properties.

(qH1) The relation B defined by the condition $B(a,b,c) \Leftrightarrow ab\Uparrow bc$ is a betweenness relation on P(c.f.[9]), elements of $\mathcal{L}$ are equivalence classes of the collinearity relation defined by B, and the Pasch Axiom holds for B (c.f.[10]).

(qH2) $\Uparrow$ is an equivalence relation in the set of non-degenerate segments; $\Uparrow$ is definable from B by the standard hyperbolic formula for the parallelity (of halflines); through every point there passes a halfline parallel to a given one. (c.f.[4],[8]).

(qH3) For every two intersecting lines k,m and every point a there exist a projection of a onto k in at least one of the two directions of m.

A precise elementary axiom system for the class of quasihyperbolic planes is to be found in [2], it is a slight modification of the system given in [3]. The class of all quasihyperbolic planes will be denoted by qH.

Let us fix $M \in qH$. For every two distinct points $a,b \in P$ there is the unique line $\overline{ab}$ joining a and b; as previously we denote by $\overrightarrow{ab}$ the halfline with origin a passing through b.

Let $V = V(M) = \{ \langle a,b\rangle \in P^2 : a \neq b\}/_{\Uparrow}$, the set V(M) will be called the horizon of M and its elements will be called ends or directions, or vanishing points. Usually points will be denoted by the letters a,b,c,... and ends by p,q,r,.... We say that a line k is incident with $q \in V$ and we write qIk iff there are a,b with $k = \overline{ab}$ and $\langle a,b\rangle \in q$. Clearly every line is incident with exactly two ends. Finally $\overline{aq}$ is the line passing through a and incident with q. In the set V(M) we define two relations, - and E, by the following formulas:

$$p-q :\Leftrightarrow (\exists k \in \mathcal{L})[p,q I k] \wedge p \neq q,$$

$$q \in E :\Leftrightarrow (\forall a,b,a',b')[\langle a,b\rangle \in q \wedge \langle a',b'\rangle \in q \Rightarrow ba \Uparrow b'a'].$$

If $q \in E$, then we call q a Euclidean direction. Notice, that if q is Euclidean, then there is the unique direction $\tilde{q}$ such that

q-$\tilde{q}$; clearly $\tilde{q} \in E$.

The relation - is the joinability relation. Note that in Strambach and in hyperbolic planes $p-q \Leftrightarrow p \neq q$, in affine planes $p-q \Leftrightarrow p = \tilde{q}$. In the case of dH_2 we assume the following two additional axioms concerning joinability:

(dH1) $(\forall q_1, q_2, q_3 \in V)\ [q_1, q_2, q_3 \notin E \wedge \sim q_1 - q_2 \wedge q_2 - q_3 \Rightarrow \sim q_1 - q_3]$,

(dH2) $|E| = 2$.

The only two Euclidean directions we denote by ω and $\tilde{\omega}$. A line k will be called isotropic if $\omega I k$, the class of isotropic lines will be denoted by $\mathcal{J}$.

In the presence of (dH1) the relation of nonjoinability is an equivalence relation in $V \setminus E$, with exactly two equivalence classes; we denote them by V^+ and V^-. Note that if $p, q \in V \setminus E$, $p, q I k$ and $p \neq q$, then the line k is determined uniquely and we write $k = \overline{pq}$.

By a collineation of M we mean any automorphism of M. Let us remind from [4] some basic properties of collineations true in every quasihyperbolic plane satisfying (dH1) and (dH2). Every collineation α can be extended to a transformation $\tilde{\alpha}: V \longrightarrow V$; the function $\alpha \longmapsto \tilde{\alpha}$ is a bijection. (In the sequel for short we shall frequently write $\alpha(q)$ instead of $\tilde{\alpha}(q)$). Every axial collineation is uniquely determined by its axis and an image of one point not belonging to the axis; no matter proper or (non-Euclidean) vanishing. If the axial collineation α has isotropic axis k and does not leave invariant V^+, then α is an involution and every invariant line distinct from k determines ω. Proofs of these facts are essentially the same as in [4] and they make use of the relation of separation defined by the condition $p|k|q \Leftrightarrow \overline{pq}\cap k \neq \emptyset$. In the sequel we shall not prove facts which follow easily from the properties (gH1), (gH3), (gH3), (dH1) and (dH2). In all the quasihyperbolic planes we shall consider below, the properties (dH1) and (dH2) are assumed.

Configurational axioms and symmetries in degenerate hyperbolic planes

Let us consider the following several configurational axioms, which are obviously true in every degenerate hyperbolic plane:

PB(k): $(\forall q_1, q_2, q_3, q_4, q_5, q_6 \in V)\ (\forall c_1, c_2, c_3 \in P)\ [c_1 = \overline{q_1q_2} \cap \overline{q_4q_5} \wedge c_2 = \overline{q_2q_3} \cap \overline{q_5q_6} \wedge c_3 = \overline{q_3q_4} \cap \overline{q_6q_1} \wedge c_1, c_2 \in k \longrightarrow c_3 \in k\ ]$.

Notice, that PB(k) describes a Pascal-Brianchon Axiom with axis k; on an affine stripe PB(k) is a usual Pappus-Pascal Axiom with axis k (cf. Fig.1)

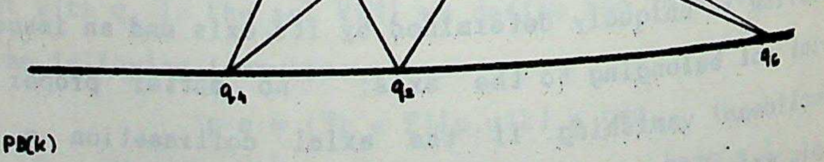


Fig. 1

PB(J): $(\forall k \in \mathcal{J})$PB(k);

PB : $(\forall k \in \mathcal{L})$PB(k);

DHS : $(\forall a_1, a_2, a_3, a_4, b_1, b_2, b_3, b_4)[a_1a_2 \Uparrow b_1b_2 \wedge a_3a_4 \Uparrow b_3b_4 \wedge$
$\wedge \overline{a_1a_2} \in J \wedge a_1a_2 \Uparrow a_3a_4 \wedge a_1a_3 \Uparrow a_2a_4 \Uparrow b_1b_3 \Uparrow b_2b_4 \wedge a_2a_3 \Uparrow b_2b_3 \wedge$
$\wedge \sim L(a_1, a_2, a_3) \wedge \sim L(b_1, b_2, b_3) \Rightarrow a_1a_4 \Uparrow b_1b_4]$

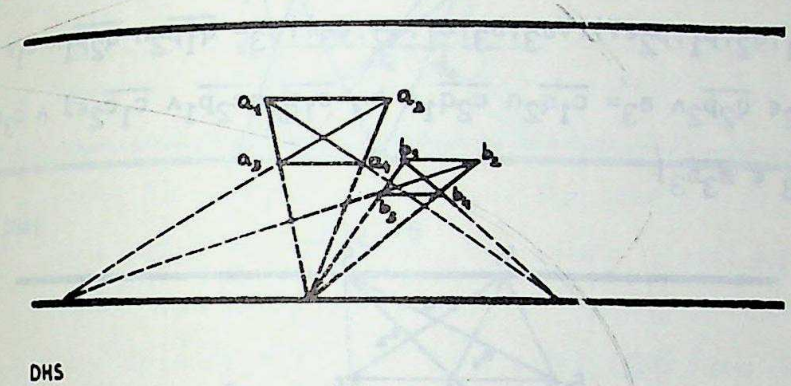


Fig. 2

The formula DHS states that if two parallel parallelograms have one pair of sides isotropic and one pair of corresponding diagonals parallel, then the second pair of diagonals is also parallel.

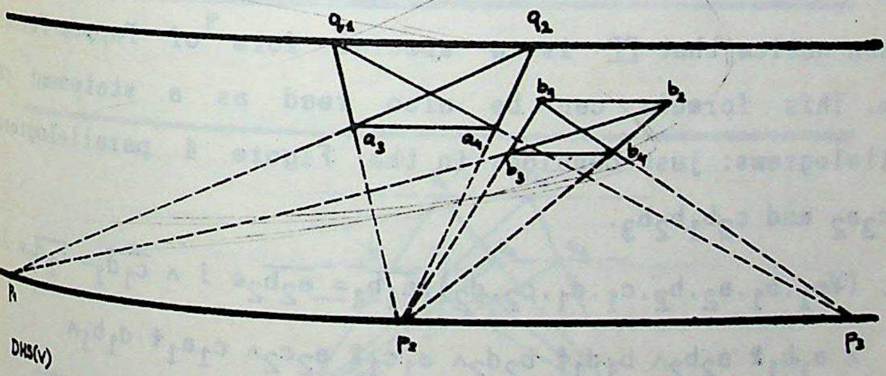


Fig. 3

DHS(V): $(\forall q_1, q_2 \in V)(\forall a_3, a_4, b_1, b_2, b_3, b_4)(\forall p_1p_2p_3 \in V)[\overline{a_3a_4} \in J \wedge$
$\wedge b_1b_2 \Uparrow b_3b_4 \Uparrow a_3a_4 \wedge a_4 \in \overline{q_2p_2} \wedge a_3 \in \overline{q_1p_2} \wedge \langle b_1b_3 \rangle,$

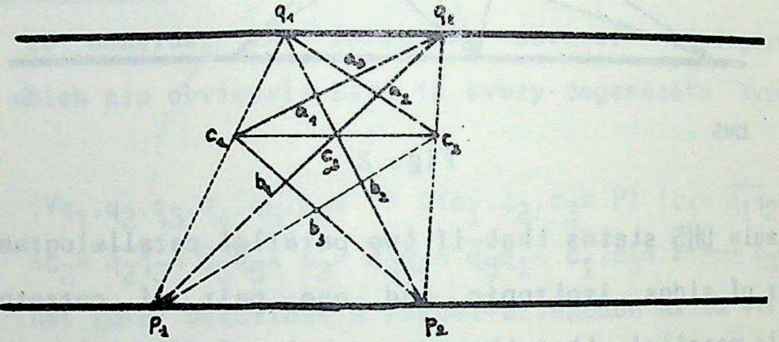
$$\langle b_2,b_4\rangle \in p_2 \wedge a_3 \in \overline{q_2p_1} \wedge \langle b_2b_3\rangle \in p_1 \Rightarrow (a_4 \in \overline{q_1p_3} \Leftrightarrow \langle b_1b_4\rangle \in p_3)$$

In fact DHS(V) can be considered as a special case of DHS, in which one isotropic side of one of parallelograms lies on the horizon.

PP: $(\forall q_1,q_2,p_1,p_2 \in V)(\forall a_3,b_3,c_1,c_2,c_3)[c_3 = \overline{q_1p_2} \cap \overline{q_2p_1} \wedge c_1 \in \overline{q_1p_1} \wedge$

$\wedge\ c_2 \in \overline{q_2p_2} \wedge a_3 = \overline{c_1q_2} \cap \overline{c_2q_1} \wedge b_3 = \overline{c_1p_2} \cap \overline{c_2p_1} \wedge \overline{c_1c_2} \in J \wedge c_3 \in \overline{c_1c_2} \Rightarrow$

$\Rightarrow c_3 \in \overline{a_3b_3}]$



PP

Fig. 4

One can notice that PP is a special form of Pappus-Pascal Axiom. This formula can be also read as a statement on parallelograms: just consider in the Figure 4 parallelograms $a_1a_3c_3a_2$ and $c_3b_1b_2b_3$.

DHR1: $(\forall a_1,b_1,a_2,b_2,c_1,d_1,c_2,d_2)[\overline{a_1b_1} = \overline{a_2b_2} \in J \wedge \overline{c_1d_1} = \overline{c_2d_2} \in J \wedge$

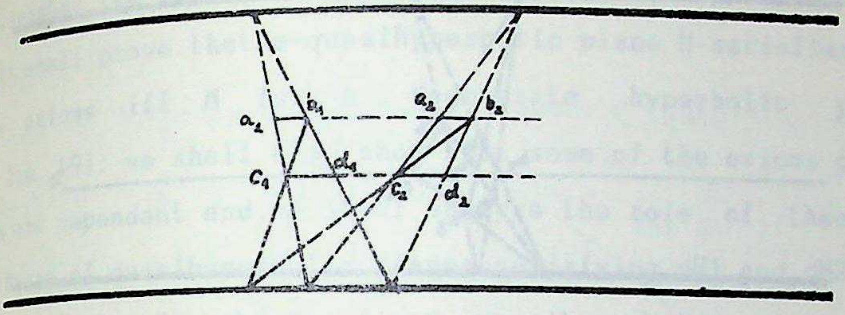
$\wedge\ a_1b_1 \Uparrow a_2b_2 \wedge b_1d_1 \Uparrow b_2d_2 \wedge a_1c_1 \Uparrow a_2c_2 \wedge c_1a_1 \Uparrow d_1b_1 \wedge$

$\wedge\ c_2a_2 \Uparrow d_2b_2 \Rightarrow b_1c_1 \Uparrow b_2c_2]$

DHR2: $(\forall a_1a_2a_3a_4b_1b_2b_3b_4)[a_1a_2 \Uparrow a_3a_4 \wedge a_1a_3 \Uparrow a_2a_4 \wedge b_1b_2 \Uparrow b_3b_4 \wedge$

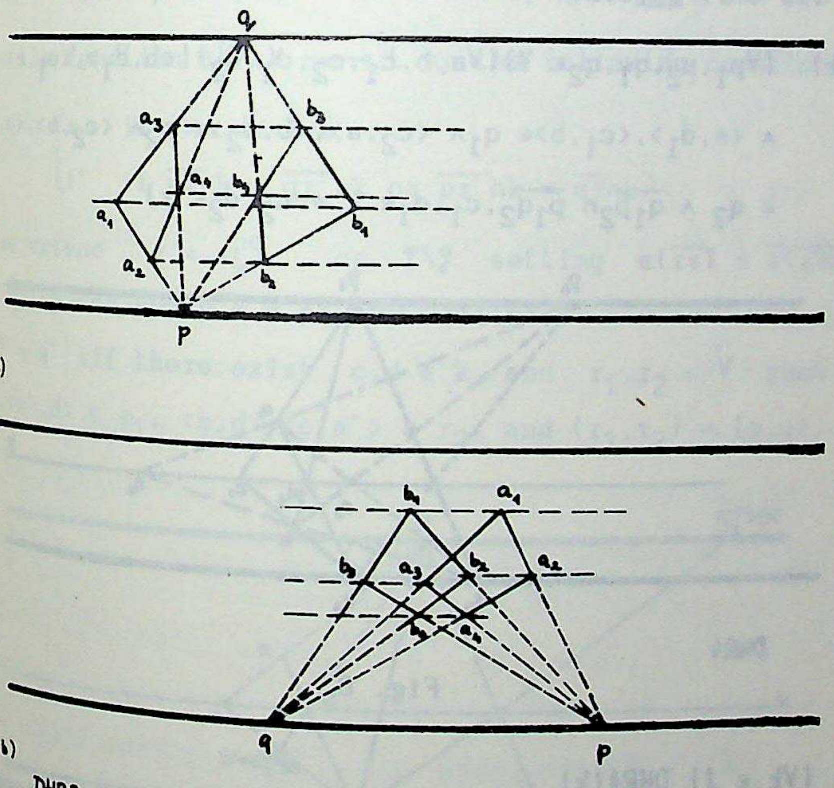
$\wedge\ b_1b_3 \Uparrow b_2b_4 \wedge \sim L(a_1,a_2,a_3) \wedge \sim L(b_1,b_2,b_3) \wedge a_1a_2 \Uparrow b_1b_2 \wedge$

$\wedge\, a_1a_3\uparrow\ b_1b_3\wedge\ \overline{a_3b_3}, \overline{a_2b_2}. \overline{a_4b_4} \in J \Rightarrow \overline{a_1b_1} \in J].$


a1
b1
a2
b2
c1
d1
c2
d2

DHR1

Fig. 5

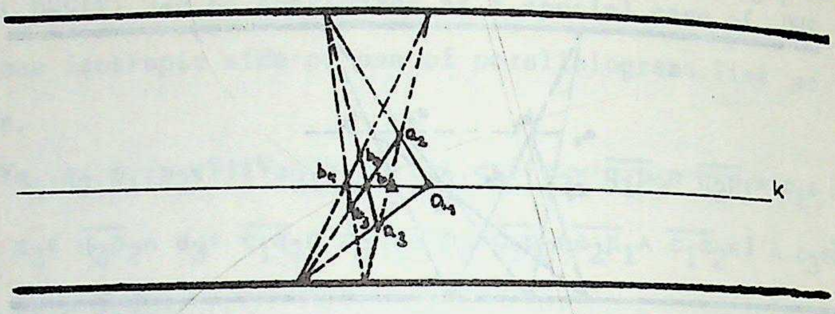

q
a3
b3
a4
b4
a1
b1
a2
b2
p
a)
b1
a1
b3
a3
b2
a2
b4
a4
q
p
b)

DHR2

Fig. 6

DHR3(k): $(\forall a_1a_2a_3b_1b_2b_3b_4)[a_1a_2\uparrow\ a_3b_1\uparrow\ b_1b_2\uparrow\ b_3b_4\wedge\ a_1a_3\uparrow\ a_2b_1\uparrow$

$$\Uparrow b_1b_3 \Uparrow b_2b_4 \wedge a_1, b_1, b_4 \in k \wedge b_4b_2 \Uparrow a_3a_2 \Rightarrow b_4b_3 \Uparrow a_2a_3]$$



DHR3

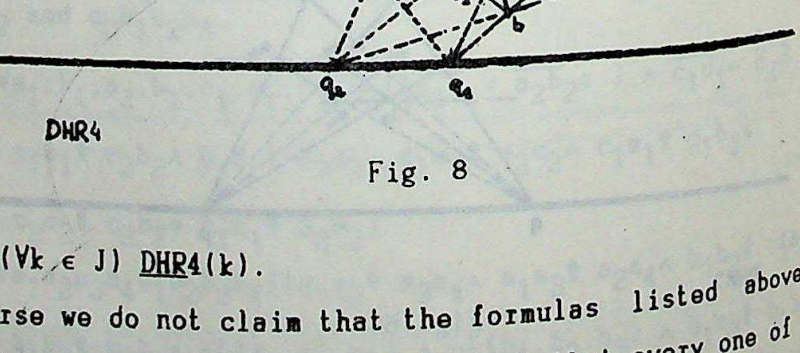
Fig. 7

DHR3: $(\forall k \in J)$ **DHR3**(k).

DHR4(k): $(\forall p_1, p_2, q_1, q_2 \in V)(\forall a, b, c_1, c_2, d_1, d_2)[\langle b, d_1\rangle, \langle c_1, a\rangle \in p_1 \wedge$

$\wedge \langle a, d_1\rangle, \langle c_1, b\rangle \in q_1 \wedge \langle c_2, a\rangle, \langle b, d_2\rangle \in p_2 \wedge \langle c_2, b\rangle, \langle a, d_2\rangle \in$

$\in q_2 \wedge \overline{q_1p_2} \cap \overline{p_1q_2}, c_1, d_1 \in k \Rightarrow c_2, d_2 \in k]$



DHR4

Fig. 8

DHR4: $(\forall k \in J)$ **DHR4**(k).

Of course we do not claim that the formulas listed above are logically simple, nevertheless we think that every one of them has a clear geometrical sense and it is relatively easy to draw and remember the pictures associated with these formulas.

We shall prove that a quasihyperbolic plane M satisfies the above axioms iff M is a degenerate hyperbolic plane (cf. Thm. 19); we shall also show that some of the axioms given above are dependent and we shall analyze the role of them in the theory of quasihyperbolic planes satisfying dH1 and dH2.

Let $M = \langle P;\Uparrow,\mathcal{L},\rangle$ be a fixed quasihyperbolic plane. Let $V = V(M)$, let ω and $\tilde{\omega}$ be the two Euclidean directions in M, $V = V^{-} \cup V^{+} \cup \{\omega,\tilde{\omega}\}$, and let $\mathcal{J}$ be the set of isotropic lines of M.

For every triple (p,q,k) such that $p,q \in V$, $k \in \mathcal{J}$, and $p|k|q$ we define the function $\sigma_k^{pq}: V \longrightarrow L$ by the following conditions

$$\sigma_k^{pq}(r) = \begin{cases} r & \text{if } r=\omega \text{ or } r=\tilde{\omega} \\ r' & \text{if } \overline{pr}\cap k = \overline{qr'}\cap k \text{ or } \overline{pr'}\cap k = \overline{qr}\cap k. \end{cases}$$

Next we extend $\alpha = \sigma_k^{pq}$ on $\mathcal{L}\setminus\mathcal{J}$ setting $\alpha(\overline{rs}) = \overline{\alpha(r)\alpha(s)}$.

Finally for every $a \notin pq$ we define

$\sigma_k^{pq}(a) = a'$ iff there exist $c,d \in k$ and $r_1, r_2 \in V$ such that $\langle c,a\rangle, \langle a',d\rangle \in r_1$, $\langle a,d\rangle, \langle c,a'\rangle \in r_2$, and $\{r_1,r_2\} = \{p,q\}$.

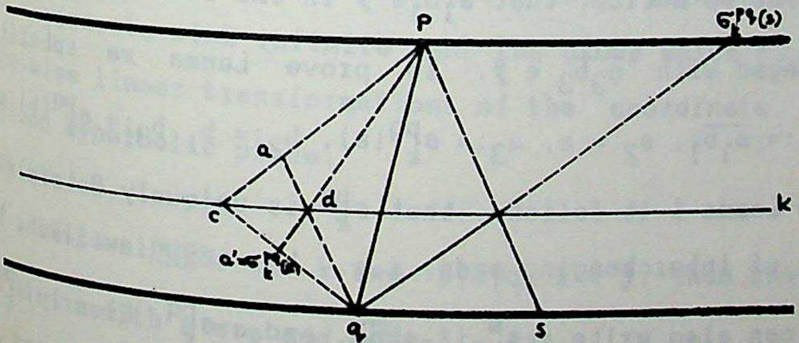


Fig. 9

From the properties of quasihyperbolic planes with Euclidean directions it follows that the above formula defines a function. The functions $\sigma_k^{pq}: V \longrightarrow V$ and $\sigma_k^{pq}: \mathcal{L} \longrightarrow \mathcal{L}$ can be defined also in hyperbolic planes. However the definition of $\sigma_k^{pq}: P \longrightarrow P$ must be given another (cf. [4])

$$\sigma_k^{pq}(a) = \begin{cases} a & \text{if } a \in k \\ a' & \text{if } a \notin k, aIm,n, m \neq n, a'I\sigma_k^{pq}(m), \sigma_k^{pq}(n), pIm,k, qIn,k \end{cases}$$

Lemma 1.

(i) If M satisfies PB(k) and $\sigma_k^{pq}(r) = s \neq r$, then $\sigma_k^{pq}|_V = \sigma_k^{rs}|_V$.

(ii) If M satisfies DHR4(k) and $\sigma_k^{pq}(r) = s \neq r$, then $\sigma_k^{pq}(a) = \sigma_k^{rs}(a)$ for every $a \notin \overline{pq} \cup \overline{rs}$.

(iii) If M satisfies DHR2, $p|k|q$, $k \in \mathcal{J}$, and $a,b \notin \overline{pq}$, then $\overline{ab} \in J$ implies $\overline{\sigma_k^{pq}(a)\sigma_k^{pq}(b)} \in J$.

Proof. (i) Follows immediately from the definition. Let q-t, r; for arbitrary $t \neq p,q,r,s$ we consider $c_1 = \overline{ps} \cap \overline{qr}$, $c_2 = \overline{qt} \cap \overline{p\sigma_k^{pq}(t)}$ and $c_3 = \overline{r\sigma_k^{pq}(t)} \cap \overline{ts}$ and we apply PB(k).

(ii) Analogously is an immediate consequence of DHR4(k).

(iii) First we notice, that $\overline{a_1b_1} \in \mathcal{J}$ in the formula DHR2 can be interchanged with $\overline{a_3b_3} \in \mathcal{J}$. To prove Lemma we substitute: $k := \overline{a_4b_4} := \overline{a_1b_1}$, $a_2 = a$, $a_3 := \sigma_k^{pq}(a)$, $b_2 := b$, $b_3 := \sigma_k^{pq}(b)$. ■

From Lemma 1 it follows, that σ_k^{pq} is uniquely determined by any pair of interchanging ends, i.e. by any invariant line; thus we can also write $\alpha = \sigma_k^m$ if $m = \overline{pq}$, and $\alpha = \sigma_k^{pq}$. Clearly σ_k^{pq} is an involution and $\sigma_k^{pq}(V^+) = V^-$. Moreover $\sigma_k^{pq}(a)$ can be defined for all $a \in P$.

C o r o l l a r y 2. If M satisfies DHR4(k), then σ_k^{pq} is a collineation of M.

P r o o f. With the help of Lemma 1(ii) we prove that $a \in m$ implies $\sigma_k^{pq}(a) \in \sigma_k^{pq}(m)$ for arbitrary line $m \in L \setminus J$. Note that for arbitrary points a, b if $a \neq b$, then $\neg(\exists m \in \mathcal{L} \setminus \mathcal{J})[a, b \in m] \leftrightarrow \overline{ab} \in \mathcal{J}$, and $\overline{ab}, \overline{ac} \in \mathcal{J}$ implies $\overline{bc} \in \mathcal{J}$. Therefore σ_k^{pq} maps isotropic lines onto isotropic lines (notice that the same can be proved with the help of Lemma 1(iii), if M satisfies DHR2). This yields that σ_k^{qp} preserves lines, thus it also preserves separation and betweenness and finally σ_k^{pq} preserves directed parallelity. Thus σ_k^{pq} is a collineation of M.■

The function σ_k^{pq} will be called a symmetry with axis k (clearly $\mathrm{Fix}(\sigma_k^{pq}) = k$) and direction pq. It may be worthwhile to notice that in hyperbolic planes not every σ_k^{pq} is a collineation; σ_k^{m} is a collineation iff m and k are orthogonal. Moreover in hyperbolic planes all the transformations $\sigma_k^{m}: V \longrightarrow V$ generate with infinite rank the group which contains all piece-wise linear transformations of the coordinate field of the given hyperbolic plane.

L e m m a 3.

(i) If M satisfies DHR3, $a \notin k$ and $p|k|q$, $k \in \mathcal{J}$, then there exists a line m with $a \in m$ and $\sigma_k^{pq}(m) = m$.

(ii) If σ_k^{pq} is a collineation and $a \notin k$, then there exists a line m with $a \in m$ and $\sigma_k^{pq}(m) = m$.

(iii) If σ_k^{pq} is a collineation, $a \in k$ and M satisfies PP, then there exists a line m such that $a \in m$ and $\sigma_k^{pq}(m) = m$.

P r o o f. (i) is an immediate consequence of DHR3; we put $m = \overline{a\sigma_k^{pq}(a)}$ and use the definition of $\sigma_k^{pq}(a)$.

(ii) is obvious.

(iii) Let us consider arbitrary two lines $m_1 = \overline{p_1q_1}$ and $m_2 = \overline{p_2q_2}$ with $a \in \overline{q_1p_1}$, which are invariant under $\alpha = \sigma_k^{pq}$; let $c_1 = m_1 \cap k$ and $c_2 = m_2 \cap k$. By PP we infer that the line $m = \overline{(\overline{q_1c_2} \cap \overline{c_1q_2})(\overline{p_1c_2} \cap \overline{p_2c_1})}$ contains a and is left invariant by α. ■

Notice, that the implications given in Lemma 1(i),(ii) and Lemma 3 (i) can be converted, thus the corresponding configurational axioms are strict geometric counterparts of suitable properties of symmetries. In particular from this follows that DHR4 implies PB and DHR3.

L e m m a 4. DHS(V) implies DHS.

P r o o f. Consider points a_i, b_i as required in DHS. Let $\langle a_3, a_1 \rangle \in q_1$, $\langle a_4, a_2 \rangle \in q_2$, $\langle a_2, a_3 \rangle \in p_1$, and $\langle a_1, a_4 \rangle \in p_3$, let $a_3' = \overline{q_1a_1} \cap \overline{q_2p_1}$ and let $a_4' = \overline{a_3'\omega} \cap \overline{q_2a_2}$. By DHS(V) it follows $a_4' \in \overline{q_1p_3}$ and $b_4 \in \overline{b_2p_3}$. ■

Let $s \in V^-$, $q \in V^+$ and $k \in \mathfrak{g}$; we define the function $\gamma_s^{k,q}: V \longrightarrow V$ as follows

$\gamma_s^{k,q}(p) = p$ if $p \in V^+$

$\gamma_s^{k,q}(p) = p'$ if $p \in V^-$ and there are $a, b \in k$ and $s' \in V^+$

with $a = \overline{qs} \cap \overline{s'p'}$, $b = \overline{pq} \cap \overline{s's}$.

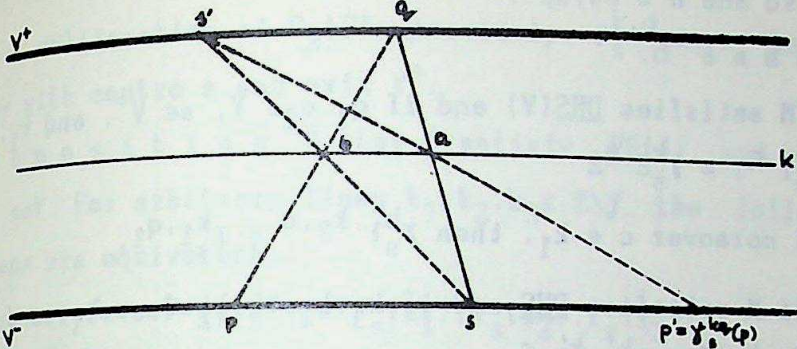


Fig. 10

It is seen that $\gamma_s^{k,q}$ is a function, moreover it is a bijection. Next we extend $\gamma_s^{k,q}$ on $\mathcal{L}\setminus\mathcal{J}$ by the condition $\gamma_s^{k,q}(\overline{pr}) = \overline{\gamma_s^{k,q}(p)\gamma_s^{k,q}(r)}$.

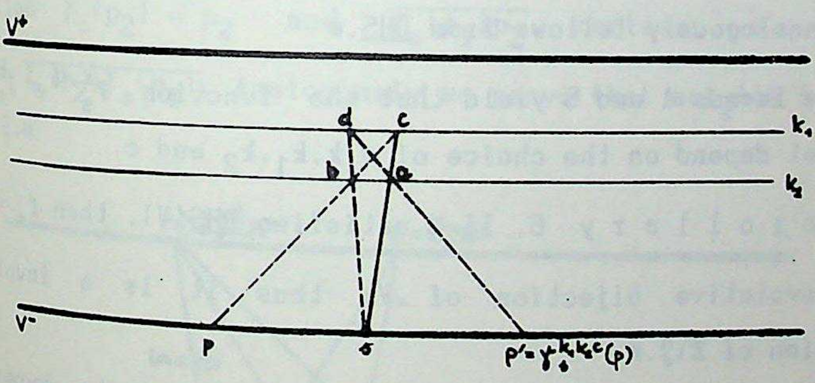


Fig. 11

Analogously with given lines $k_1, k_2 \in \mathcal{J}$, $s \in V^-$ and $c \in k_1$ such that $\overrightarrow{cs} \cap k_2 \neq \emptyset$ we correlate the function $\alpha = \gamma_s^{k_1,k_2,c} : V \longrightarrow V$ defined by the conditions

$\alpha(p) = p$ for $p \in V^+ \cup \{\omega, \tilde{\omega}, s\}$;

$\alpha(p) = p'$ if $p \in V^-$ and there are $a, b \in k_2$, $d \in k_1$ such that $b \in \overline{cp}, \overline{sd}$ and $a \in \overline{cs}, \overline{dp'}$.

Lemma 5.

(i) If M satisfies DHS(V) and if $q_1, q_2 \in V$, $s \in V^+$, and $k_1, k_2 \in J$, then $\gamma_s^{k_1, q_1} = \gamma_s^{k_2, q_2}$.

(ii) If moreover $c \in k_1$, then $\gamma_s^{k_1, k_2, c} = \gamma_s^{k_1, q_1}$.

(iii) If M satisfies DHS and if $k_1, k_2, k_1', k_2' \in J$, $c \in k_1$, and $c' \in k_1'$, then $\gamma_s^{k_1, k_2, c} = \gamma_s^{k_1', k_2', c}$.

Proof. (i) It remains to notice that DHS(V) implies its modification in which points b_1, b_2 are replaced by ends; still it is an easy exercise.

(ii) directly follows from DHS(V) if we substitute $p_2 := s$, $q_1 := q$, $p_3 := p$, $p_1 := p'$, $b_1 := c$, $b_3 := k_2 \cap \overline{cs}$.

(iii) analogously follows from DHS. ∎

Now lemmas 4 and 5 yield that the function $\gamma_s^{k,q} = \gamma_s^{k_1, k_2, c}$ does not depend on the choice of q, k, k_1, k_2 and c.

Corollary 6. If M satisfies DHS(V), then $\gamma_s = \gamma_s^{k,q}$ is a involutive bijection of V; thus it is a involutive bijection of $\mathcal{L} \setminus \mathcal{J}$. ∎

If we compare hyperbolic planes with degenerate hyperbolic planes we see that it is impossible to define the function $\gamma_s^{k,q}$ in hyperbolic planes. In the theory of degenerate hyperbolic planes functions $\gamma_s^{k,q}$ play the role of polarity induced by orthogonality in hyperbolic planes - each $\gamma_s^{k,q}$ transforms $\Omega(F)$ onto its complement (minus V) in the projective plane $P_2(F)$. However in the case of degenerate hyperbolic plane the complement of $\Omega(F)$ is again a degenerate hyperbolic plane and $\gamma_s^{k,q}$ is a collineation of $P_2(F)$; precisely: $\gamma_s^{k,q}$ is a harmonic homology with centre s and axis V^+.

Proposition 7. Let M satisfy PB(J) and DHS(V) and let $s\in V$. For arbitrary lines $k_1,k_2,k_3\in \mathcal{L}\setminus\mathcal{J}$ the following conditions are equivalent:

(i) there exists $a\in P$ with $a\in\gamma_s(k_1),\gamma_s(k_2),\gamma_s(k_3)$,

(ii) there exist $m,k\in\mathcal{L}$ such that $k\in\mathcal{J}$ and $\sigma_k^m(k_i)=k_i$ for i=1,2,3.

Proof. Let $k\in\mathcal{J}$, we consider $\alpha=\sigma_k^m$. Let $s'=\alpha(s)$, $n=\overline{ss'}$, and $a=n\cap k$. For arbitrary line $l\in\mathcal{L}\setminus\mathcal{J}$ it holds $a\in\gamma_s(l)\Leftrightarrow\alpha(l)=l$. Indeed, let $l=\overline{p_1p_2}=\alpha(l)$ and $p_1\in V^-$; we can consider γ_s as $\gamma_s^{k,s'}$ and α as $\sigma_k^{ss'}$. Let $b=\overline{p_1s'}\cap\overline{p_2s}$; $\alpha(l)=l$ implies that $b\in k$. From the definition $\gamma_s(p_2)=p_2$ and $\overline{\gamma_s(p_1)p_2}$ contains a, thus $a\in\gamma_s(l)=\overline{\gamma_s(p_2)\gamma_s(p_1)}$. Analogously we prove that $a\in\gamma_s(l)$ implies $\alpha(l)=l$.■

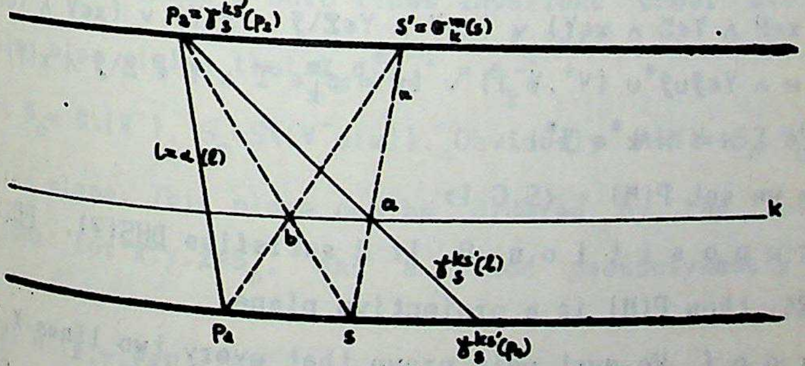


Fig. 12

The transformation γ_s will be called a pseudosymetry. We shall introduce three classes of functions. We set for given $k \in \mathcal{J}$, $m \in \mathcal{L}\setminus\mathcal{J}$

$$\Sigma_k := \{\sigma_k^{m'} : m' \in \mathcal{L}\setminus\mathcal{J}\},$$

$$\Sigma^m := \{\sigma_{k'}^{m} : k' \in \mathcal{J}\},$$

$$\mathbf{I} := \{\sigma_{k'}^{m'} : m' \in \mathcal{L}\setminus\mathcal{J},\ k' \in \mathcal{J}\}.$$

For a given end $s \in V$ the function γ_s is a bijection of the set $L\setminus J$. By Proposition 7, it is possible to construct the bijection $\gamma_s^* : \Sigma \longrightarrow P$ defined by the condition

$$\gamma_s^*(\sigma) = a \Leftrightarrow (\exists l, m)[a \in \gamma_s(l), \gamma_s(m) \wedge \sigma(l) = l \wedge \sigma(m) = m \wedge l \neq m].$$

Then we get $\gamma_s^*(\sigma) \in l \Leftrightarrow \sigma(l) = l$, for every $\sigma \in \Sigma$ and $l \in L\setminus J$.

We pass to the construction of a projective plane . To this aim we shall define several new objects. Let $\mathcal{J}^*$ be any set disjoint with $\mathcal{J}$ such that $\operatorname{card}(\mathcal{J}) = \operatorname{card}(\mathcal{J}^*)$, let the transformation $k \longmapsto k^*$ be any bijection of $\mathcal{J}$ onto $\mathcal{J}^*$. We set $S := P \cup \Sigma \cup V^+ \cup V^- \cup \{\omega\}$ and $\mathcal{G} := \mathcal{L} \cup \mathcal{J}^* \cup \{V^+, V^-\}$.

The relation $I \subseteq S \times \mathcal{G}$ is defined as follows

$$xIY \Leftrightarrow (x \in P \wedge Y \in \mathcal{L} \wedge x \in Y) \vee (x \in V \wedge Y \in \mathcal{L}\setminus\mathcal{J} \wedge xIY) \vee (x \in V \wedge Y \in \{V^+, V^-\}) \vee (x = \omega \wedge Y \in \mathcal{J} \cup \mathcal{J}^* \cup \{V^+, V^-\}) \vee (x = \sigma_k^m \in \Sigma \wedge Y \in \mathcal{L}\setminus\mathcal{J} \wedge x(Y) = Y) \vee (x = \sigma_k^m \in \Sigma \wedge Y = k^* \in \mathcal{J}^*).$$

Finally we set $P(M) = \langle S, G; I \rangle$.

Proposition 8. If M satisfies DHS(V), PB(J), PP, and DHR4, then $P(M)$ is a projective plane.

Proof. We must only prove that every two lines X, Y from $\mathcal{G}$ have a common point in S, and every two points $x, y \in S$ have a common line Z. Let $x, y \in S$ and $x \neq y$. If $x, y \in P \cup V^+ \cup V^-$ we consider $Z = \overline{xy}$; or $Z = V^+ (V^-)$, if $\sim x\text{-}y$ and $x, y \in V^+(V^-)$. If $x \in P$ and $y = \sigma_k^m \in \Sigma$, then we use Lemma 3; if $x\in V$, then we set $Z = \overline{xy(x)}$. Now assume $x=\sigma_{k_1}^{m_1}$, $y=\sigma_{k_2}^{m_2}$, and $x,y\in\Sigma$. Let us consider a pseudosymmetry $\gamma=\gamma_s$, let $a=\gamma^*(x)$, $b=\gamma^*(y)$, and $l=\overline{ab}$. If $l\notin\mathcal{J}$, then $Z=\gamma(l)$; otherwise $k_1=k_2$ and $Z=k_1^*$. Other cases are trivial. Now assume $X,Y\in\mathcal{G}$. In the first case if $X,Y\in\mathcal{L}\setminus\mathcal{J}$, then either $X\cap Y\neq\emptyset$, either X and Y have a common end, or $X=\overline{p_1p_2}$, $Y=\overline{q_1q_2}$, and $\overline{p_1q_2}\cap\overline{p_2q_1}\neq\emptyset$. In the last case we consider the line $k=\overline{(\overline{p_1q_2}\cap\overline{p_2q_1})\omega}$ and we set $z=\sigma_k^X$. Clearly other cases are obvious.■

Immediately from the construction it follows that if $s\in V^-$, then γ_s is a harmonic homology in $P(M)$; moreover we can prove that every $\sigma\in\Sigma$ can be extended to a collineation (a symmetry) in $P(M)$. To do so one can notice, that σ_k^m is definable in M with parameters k and m; therefore $\alpha(\sigma_k^m)=\alpha\sigma_k^m\alpha^{-1}=\sigma_{\alpha(k)}^{\alpha(m)}$ for every collineation α, and, consequently $\sigma\in\Sigma$ implies $\alpha(\sigma)\in\Sigma$ and α maps lines invariant under σ onto lines invariant under $\alpha(\sigma)$. This reasoning also yields that $\gamma_s\sigma_k^m\gamma_s^{-1}=\sigma_{k^*}^{\gamma_s(m)}$.

Let $\mathcal{G}_o=\mathcal{G}\setminus\{V^-\}$, $S_o=S\setminus(V^-\cup\{\omega\})$. Obviously $A(M)=\langle S_o,\mathcal{G}_o;I\rangle$ is an affine plane. This plane can be ordered by the following conditions: for $x,y,z\in S_o$, and a fixed pseudosymmetry γ we define

$$B(x,y,z)\Leftrightarrow(x,y,z\in P\wedge B(x,y,z))\vee(x,y,z\in\Sigma\wedge B(\gamma(x),\gamma(y),\gamma(z)))$$
$$\vee[(x\in\Sigma\wedge y\in V^{+}_{\sqcup}\wedge z\in P)\vee(x\in P\wedge y\in V^{+}_{\sqcup}\wedge z\in\Sigma)]\wedge(\exists U\in\mathcal{G}_o)[x,y,z\in U]\vee$$
$$\vee(x\in V^{+}\wedge\langle z,y\rangle\in x)\vee(z\in V^{+}\wedge\langle x,y\rangle\in z)\vee(x\in V^{+}\wedge z,y\in\Sigma\wedge$$

$\wedge \langle\gamma(z),\gamma(y)\rangle \in x) \vee (x,y\in\Sigma \wedge z\in P \vee x,y\in P \wedge z\in\Sigma\ ) \wedge (\exists u\in V^{+})$
$[B(x,y,u) \wedge B(y,u,z)] \vee (x\in\Sigma \wedge y,z\in P \vee x\in P \wedge y,z\in\Sigma\ ) \wedge B(z,y,x)$
$\vee\ (z\in V^{+}\wedge x,y\in\Sigma \wedge \langle\gamma(x),\gamma(y)\rangle\in z) \vee (x,y,z\in V^{+}\wedge B^{`}_{o}(x,y,z)),$

where $B_o(x,y,z)$ means that the ends x,y,z can be obtained by a parallel projection of points a,b,c such that $B(a,b,c)$ and $a,b,c \in k \in \mathcal{J}$, for some line k.

It is a long and clumsy proof (though rather easy) that B is a betweenness relation in $A(M)$. All these considerations result in

Corollary 9. M is isomorphic with an open halfplane in $A(M)$, with boundary V^{+}.■

To close our considerations we shall study in more details the family of symmetries in M.

End calculus in a degenerate hyperbolic plane

At the beginning we recall a general property of structures from qH satisfying (qH_1) and (qH_2):
if α is a collineation of M, α fixes point-wise a line k, $q\neq\omega$, $\tilde{\omega}$, $\neg qIk$, and $q|k|\alpha(q)$, then α is a symmetry in k; precisely then $\alpha = \sigma_{k}^{\overline{q\alpha(q)}} = \sigma_{k}^{q\alpha(q)} \in \Sigma_{k}$. Thus the class Σ_k with $k\in\mathcal{J}$ coincides with the class of all collineations of M such that $q|k|\alpha(q)$ for some (hence for every) $q\in V$ with $\neg qIk$, i.e. with the class of all involutive collineations of M which point-wise fix the line k. As an immediate consequence of this property we obtain

Lemma 10. If M satisfies DHR4 and PB(J) and $k\in\mathcal{J}$, then $\Sigma_k\Sigma_k\Sigma_k \subseteq \Sigma_k$.■

In this section (unless explicitly stated) we shall always consider symmetries as transformation of V. Let $D_k = \Sigma_k\Sigma_k$. . By Lemma 10 the set D_k is an abelian transformation group.

Lemma 11. If M satisfies PB(J) and DHR1, and $k_1, k_2 \in J$, then $D_{k_1}|V^+ = D_{k_2}|V^+$ and $D_{k_1}|V^- = D_{k_2}|V^-$.

Proof. We must prove that for every $\alpha \in D_{k_1}$ there exists $\beta \in D_{k_2}$ with $\alpha|V^+ = \beta|V^+$ (the same arguments can be applied for V^-). Let $\alpha = \sigma^n_{k_1}\sigma^m_{k_1}$, $p \in V^+$. One can assume that pIm, $\sigma^m_{k_1}(p) = p'$ In. We set $\beta = \sigma^n_{k_2}\sigma^m_{k_2}$. Obviously $\alpha(p) = \beta(p)$; let $r \in V^+$ and $r \neq p$, $p'' = \alpha(p)$. We consider points and ends constructed as follows $b_2 = k_1 \cap \overline{p'r}$, $c_2 = k_2 \cap \overline{p'r}$, $r'_1 : \overline{pr'_1} \ni b_2$, $d_2 = \overline{pr'_1} \cap k_2$, $r'_2 : c_2 \in \overline{pr'_2}$, $a_2 = \overline{pr'_2} \cap k_1$, $d_1 = \overline{r'_1p''} \cap k_2$, $b_1 = r_1\overline{p'' \cap k_1}$, $a_1 = r'_2\overline{p'' \cap k_1}$, $c_1 = r'_2\overline{p'' \cap k_2}$. By DHR1 we obtain $p' I \overline{b_1c_1}$, which implies that $\beta(r) = \alpha(r)$. ■

Lemma 12. If M satisfies PB($\mathcal{J}$) and DHR2, then $\Sigma^m\Sigma^m\Sigma^m \subseteq \Sigma^m$.

Proof. Let, $k_1, k_2, k_3 \in \mathcal{J}$, $a \notin m$, and $a \in k_1$. We put $a = a_1$ and find the points a_2, a_3, a_4 such that $a_2 \in k_2$, $a_3 \in k_3$ and $a_4a_3a_1a_2$ is a parallelogram with directions of m; let $k = \overline{a_4\omega}$. By DHR2 (cf. Figure 6a) we get that $\sigma^m_{k_3}\sigma^m_{k_2}\sigma^m_{k_1} = \sigma^m_k$. ■

Let $D^m = \Sigma^m\Sigma^m$; Lemma 12 yields that D^m is an abelian transformation group. For $p_1, p_2, p_3, p_4 \in V^+$ we define

$$p_1p_2 \Uparrow p_3p_4 \Leftrightarrow (\exists a_1, a_2, a_3, a_4)(\exists t \in V^-)[(\forall i = 1, \ldots, 4)\ \overline{p_it} \ni a_i \wedge$$
$$\wedge\ a_1a_2 \Uparrow a_3a_4 \wedge (\overline{a_1a_2} \in J \vee p_1 = p_2)].$$

One can prove, that $\Uparrow$ is a linear oriented parallelity in V^+

(cf. [3]). Analogously we define $\Uparrow$ in V^-. Let us fix two ends $q, o \in V^+$ with $q \neq o$, and a line $h = \overline{oo'}$, we set $V_{oq} = \{p' \in V^+ : op' \Uparrow oq\}$. Clearly if $\alpha \in D^m$ or $\alpha \in D_k$, then $\alpha(V^+) = V^+$. But we have even more. As an easy observation we note

Lemma 13. If $m = \overline{p_1p_2}$, $k \in \mathcal{J}$, and $r \in V$, then $m \cap \overline{\sigma_k^m(r) r} = \emptyset$. Therefore for every $\alpha \in D^m$ and every r such that $r - p_2$ it holds $p_1 r \Uparrow p_1 \alpha(r)$. ■

With the help of the above lemmas we prove now.

Proposition 14. Let M satisfy $PB(\mathcal{J})$, DHR1, and DHR2. Then

(i) D_k acts sharply transitive on V^+,

(ii) D^h acts sharply transitive on $V_{oq} \setminus \{o\}$, and

(iii) D_k is invariant under D^h.

Proof. (i) Transitivy is clear; indeed, for given $p_1, p_2 \in V^+$ we consider arbitrary $p \in V^-$ and set $\alpha = \sigma_k^{p_2p} \sigma_k^{p_1p} \in D_k$, obviously $\alpha(p_1) = p_2$. Now, if $\sigma_k^{m_1} \sigma_k^{m_2}(p) = \sigma_k^{m_3} \sigma_k^{m_4}(p)$ for some $p \in V^+$, then $\sigma_k^{m_3} \sigma_k^{m_1} \sigma_k^{m_2}(p) = \sigma_k^{m_4}(p)$ and since $\sigma_k^{m_3} \sigma_k^{m_1} \sigma_k^{m_2} \in \Sigma_k$, we infer $\sigma_k^{m_3} \sigma_k^{m_1} \sigma_k^{m_2} = \sigma_k^{m_4}$.

(ii) For given ends p_1, p_2 with $op_1 \Uparrow op_2$ and $o \neq p_1, p_2$ we consider arbitrary point $a \in \overline{p_1 o'}$ and $b = \overline{p_2 o'} \cap \overline{oa}$ and we set $k_1 = \overline{a\omega}$, $k_2 = \overline{b\omega}$, and $\alpha = \sigma_{k_2}^h \sigma_{k_1}^h$. Then clearly $\alpha(p_1) = p_2$ and $\alpha \in D^h$. To prove rigidity we calculate as in (i). We obtain $\sigma_{k_3}^h \sigma_{k_1}^h \sigma_{k_2}^h = \sigma_k^h$ and $\sigma_k^h(p) = \sigma_{k_4}^h(p)$

for some $p\neq o,o'$. Therefore the line $m=\overline{p\sigma_k^h(p)}\neq h$ is invariant under both symmetries σ_k^h and $\sigma_{k_1}^h$ and thus $k=k_1$ and $\sigma_k^h=\sigma_{k_1}^h$.

(iii) follows immediately from Lemma 11.∎

We have observed that $\gamma_o(V_{oq})=(V^+\setminus V_{oq})\cup\{o\}$. For $p\in V^+$ we denote $-p=\gamma_o(p)$. Using parameters o,q,h we shall define two operations on V^+:

let $p_1,p_2,p\in V^+$ $p_1+p_2=p \Leftrightarrow (\exists\alpha\in D_k)[\alpha(o)=p_1 \wedge \alpha(p_2)=p_2]$;

let $p_1,p_2,p\in V_{oq}$ $p_1\cdot p_2=p \Leftrightarrow (\exists\beta\in D^h)[\beta(q)=p_1 \wedge \beta(p_2)=p] \vee p_1=p_2=0$;

finally we extended the operation upon V^+ by the conditions $(-p_1)\cdot p_2=p_1\cdot(-p_2)=-(p_1\cdot p_2)$ and $(-p_1)\cdot(-p_2)=p_1\cdot p_2$.

In the Proposition 15, Corollary 16, and Theorem 18 below we assume that M is a fixed quasihyperbolic plane which satisfies PB(J), DHR1, DHR2, and DHS(V). Then we get

Proposition 15.

(i) $\langle V^+;+,o\rangle$ is an abelian group;

(ii) $\langle V_{oq};\cdot,q\rangle$ is an abelian group, V_{oq} is closed under +,

(iii) $p+(-p)=o$, for every $p\in V^+$

(iv) $p\cdot(p_1+p_2)=p\cdot p_1+p\cdot p_2$ for all $p,p_1,p_2\in V^+$.

Proof. (i) follows from Proposition 14(i). Proposition 14(ii) yields that $\langle V_{oq};\cdot,q\rangle$ is an abelian group. From the theory of quasihyperbolic planes we immediately infer $op_1\uparrow op_2 \Rightarrow op_1\uparrow o(p_1+p_2)$ which proves that V_{oq} is +-closed. To prove (iii) we consider arbitrary end $r\in V$ and arbitrary line $k\in q$ and we set $\alpha=\sigma_k^{rp}\sigma_k^{ro}\in D_k$; then $\alpha(o)=p$. Let $a=k\cap\overline{or}$, $b=k\cap\overline{pr}$, and $s, s' \in V^-$ with $\overline{os} \ni b$, $\overline{os'} \cap \overline{r\gamma_o(p)} \in k$. From PB we infer that $a \in \overline{ps'}$, from the definition $a \in \overline{s\gamma_o(p)}$. This implies that $\alpha(\gamma_o(p)) = o$, i.e. $p+(-p)=o$. Finally (iv) is a direct consequence of Proposition 14(iii).■

Let $F(o,q,h) = \langle V^+; +, \cdot . o, q, V_{oq}\rangle$. Proposition 15 immediately yields.

Corollary 16. $F(o,q,h)$ is a ordered commutative field, V_{oq} is a positive cone in this field.■

Remark 17. Let $k \in J$, $\beta = \sigma^h_k$, and $q' = \beta(q)$. Then DHR4 yields that β is an isomorphism of $F(o,q,h)$ and $F(o',q',h)$. Indeed, from Lemma 12 we infer that $\beta(\Sigma^m) = \Sigma^m$; analogously $\beta(D_k) = D_k$. Since β is an automorphism of M the thesis follows from the definability of $F(o,q,h)$ in M. Every nonisotropic line in M can be associated now with a pair of elements of $F(o,q,h)$: with $n = \overline{p_1 p_2}$, where $p_1 \in V^+$ we correlate the pair $\langle p_1, \beta(p_2)\rangle$. This results in a direct analytical description of M in terms of the field $F(o,q,h)$.■

Let us consider again the affine plane A(M), let $k \in \mathcal{J}$ $e = \overline{qo'} \cap k$. It is seen (c.f. Figures 13,14) that $F = F(o,q,h)$ is a coordinate quasifield of A(M), as constructed in [10]. But F is a commutative field, therefore A(M) is Pappian. Moreover if $\overline{ab} \notin \mathcal{J}$, then $B(a,b,c)$ is equivalent with

$$B(\overline{a\omega} \cap h, \overline{b\omega} \cap h, \overline{c\omega} \cap h) \wedge B(p,r,s) \wedge a \in \overline{po'} \wedge b \in \overline{ro'} \wedge c \in \overline{so'} \wedge L(abc).$$

This implies that $\langle S_o, \mathcal{G}_o, B\rangle$ is an ordered affine plane coordinatized by F. Clearly P is an open half-plane in A(M). Since an open affine half-plane is isomorphic to $\Omega(F)$ we obtain as a consequence

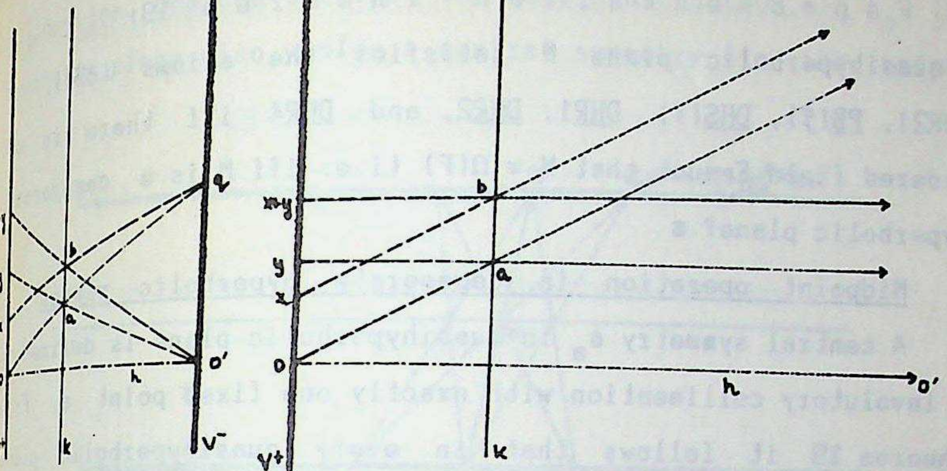

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Fig. 13

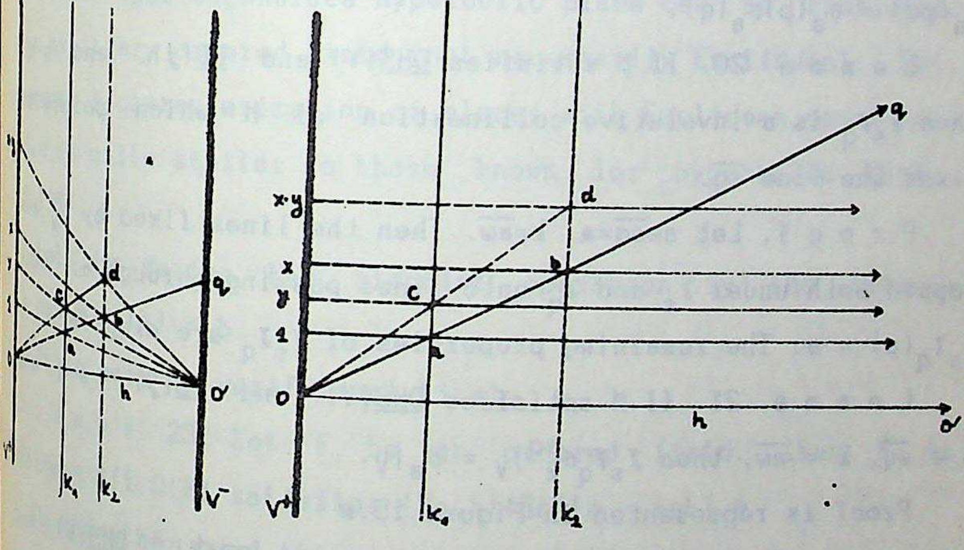
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Fig. 14

Theorem 18. $M \cong \Omega(F(o,q,h))$.■

The last theorem immediately implies our main

Representation Theorem 19: A quasihyperbolic plane M satisfies the axioms (dH1), (dH2), PB($\mathcal{J}$), DHS(V), DHR1, DHR2, and DHR4 iff there is an ordered field F such that $M \cong \Omega(F)$ (i.e. iff M is a degenerate hyperbolic plane).■

Midpoint operation in degenerate hyperbolic planes

A central symmetry σ_a in quasihyperbolic plane is defined as a involutory collineation with exactly one fixed point a. From Theorem 19 it follows that in every quasihyperbolic plane satisfying appropriate axioms there exists a symmetry in every point. In fact it is easy to define directly σ_a. For every $p \in V \setminus E$ we consider $p' \in V$ with $a \in \overline{pp'}$ and we set $\sigma_a(p) = p'$, $\sigma_a(\overline{pq}) = \overline{\sigma_a(p)\sigma_a(q)}$.

Lemma 20. If M satisfies DHS(V) and PB($\mathcal{J}$), and s-q, then $\gamma_s\gamma_q$ is a involutive collineation of M which pointwise fixes the line $\overline{sq}$.

Proof. Let $a \in \overline{sq} = m$, $k = \overline{a\omega}$. Then the lines fixed by σ_k^m are mapped both under γ_s and γ_q onto lines passing through a; thus $\gamma_s\gamma_q(a) = a$. The remaining properties of $\gamma_s\gamma_q$ are clear.■

Lemma 21. If M satisfies DHS(V) and PB($\mathcal{J}$), s-q and $a \in \overline{sq}$, $k = \overline{a\omega}$, then $\gamma_s\gamma_q\sigma_k^{sq}|_V = \sigma_a|_V$.

Proof is represented in Figure 15.■

Corollary 22. If M satisfies PB($\mathcal{J}$), DHS(V) and DHR4, then for every point a there exists a central symmetry σ_a.■

In other words Corollary 22 states that, under given assumptions the function σ_a defined above yields a collineation. One can see that the same construction of σ_o in hyperbolic plane also yields a central symmetry (in the point o).

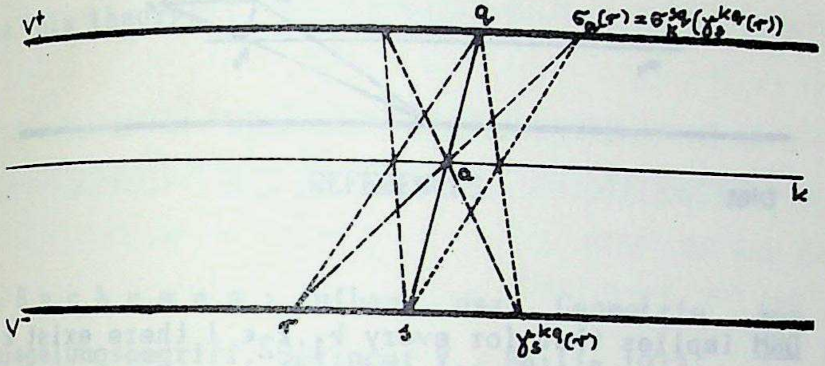


Fig. 15

Notice that degenerate hyperbolic plane can be considered over arbitrary ordered field (not necessarily Euclidean). But, however, a characterization of planes with Euclidean coordinate field is quite similar to those known for hyperbolic planes (cf.[1]).

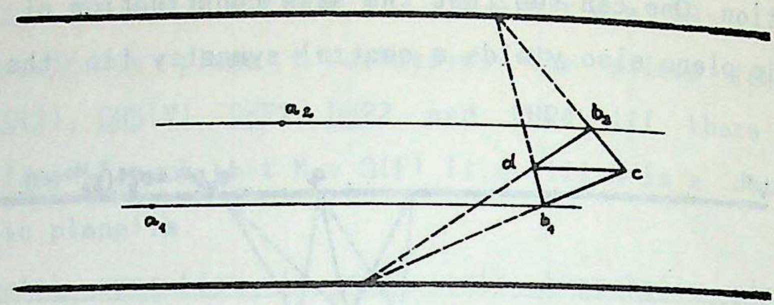
Let us define the midpoint operation by the formula $c := a \oplus b \Leftrightarrow \sigma_c(a) = b$.

It is easy to verify analyticaly

Lemma 23. Let F be an ordered field. Then F is Euclidean iff $\Omega(F)$ satisfies $(\forall a,b)(\exists c)[c = a \oplus b]$. ■

On the other hand the existence of a midpoint of every segment can be guaranteed by the following configurational xiom

[illegible]: $(\forall a_1,a_2)[\overline{a_1a_2} \notin \mathcal{J} \Rightarrow (\exists b_1,b_2,c,d)[b_1d \uparrow cb_2 \wedge b_2d \uparrow cb_1 \wedge a_1b_1 \uparrow b_1b_2 \wedge \overline{a_1b_1}, \overline{cd} \in J]]$.



DHM

Indeed, DHM implies that for every $k_1, k_2 \in J$ there exist k and m with $\sigma_k^m(k_1) = k_2$, which forces the coordinate field to be Euclidean. Some other formulas equivalent with DHM can be found if we consider an explicit construction of σ_c and if we consider the multiplication in F(o,q,h).

Obviously the list of interesting problems arising in the theory of degenerate hyperbolic planes is not exhausted. In particular we are interested in the following questions: M satisfies PB(k) for every line $k \in \mathcal{L}$, it would be nice to give a direct proof of this fact; it would be nice to prove directly PP from PB($\mathcal{J}$) and DHR4; the formula DHR2 was used only once (to prove the reduction principle for Σ^m), and only for joinable directions p,q of considered parallelograms (cf. Fig 6a), it would be nice to know if there is an interesting application of DHR2 when ~p-q.

Finally we notice that all the axioms we propose are special versions of the Desargues and Pappos Axioms. There are also some further configurations obtained in this way which look natural and nice in the context of the theory of directed parallelity. It seems of some interest to know what is their role in this theory.

REFERENCES

[1] F. Bachmann : Aufbau der Geometrie aus dem Spiegelungsbegriff, Springer V., Berlin 1973.

[2] M. Grochowska : A proof of Pasch Axiom in the absolute theory of Oriented Parallelity (submitted to;) J. of Geometry.

[3] M. Grochowska : An axiom system for oriented Euclidean parallelity, Demonstratio Math. 16(1983) 937-943.

[4] K. Prażmowski : An axiomatic description of the Strambach planes (submitted to) Geometriae Dedicata.

[5] K. Prażmowski : On the structure of the group of isometries of degenerate hyperbolic plane and a geometry arising from it (to appear in:) Glasn. Mat. Fiz.

[6] K. Strambach : Salzmann-Ebenen mit hinreichend vielen Punkt- oder Geradenspiegelungen, Math.Z.99(1967).

[7] H.&R. Struve : Eine synthetische Characterizirung der ko-Minkowskische Ebenen, mimeographed.

[8] L.W. Szczerba, A. Tarski : Metamathematical discussion of some affine geometries, Fund. Math.104(1979) 155-192.

[9] W. Szmielew: On n-ary equivalence relations, Diss.Math.PWN Warsaw 1981.

[10] W. Szmielew: From affine to Euclidean geometry, PWN Warsaw, D.Reidel Publ.Co, Dordrecht 1983.

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ON SOME REPRESENTATIONS OF GROUPOID MODES SATISFYING THE REDUCTION LAW

Dedicated to the memory of Professor Edward Otto

1. Introduction

The aim of this paper is to describe the structure of groupoids satisfying the following axioms:

(L) $(x\cdot y)\cdot z = (x\cdot z)\cdot y$ (Left normal law),

(I) $x\cdot x = x$ (Idempotent law),

(R) $x\cdot(y\cdot z) = x\cdot y$ (Reduction law).

In the sequel we write $xx_1\ldots x_n$ for $(\ldots((x\cdot x_1)\cdot x_2\cdot\ldots)\cdot x_n$ and simply xy^n if $x_1 = x_2 = \ldots = x_n = y$).

We call groupoids satisfying the above identities LIR-groupoids. As was shown in [RR1], LIR-groupoids may be defined alternatively as groupoids satisfying the identities (I), (R) and

(E) $(x\cdot y)\cdot(z\cdot t) = (x\cdot z)\cdot(y\cdot t)$ (Entropic law).

Hence they are groupoid modes as defined in [RS]. The investigations presented here belong to the recently developed theory of groupoid modes. (See e.g. [JK] and references there, 2], [R], [RR1], [RR2] and [RS]). On the other hand, the class

LIR of all LIR-groupoids contains 2-cyclic, and more generally n-cyclic, groupoids ([P1], [P2], [RR1], [RR2]). All varieties of LIR-groupoids were described in [RR1], and it was shown there that each nontrivial subvariety of LIR is defined by one additional identity

$$(G_{ij}) \qquad xy^{i+j} = xy^{i},$$

for some non-negative i and positive j. It was shown as well in the same paper that all IR-groupoids satisfying the identities (I) and (R) may be characterized as sums of left-zero bands over a left-zero band (with some additional condition). In this paper further consequences of this fact for LIR-groupoids are investigated. In particular, it is shown that all decompositions of an LIR-groupoid $(G,\cdot)$ into such sums form an interval in the congruence lattice $C(G)$ of $(G,\cdot)$ (see Section 2), and that the least such decompositions are the most essential for an understanding of the structure of LIR-groupoids. We show that the classes of the least decomposition of $(G,\cdot)$ are in fact unions of commutative monoids, say $(G_i,*)$ for i in I, and are constructed from G_i in a similar way as flowers are built from shreds (Section 3). This fact builds an interesting "gangway" between LIR-groupoids and commutative monoids, and makes it possible to use some known results about commutative monoids in investigations of LIR-groupoids. Some examples of this are given in Section 4. The representation of an LIR-groupoid as the sum of unions of commutative monoids is closely connected to a representation by means of some directed graphs. The latter is discussed in Section 5.

2. Representation of LIR-groupoids as sums of left-zero bands over a left-zero band

It was shown in [RR1] that for each IR-groupoid $(G,\cdot)$, there is a congruence relation ϕ such that the quotient $I:=G/\phi$ and the ϕ-classes G_i, $i \in I$, are left-zero bands, and for each a_i in G_i, b_j, b'_j in G_j,

(2.1) $$a_i\cdot b_j = a_i\cdot b'_j .$$

In other words, the variety of IR-groupoids is a subclass of the Mal'cev product $LZ\circ LZ$ of the variety LZ of left-zero bands by itself, consisting of groupoids satisfying the condition (2.1). (See [M] or [G] for the definition of a Mal'cev product.) We say that $(G,\cdot)$ is the sum of the left-zero bands $(G_i,\cdot)$ over the left-zero band $(I,\cdot)$, or briefly that $(G,\cdot)$ is an LZ-LZ-sum. (See [RR1].) On the other hand, it was shown in [RR1] that each LZ-LZ-sum is an IR-groupoid. In particular, each LIR-groupoid can be decomposed into an LZ-LZ-sum, and each LZ-LZ-sum, satisfying (L) is a LIR-groupoid.

In general, there are many congruence relations decomposing a LIR-groupoid into an LZ-LZ-sum. For instance, each equivalence relation of a left-zero band decomposes it into an LZ-LZ-sum. One of the congruences decomposing an LIR-groupoid $(G,\cdot)$ into an LZ-LZ-sum may be defined as follows:

(2.2) $$a \varrho b \text{ if and only if } xa = xb \text{ for each } x \text{ in } G.$$

(See [P1],[P2] and [RR1].) For example, the congruence relation

$\mathcal{G}$ on the LIR-groupoid $(G,\cdot)$ given by the table

	a_1	a_2	b_1	b_2	c
a_1	a_1	a_1	a_1	a_1	a_2
a_2	a_2	a_2	a_2	a_2	a_1
b_1	b_1	b_1	b_1	b_1	b_2
b_2	b_2	b_2	b_2	b_2	b_1
c	c	c	c	c	c

partitions G into two classes $\{a_1,a_2,b_1,b_2\}$ and $\{c\}$. It is easy to see that $\mathcal{G}$ is not the unique congruence that decomposes $(G,\cdot)$ into an LZ-LZ-sum. Another one is given by the partition of G into three classes $\{a_1,,a_2\}$, $\{b_1,b_2\}$ and $\{c\}$.

2.3 Proposition. The relation $\mathcal{G}$ is the greatest congruence relation on an LIR-groupoid $(G,\cdot)$ decomposing it into an LZ-LZ-sum.

Proof. Let $\mathcal{C}$ be a congruence relation decomposing $(G,\cdot)$ into an LZ-LZ-sum of $(G_i,\cdot)$ over a left-zero band $(I,\cdot)$. For any i,j in I, a_i in G_i, b_j and c_j in G_j, $a_ib_j=a_ic_j$, whence $b_j \,\mathcal{G}\, c_j$. Consequently $\mathcal{C} < \mathcal{G}$.■

We will define another congruence relation that seems to be crucial for understanding the structure of LIR-groupoids.

Let $\{g_j \mid j\in J\}$ be a set of generators of $(G,\cdot)$. The subset $G_j := \{g_jg_{j_1}^{p_1} \dots g_{j_k}^{p_k} \mid j,j_1,\dots,j_k\in J\}$ is said to be an orbit of g_j. We define the following relation $\mathcal{L}$ on the groupoid $(G,\cdot)$:

$$(2.4)\quad \begin{cases} a \,\mathfrak{L}\, b \text{ if and only if } a \text{ is in } G_i, \\ b \text{ is in } G_j \text{ for some } i,j \text{ in } J \\ \text{and the intersection } G_i \cap G_j \text{ is not empty.} \end{cases}$$

Note that in particular it can happen that $i = j$.

2.5 **Proposition.** The relation $\mathfrak{L}$ on an LIR-groupoid $(G,\cdot)$ is a congruence relation decomposing $(G,\cdot)$ into an LZ-LZ-sum.

Proof. The reflexivity and symmetry of $\mathfrak{L}$ are obvious. Now we prove transitivity. Let $a \,\mathfrak{L}\, b$ and $b \,\mathfrak{L}\, c$. Let a be in G_i, b in G_j and c in G_k, and let

$g_i g_{p_1}^{i_1} \dots g_{p_x}^{i_x} = g_j g_{r_1}^{j_1} \dots g_{r_j}^{j_y}$ be in $G_i \cap G_j$ and

$g_j g_{q_1}^{k_1} \dots g_{q_z}^{k_z} = g_k g_{s_1}^{m_1} \dots g_{s_t}^{m_t}$ be in $G_j \cap G_k$. Then

$$g_j g_{r_1}^{j_1} \dots g_{r_y}^{j_y} g_{q_1}^{k_1} \dots g_{q_z}^{k_z} = g_i g_{p_1}^{i_1} \dots g_{p_x}^{i_x} g_{q_1}^{k_1} \dots g_{q_z}^{k_z}$$

is in G_i. On the other hand, by (L)

$$g_j g_{r_1}^{j_1} \dots g_{r_y}^{j_y} g_{q_1}^{k_1} \dots g_{q_z}^{k_z} = g_j g_{q_1}^{k_1} \dots g_{q_z}^{k_z} g_{r_1}^{j_1} \dots g_{r_y}^{j_y} =$$

$$= g_k g_{s_1}^{m_1} \dots g_{s_t}^{m_t} g_{r_1}^{j_1} \dots g_{r_y}^{j_y}$$

is in G_k. It follows that $\mathfrak{L}$ is an equivalence relation on $(G,\cdot)$.

To prove that $\mathfrak{L}$ is a congruence relation, let $a = g_i g_{p_1}^{i_1} \dots g_{p_u}^{i_u}$, $b = g_j g_{q_1}^{j_1} \dots g_{q_v}^{j_v}$ and $c = g_k g_{r_1}^{k_1} \dots g_{r_w}^{k_w}$. Let $G_i \cap G_j$ be non-empty. Then it is easy to see that

$$ca = g_k g_{r_1}^{k_1} \dots g_{r_w}^{k_w} g_i \,\mathfrak{L}\, g_k g_{r_1}^{k_1} \dots g_{r_w}^{k_w} g_j = cb, \text{ and}$$

$ac = g_i g_{p_1}^{i_1} \dots g_{p_u}^{i_u} g_k \,\mathcal{L}\, g_j g_{q_1}^{j_1} \dots g_{q_w}^{j_w} g_k = bc$, whence $\mathcal{L}$ is a congruence relation.

Now since $a^{\mathcal{L}} b^{\mathcal{L}} = (ab)^{\mathcal{L}} = (g_i g_{p_1}^{i_1} \dots g_{p_u}^{i_u} g_j)^{\mathcal{L}} = a^{\mathcal{L}}$, it follows that $(G,\cdot)/\mathcal{L}$ is in the variety LZ of left-zero bands.

Next we show that $\mathcal{L} \le \mathcal{G}$. Indeed, if a and b are defined as above then there are $g_{w_m}^{a_m}$, $g_{v_n}^{b_n}$ such that

$$g_i g_{w_1}^{a_1} \dots g_{w_x}^{a_x} = g_j g_{v_1}^{b_1} \dots g_{v_y}^{b_y} .$$

Since $a \,\mathcal{G}\, g_i g_{w_1}^{a_1} \dots g_{w_x}^{a_x}$ and $b \,\mathcal{G}\, g_j g_{v_1}^{b_1} \dots g_{v_y}^{b_y}$, it follows that $a \,\mathcal{G}\, b$.

Obviously each $\mathcal{L}$-class is contained in some $\mathcal{G}$-class. It follows that $\mathcal{L}$-classes are left-zero bands, and condition (2.1) is satisfied as well.■

2.6. **Lemma.** Let $\mathcal{B}$ be the least congruence relation on an LIR-groupoid $(G,\cdot)$ such that the quotient $(G,\cdot)/\mathcal{B}$ is a left-zero band. Then $\mathcal{B}$ coincides with $\mathcal{L}$.

Proof. First note that $\mathcal{B} \le \mathcal{L}$. It follows that each $\mathcal{B}$-class is contained in an $\mathcal{L}$-class. Now if g_i is in $a^{\mathcal{B}}$, then since $(G,\cdot)/\mathcal{B}$ is a left-zero band, it follows that $G_i \subseteq a^{\mathcal{B}}$. If g_i, g_j are both in $a^{\mathcal{B}} \le a^{\mathcal{L}}$, then $G_i \subseteq a^{\mathcal{B}}$, $G_j \subseteq a^{\mathcal{B}}$, and $G_i \cap G_j$ is non-empty. It follows that $a^{\mathcal{B}} = a^{\mathcal{L}}$, whence $\mathcal{B} = \mathcal{L}$.■

2.7. **Proposition.** A congruence relation $\mathcal{C}$ on an LIR-groupoid $(G,\cdot)$ decomposes $(G,\cdot)$ into an LZ-LZ-sum if and only if

$$\mathcal{L} \le \mathcal{C} \le \mathcal{G}. \tag{2.8}$$

Proof. By Proposition 2.3 and Lemma 2.6, each congruence relation $\mathcal{C}$ decomposing $(G,\cdot)$ into an LZ-LZ-sum must satisfy the inequality (2.8).

On the other hand, if $\mathcal{C}$ is a congruence satisfying (2.8), then since $\mathcal{C} \leq \mathcal{G}$, it follows that $\mathcal{C}$-classes are left-zero bands and the condition (2.1) is satisfied, and since $\mathcal{L} \leq \mathcal{C}$ it follows that the quotient $(G,\cdot)/\mathcal{C}$ is a left-zero band.■

In the case that $(G,\cdot)$ is a left-zero band, the relation $\mathcal{G}$ "decomposes" $(G,\cdot)$ into just one class consisting of all elements of G, and the relation $\mathcal{L}$ is just the equality relation. In the example following the definition of $\mathcal{G}$, the relation $\mathcal{L}$ is given by the decomposition of $(G,\cdot)$ into the three classes $\{a_1,a_2\}$, $\{b_1,b_2\}$ and $\{c\}$.

3. Monoids induced by LIR-groupoids

In this section we show that each orbit of an LIR-groupoid has a much richer structure than one might expect, namely it is a commutative monoid.

Let G_j be an orbit in an LIR-groupoid $(G,\cdot)$ satisfying (G_{ik}). Let $a = g_j g_{j_1}^{p_1} \ldots g_{j_r}^{p_r} g_{j_{r+1}}^{p_{r+1}} \ldots g_{j_s}^{p_s}$ and

$$b = g_j g_{j_1}^{q_1} \ldots g_{j_r}^{q_r} g_{j_{s+1}}^{q_{s+1}} \ldots g_{j_t}^{q_t}$$

be elements of G_j. Note that $0 \leq p_1,\ldots,p_s,q_1,\ldots,q_t < i+k$ and any of the sets $\{j_1,\ldots,j_r\}$, $\{j_{r+1},\ldots,j_s\}$ and $\{j_{s+1},\ldots,j_t\}$ may be empty. We define the multiplication $*$ on G_j as follows

(3.1) $$a * b := g_j g_{j_1}^{p_1} \ldots g_{j_s}^{p_s} g_{j_1}^{q_1} \ldots g_{j_t}^{q_t} = g_j g_{j_1}^{u_1} \ldots g_{j_r}^{u_r} g_{j_{r+1}}^{p_{r+1}}$$

$$\ldots g_{j_s}^{p_s} g_{j_{s+1}}^{q_{s+1}} \ldots g_{j_t}^{q_t},$$

where for $w = 1,\ldots,r$

$$u_w = \begin{cases} p_w + q_w & \text{if } p_w + q_w < i+k \\ i+(p_w+q_w-i) \bmod k & \text{otherwise} \end{cases}$$

In the sequel we write y^i for $y * y * \ldots * y$, where y is repeated i times.

3.2. Proposition. For each orbit G_j in an LIR-groupoid $(G,\cdot)$ satisfying the identity

$$x\, y^i = x\, y^{i+k} \tag{3.3}$$

$(G_j,*)$ is a commutative monoid satisfying the identity

$$y^i = y^{i+k}. \tag{3.4}$$

Proof. The associative and commutative laws for $*$ follow directly by the definition of $*$ and the left-normal law for $\cdot$. The identity ((3.4) follows by the left-normal law and (3.3). The element g_j plays the role of the identity element in $(G_j,*)$.■

It was proved in [RR2] that in the case $(G,\cdot)$ is an n-cyclic groupoid, i.e. it satisfies the identity

$$x\, y^n = x \tag{3.5}$$

for some positive number n, each $\mathcal{L}$-class L_i contains just one generator g_j, whence it coincides with its orbit G_j. In this case the commutative monoid $(G_j,*)$ is an abelian group. In general, each $\mathcal{L}$-class L_i of an LIR-groupoid $(G,\cdot)$ is the union of orbits it contains, and it is easy to see that the intersection of two orbits G_j and G_k contained in L_i is an ideal in both monoids $(G_j,*)$ and $(G_k,*)$. In general, the monoid structure on all G_j contained in L_i cannot be naturally extended to a semigroup structure on the whole L_i. In other words, the orbits G_j are maximal subsets of G having a natural monoid structure.

Let us summarize the results of the second and third sections as follows.

3.6. Theorem. Let $(G,\cdot)$ be an LIR-groupoid. The congruence relation $\mathcal{L}$ induces the smallest decomposition of $(G,\cdot)$ into an LZ-LZ-sum, namely the sum of its $\mathcal{L}$-classes over the left-zero band $(I,\cdot)$ isomorphic with $(G,\cdot)/\mathcal{L}$.

For any set $\{g_j \mid j\in J\}$ of generators of $(G,\cdot)$, each $\mathcal{L}$-class L_i is the union of orbits G_j of generators g_j belonging to L_i. Each orbit G_j has the structure of a commutative monoid defined by (3.1).■

4. Some applications

It was shown in [RR1] that in the variety LIR, each groupoid word depending on variables $x_1,\dots,x_n$ with x_1 as the first variable is equal to a word of the form

(4.1)
$$x_1\, x_2^{p_2} \dots x_n^{p_n}$$

for some positive $p_2,\dots,p_n$. In the case that the additional identity

(4.2)
$$x\, y^{i+j} = x\, y^{i}$$

is satisfied as well, $p_2,\dots,p_n < i+j$. This gives a canonical form for elements in free LIR-groupoids (possibly satisfying (4.2)). These facts together with the results of the previous sections give the following.

4.3. Theorem. (i) The free LIR-groupoid F(X) on a set X of free generators is an LZ-LZ-sum of $|X|$ $\mathcal{L}$-classes L_i, each containing exactly one free generator. Each $(L_i,*)$ is the free commutative monoid on $|X|-1$ generators.

(ii) The free LIR-groupoid $F_{i,j}(X)$ on a set X of free generators, satisfying the identity (4.2), is an LZ-LZ-sum of $|X|$ $\mathcal{L}$-classes L_i, each containing exactly one free generator. Each $(L_i,*)$ is the free commutative monoids on $|X|-1$ generators satisfying the identity

(4.4) $$y^{i+j} = y^i . \blacksquare$$

Now, we will use the structure theorems of previous sections to describe some properties of congruence relations of LIR-groupoids.

4.5. Proposition. Let $\mathcal{C}$ be a congruence relation of an LIR-groupoid $(G,\cdot)$. Then for each $\mathcal{L}$-class L_i, the restriction $\mathcal{C}|_{L_i}$ of $\mathcal{C}$ to L_i coincides with the restriction $\mathcal{C}'|_{L_i}$ of some congruence relation $\mathcal{C}' \leq \mathcal{L}$.

Proof. In the case that $\mathcal{C}$ is not less than $\mathcal{L}$, the congruence $\mathcal{C}'$ is defined as follows. On the set L_i, $\mathcal{C}$ and $\mathcal{C}'$ coincide. On the set $G - L_i$, $\mathcal{C}'$ coincides with $\mathcal{L}$.$\blacksquare$

Let us say that semigroup congruences $\mathcal{A}$ on G_j and $\mathcal{B}$ on G_k, where both orbits G_j and G_k are in the same $\mathcal{L}$-class of $(G,\cdot)$, are compatible if their restrictions $\mathcal{A}|_{G_j \cap G_k}$ and $\mathcal{B}|_{G_j \cap G_k}$ to $G_j \cap G_k$ respectively, are equal.

4.6 Proposition. Let $\mathcal{C}$ be a congruence relation of an LIR-groupoid $(G,\cdot)$ such that $\mathcal{C} \leq \mathcal{L}$. Then for each j in J, the restriction $\mathcal{C}|_{G_j}$ of $\mathcal{C}$ to G_j is a congruence relation of the semigroup $(G_j, *)$. Moreover, if G_j and G_k are contained in the same $\mathscr{L}$-class, then $\mathscr{C}|_{G_j}$ and $\mathscr{C}|_{G_k}$ are compatible.

On the other hand, if for each j in J, $\mathscr{C}_j$ is a semigroup congruence on $(G_j, *)$, and for G_j, G_k in the same $\mathscr{L}$-class of $(G,\cdot)$, the congruences $\mathscr{C}_j$ and $\mathscr{C}_k$ are compatible, then the collection of all $\mathscr{C}_j$ determines uniquely a congruence $\overline{\mathscr{C}} \leq \mathscr{L}$ of the groupoid $(G,\cdot)$ such that $\overline{\mathscr{C}}|_{G_j} = \mathscr{C}_j$.

Proof. We observe that the restriction $\mathscr{C}|_{G_j}$ may be characterized by the following property.

$$(4.7)\quad \begin{cases} \text{If for } a,b \text{ in } G_j \quad a\ \mathscr{C}|_{G_j}\ b, \\ \text{then } \ a g_i\ \mathscr{C}|_{G_j}\ b g_i \ \text{ for each } i \text{ in } J \text{ with } i \neq j. \end{cases}$$

Now the first part of the proposition follows easily by this observation, the definition of $*$, commutativity of $*$ and the identity (L).

To prove the other part, let $(G,\cdot)$ be the sum of $\mathscr{L}$-classes G_i, i in I, and let L_i be the union of orbits G_k, where k is in $K \subseteq J$. Let $\overline{\mathscr{C}}_i$ be a relation on $(G,\cdot)$ defined as follows

$$(4.8)\qquad \overline{\mathscr{C}}_i|_{G_k} = \mathscr{C}_k \ \text{ for all } \ k \ \text{ in } K,$$

$$\overline{\mathscr{C}}_i|_{G-L_i} = \mathscr{L}|_{G-L_i}.$$

It is easy to see that $\overline{\mathscr{C}}_i$ is a congruence relation, whence

$$\overline{\mathscr{C}} = \cap(\overline{\mathscr{C}}_i \mid i \in I)$$

is also a congruence relation, satisfying the conditions of the proposition.■

By Proposition 4.5, a congruence relation $\mathscr{C}$ on a

LIR-groupoid $(G,\cdot)$ is determined by its restrictions to $\mathcal{L}$-classes L_i, i in I, and a corresponding partition of the set I. If $(G,\cdot)$ is a free LIR-groupoid, each $\mathcal{L}$-class L_i is equal to some orbit G_i and the restriction $\mathcal{C}|_{G_i}$ is just a congruence of the semigroup $(G_i,*)$. This fact makes it easy to describe the congruence lattice of $(G,\cdot)$.

Now for lattice L, let $L_{01}(L_1)$ be the lattice obtained from L by adding new least and greatest elements (a new greatest element). Let C_ω be a chain isomorphic to the chain of natural numbers with natural ordering, and N the lattice isomorphic to the lattice of positive integers with division as ordering relation. Now let $L = C_\omega \times N$. It was proved in [RR1] that the lattice of all varieties of LIR-groupoids is isomorphic to the lattice L_{01} and each nontrivial subvariety of LIR is defined by exactly one additional identity of the form

$$x\, y^{i+j} = xy^i.$$

Let us denote such a variety by $\underline{\underline{G}}_{i,j}$. On the other hand it was proved by Head [H] that the lattice of all varieties of commutative monoids is isomorphic to L_1 and each non-trivial variety is defined by one additional identity of the form

$$y^{i+j} = y^i.$$

Let us denote such a variety by $\underline{\underline{M}}_{i,j}$, and the variety of all commutative monoids by CM. It is easy to see that there is a one-to-one correspondence between the lattice of non-trivial subvarieties of LIR and the lattice of subvarieties of CM. In fact, using 4.3, 4.5 and 4.6, one could easily produce an alternative proof of the theorem about the lattice of subvarieties of LIR, by showing that the lattice of fully invariant congruences of the free groupoid on countably many generators in LIR (without the least element) is isomorphic to the lattice of fully invariant congruences of the free commutative monoid on countably many generators in CM.

5. Representation of LIR-groupoids by graphs

Decomposition of an LIR-groupoid $(G,\cdot)$ into a sum of commutative monoids suggest one more description and characterization of these groupoids, especially useful in the finite case.

To each LIR-groupoid $(G,\cdot)$ with $\mathcal{L}$-classes L_i, where i is in I, we assign a directed graph

$$\underline{G}(G) = (G, A(G))$$

with the set G of vertices and the set $A(G)$ of arcs labelled by elements of I. A triplet (a,b,i), where a,b are in G and i is in I, is in $A(G)$, if for any c in L_i, $b = ac$.

5.1. **Proposition.** The directed graph $\underline{G}(G)$ of an LIR-groupoid $(G,\cdot)$ has the following properties:

(i) $\underline{G}(G)$ has exactly $|I|$ connected components.

(ii) Each element a of L_i is an initial point of exactly $|I|$ arcs labelled by elements of I. Different arcs from a are labelled by different elements from I.

(iii) From each a in G and any j,k in I, the following diagram commutes:

$$\begin{tikzcd}
 & a \arrow[dl, "j"'] \arrow[dr, "k"] & \\
a\,g_j \arrow[dr, "k"'] & & a\,g_k \arrow[dl, "j"] \\
 & a\,g_j \circ g_k &
\end{tikzcd}$$

Now a directed graph (G,A) with labelled arcs is called an LIR-graph whenever:

(i) It is a disjoint sum of α connected components L_i, i in I, where $|I|=\alpha$.

(ii) Each point a of L_i is an initial point of exactly α arcs labelled by elements of I. Different arcs from a are labelled by different elements from I.

(iii) For each a in L_i and j,k in I the following diagram commutes

$$\begin{tikzcd}
 & a \arrow[dl, "j"'] \arrow[dr, "k"] & \\
\bullet \arrow[dr, "k"'] & & \bullet \arrow[dl, "j"] \\
 & \bullet &
\end{tikzcd}$$

To each LIR-graph (G,A) we assign an LIR-groupoid $\underline{L}(G) = (G,\cdot)$ as follows. For a_i and b_i in a connected component L_i of (G,A), we define

$$a_i \cdot b_i = a_i$$

For b_j and c_j in L_j, we define

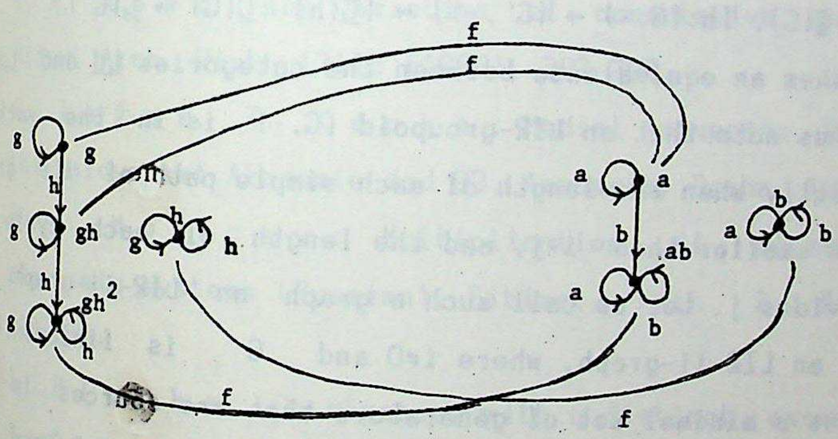
$$a_i \cdot b_j = a_i \cdot c_j = d_i,$$

where (a_i, d_i, i) is in A. It is easy to see that the groupoid $\underline{L}(G)$ we obtain is an LIR-groupoid, and the L_i are $\mathcal{L}$-classes of $\underline{L}(G)$. The proof of the following theorem is obvious.

5.2 **Theorem.** (i) If $(G,\cdot)$ is an LIR-groupoid, then $\underline{G}(G)$ is an LIR-graph, and moreover $\underline{L}(\underline{G}(G))$ is isomorphic to $(G,\cdot)$.

(ii) If (G,A) is an LIR-graph, then $\underline{L}(G)$ is an LIR-groupoid, and moreover $\underline{G}(\underline{L}(G))$ is isomorphic to (G,A).■

It can be easily verified that if $h:G \to G'$ is a homomorphism between LIR-groupoids $(G,\cdot)$ and $(G',\cdot)$, then h is a homomorphism between the graphs $\underline{G}(G)$ and $\underline{G}(G')$. However the converse need not be true. For example, if $G = \{g, gh, gh^2, h\}$ and $G' = \{a, ab, b\}$, and the corresponding graphs are as follows



then the mapping f indicated on the picture is a graph homomorphism, but not a groupoid homomorphism. (Indeed, $(gh)f = a$, but $gf \cdot hf = ab \neq a$.) Now we describe some special homomorphisms of LIR-graphs that induce homomorphisms of corresponding LIR-groupoids. Let $f:(G,A) \to (G',A')$ be a homomorphism of LIR-graphs. Let a,b be in G, af in a connected component L_i and bf in a connected component L_j of (G',A'). The homomorphism f is called an LIR-homomorphism, if the arc from af to $(ab)f$ is labelled by j. Now if f is a LIR-homomorphism, then f is a homomorphism from the groupoid $(G,\cdot)$ into the groupoid $(G',\cdot)$. Indeed, since f is a graph homomorphism, $(ab)f$ is in L_i, and since the arc from af to $(ab)f$ is labelled by j, $(ab)f = af\cdot c$ for each element c in L_j. In particular $(ab)f = af\cdot bf$.

Let Ld denote the category of LIR-groupoids and their homomorphisms, and Lph the category of LIR-graphs and LIR-homomorphisms.

5.4. Theorem. The functor $\underline{G}$: Ld → Lph given by $(G,\cdot) \to \underline{G}(G)$, $[h:(G,\cdot) \to (G',\cdot)] \to [\underline{G}(h) : \underline{G}(G) \to \underline{G}(G')]$ establishes an equivalence between the categories Ld and Lph.∎

Let us note that an LIR-groupoid $(G,\cdot)$ is in the variety $\underline{\underline{G}}_{i,j}$ exactly when the length of each simple path of its graph $\underline{G}(G)$ is smaller than $i+j$, and the length of each cycle in $\underline{G}(G)$ divides j. Let us call such a graph an LIR-ij-graph. If $\underline{G}(G)$ is an LIR-ij-graph, where $i \neq 0$ and G is finite, then $(G,\cdot)$ has a minimal set of generators that are sources in the graph $\underline{G}(G)$. It is easy to see that both Theorems 5.2 and 5.4 remain true if we replace LIR-groupoids by LIR-groupoids in the variety $\underline{\underline{G}}_{i,j}$ and LIR-graphs by LIR-ij-graphs.

More generally, IR-groupoids have a graph representation very similar to that of LIR-groupoids, though the orbits are no longer commutative monoids. IR-graphs are defined similarly to LIR-graphs; we only drop the condition (iii) of the definition. Both Theorems 5.2 and 5.4 hold true in the case of IR-groupoids and IR-graphs as well.

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REFERENCES

[G] G. Grätzer : Universal Algebra, 2nd edition, Springer Verlag, New York, 1979.

[H] T. J. Head : The varieties of commutative monoids, Nieuw. Arch. Wisk. (3)16 (1968), 203-206.

[JK] J. Ježek, T. Kepka : Medial groupoids, Rozpravy Ceskoslovenské Akademie Ved 93, Academia, Praha, 1983.

[M] A. I. Mal'cev : Multiplication of classes of algebraic systems (Russian), Sibirsk. Math. Z. 8(1967), 346-365.

[P1] J. Płonka : On algebras with n distinct essentially n-ary operations, Algebra Universalis 1(1971), 73-79.

[P2] J. Płonka : On k-cyclic groupoids, Math. Japonica 30 (1985), 371-382.

[R] B. Roszkowska : The lattice of varieties of symmetric idempotent entropic groupoids, to appear in Demonstratio Math.

[RR1] A. Romanowska, B. Roszkowska : On some groupoid modes, to appear in Demonstratio Math.

[RR2] A. Romanowska, B. Roszkowska: Representations of n-cyclic groupoids to appear in Algebra Universalis.

[RS] A. Romanowska, J. D. H. Smith: Modal Theory - an algebraic approach to order, geometry and convexity, Helderman Verlag, Berlin, 1985.

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STABILIZED SEQUENCES OF ALGEBRAS

*Dedicated to the memory
of Professor Edward Otto*

1. Introduction

Stabilized sequences of algebras are introduced as a generalization of increasing sequences of subalgebras of a given algebra. Using the fact that the limit of such a sequence, say (A_n), belongs to the variety generated by all the A_n we prove that every modular ortholattice of length 3 belongs to the variety generated by all finite orthomodular lattices.

Although our construction of the limit of stabilized sequences of algebras is used for to study the lattice of subvarieties of the variety of orthomodular lattices, we hope that it will be useful in studying lattices of other varieties as well.

2. Stabilized sequences of algebras

Definition 1. Let $\mathcal{A}_n = \langle A_n , \{f_{n,t} : t\in T\} \rangle$ be a sequence of algebras of the same type τ. The sequence $(\mathcal{A}_n)$ is said to be stabilized if the following two condi-tions are satisfied :

(SSA1) $\forall n \ \ A_n \subseteq A_{n+1}$.

(SSA2) $\forall k \ \ \forall t \in T \ \ \forall i \ \ \forall x_1,\dots,x_k \in A_i \ \ \exists m \ \ \forall n \geq m$

$$f_{n,t}(x_1,\dots,x_k) = f_{m,t}(x_1,\dots,x_k) .$$

An algebra $\mathcal{A} = \langle A , \{f_t : t \in T\} \rangle$ of type τ is said to be a limit of the sequence ($\mathcal{A}_n$) if the following two conditions hold :

(LS1) $A = \bigcup_{n=1}^{\infty} A_n$,

(LS2) $\forall k \ \ \forall t \in T \ \ \forall i \ \ \forall x_1,\dots,x_k \in A_i$

$$f_t(x_1,\dots,x_k) = f_{m,t}(x_1,\dots,x_k)$$

where m is given by the above condition (SSA2) .

Clearly, for any stabilized sequence of algebras there exists a unique limit. If, for any natural number n, A_n is a subalgebra of A_{n+1} then the sequence is stabilized. Another example is given by the sequence $(\mathcal{Z}_n) = (\langle Z_n , + , \cdot \rangle)$ The limit of the sequence $(\mathcal{Z}_n)$ is equal to $\langle N , + , \cdot \rangle$.

Proposition 1. Let $\mathcal{A}$ be a limit of stabilized sequence $(\mathcal{A}_n)$ and let F be a nontrivial ultrafilter on N. Then $\mathcal{A}$ is isomorphic to a subalgebra of the ultraproduct $U = {}^{\prod \mathcal{A}_n}/_{F}$.

Proof. Let x be a fixed element of A_1. If $a \in A$ then there exists $i \in N$ such that $a \in A_i$. Put

$$h (a) = \Big[(x , \dots , x , a , a , a , \dots\dots) \Big]_F$$

where x occurs i-1 times. A simple calculation shows that :

1) h is well-defined map from A to U ,

2) h is an injection .

Then we have to prove that h preserves any operation f_t , i.e. that for any k-ary f_t and any $a_1,\dots,a_k \in A$ the equality

(P) $h(\,f_t(a_1,\dots,a_k)\,) = f_t(\,h(a_1),\dots,h(a_k)\,)$ holds,

where f_t denotes an operation on U as in the definition 1.

Let $a_1,\dots,a_k \in A$ and let f_t be k-ary operation on A. It follows from Definition 1 that :

$$\exists\, m \ \forall\, n \geq m \quad f_{n,t}(a_1,\dots,a_k) = f_t(a_1,\dots,a_k)$$

and, of course, $a_1,\dots,a_k \in A_m$. Put :

$$y_i = f_{i,t}(x,\dots,x)\ , \quad i = 1,2,\dots,m-1$$

$$z = f_t(a_1,\dots,a_k)$$

Now $h(\,f_t(a_1,\dots,a_k)\,) = h(z) = \left[\,(x,\dots,x,z,z,z,\dots\dots)\,\right]_F$,

$h(a_i) = \left[\,(x,\dots,x,a_i,a_i,a_i,\dots\dots)\,\right]_F$, $i = 1,2,\dots,k$,

where x occurs m-1 times. On the other hand :

$f_t(\,h(a_1),\dots,h(a_k)\,) = \left[\,(y_1,\dots,y_{m-1},z,z,z,\dots\dots)\,\right]_F =$

$= \left[\,(x,\dots,x,z,z,z,\dots\dots)\,\right]_F$.

Therefore the equality (P) holds. Thus h is a monomorphism from A to U.

Corollary. The limit $\mathcal{A}$ of stabilized sequence of algebras $\mathcal{A}_n$ belongs to the variety generated by them.

3. Greechie diagrams for orthomodular lattices of length 3

Now we study orthomodular lattices of length 3. It means that every maximal chain contains exactly 4 elements. A Greechie diagram (see [1]) associated with L consists of a set of points and a set of lines. Every line is a nonempty set of points. The points are in one-to-one correspondence with the atoms of L; the lines are in one-to-one corres-pondence with the bloks (maximal Boolean subalgebras) of L.

A family G consisting of points and lines is a Greechie diagram of an orthomodular lattice of length 3 iff the following conditions hold :

(G1) every point lies on (belongs to) some line ,

(G2) every line contains exactly 3 points ,

(G3) every two different lines are either disjoint or neighbouring, i.e. having a one-point itersection ,

(G4) triangles and quadrangles do not occur in the diagram G .

For details see [1] .

Next Proposition gives a necessary and sufficient condition for a Greechie diagram to be diagram of a modular ortholattice of length 3 .

Proposition 2. Let L be an orthomodular lattice of length 3 with the associated Greechie diagram G. Then the following conditions are equivalent :

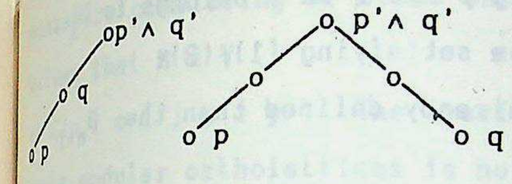
1) L is modular .

2) The join a v b of any two atoms is strictly less then 1.

(G5) Any two points of the Greechie diagram of L lie eitheron the same line or on neighbouring lines.

The proof is straightforward and we omit it.

Both situations described in (G5) are depicted below:



Two neighbouring lines in the right picture will be called an arm connecting p and q. It follows from (G4) that there is exactly one such an arm .

4. Modular limits of orthomodular lattices

Let us denote by MOL the variety of all modular ortholattices, by [FOML] the variety generated by all finite orthomodular lattices, and by $[MO_\omega]$ the variety generated by all finite modular ortholattices. The natural question arises : is the equation $[MO_\omega] = [FOML] \wedge MOL$ true ? We present here a negative answer to this question (Corollary of Theorem 1) .

L e m m a 1. Let $\mathcal{L} = \langle L, \vee, \wedge, ', 0, 1 \rangle$ be a modular ortholattice of length 3 generated by a finite set $P = \{ p_1, \dots, p_n \}$, $n \geq 3$, of (distinct) atoms. Then $\mathcal{L}$ is a limit of stabilized sequence of finite orthmodular lattices.

P r o o f . Let D be a Greechie diagram associated with $\mathcal{L}$. We define by induction a sequence (D_n) of Greechie diagrams. The diagram D_1 is defined as follows :

(1) every element of P is a point of D_1 ,

(2) if $p, q \in P$ are orthogonal atoms in L then the atom $p' \wedge q'$ is a point of D_1 and the line $\{ p, q, p' \wedge q' \}$

from D belongs to D_1 ,

(3) if $p, q \in P$ are different, non-orthogonal atoms in $\mathcal{L}$ then the arm connecting p and q in D belongs to D_1 ,

(4). D_1 is a minimal diagram satisfying (1)-(3).

If the Greechie diagram D_n is already defined then the D_{n+1} is constructed as follows :

(D1) $D_n \subseteq D_{n+1}$,

(D2) if $p, q \in D_n$ are orthogonal in $\mathcal{L}$ then the atom $p' \wedge q'$ is a point of D_{n+1} and the line $\{ p, q, p' \wedge q' \}$ from D belongs to D_{n+1},

(D3) if $p, q \in D_n$ are different, non-orthogonal atoms in $\mathcal{L}$ then the arm connecting p and q in D belongs to D_{n+1} ,

(D4) D_{n+1} is a minimal diagram satisfying the above conditions

It follows immediately from the above construction that D_n is a Greechie diagram of a finite orthomodular lattice $\mathcal{L}_n = \langle L_n , \vee , \wedge , ' , 0 , 1 \rangle$, and we can assume that $L_1 \subseteq L_2 \subseteq \ldots \subseteq L$. Now if $a \in L_n$ then $a' \in L_n$, too. For any $a, b \in L_n$ the join $a \vee b$ and the meet $a \wedge b$ in $\mathcal{L}$ belong to L_{n+1} . This shows that $L = \bigcup_{n=1}^{\infty} L_n$ and $(\mathcal{L}_n)$ is stabilized sequence of algebras (Put m = i+1 in the con-dition (SSA2)) . Thus $\mathcal{L}$ is a limit of the sequence $(\mathcal{L}_n)$.

Theorem 1. Let $\mathcal{L} = \langle L , \vee , \wedge , ' , 0 , 1 \rangle$ be a modular ortholattice of length 3. Then $\mathcal{L}$ belongs to the variety [FOML] generated by all finite orthomodular lattices.

Proof . It is sufficient to prove that every finitely generated subalgebra M of $\mathcal{L}$ belongs to V. If M is finite then M belongs to V. Otherwise M is of length 3. Let S be a set of generators of M. We can assume that $0, 1 \notin S$. Replacing every coatom $p' \in S$ by an atom p we get another finite set, say P_1 of generators consisting of atoms only. By Lemma 1 and Corollary it follows that $M \in V$.

Corollary. The variety $[MO_\omega]$ generated by all finite modular ortholattices is not equal to an intersection of the variety MOL of all modular ortholattices and the variety [FOML] generated by all finite orthomodular lattices.

Proof. It is a well known fact that every subdirectly irreducible algebra from $[MO_\omega]$ has length 2. On the other hand the modular ortholattice $\mathcal{L}(\mathbb{R}^3)$ of all subspaces of the Hilbert space $\mathbb{R}^3$ is subdirectly irreducible and has length 3. Thus $\mathcal{L}(\mathbb{R}^3)$ belongs to the intersection of the MOL and the [FOML], but does not belongs to the $[MO_\omega]$.

REFERENCES

[1] G. Kalmbach: Orthomodular lattices, Academic Press, London 1983.

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Janusz Wąsowski

SOME REMARKS ON THE BALANCED SITUATION IN THE THEORY OF CONFLICTS

Dedicated to the memory
of Professor Edward Otto

Introduction

The notion of the balanced situation in the theory of conflicts is introduced in [1] and investigated in [2],[3]. Such situation exists iff some system of linear algebraic equations as a non-negative solution. The necessary and sufficient condition for the existence of such a solution, introduced in 2], is investigated. The notion of stable and unstable balanced situation is introduced and investigated.

Let us quote here some definitions from [3]. Let X be a inite set with card $X = n \geq 2$. Elements of X will be called bjects.

By

$$\varphi : X^2 \mathrel{-\!\!\!\circ\!\!\!\rightarrow} \{-1,+1\}$$

denote a partial function describing relation between objects $y \in X$: a conflict iff $\varphi(x,y) = -1$, a cooperation iff $\varphi(x,y) = +1$ a neutrality iff $(x,y) \in X^2 - D\varphi$, where $D\varphi$ denotes the domain of . We assume that $D\varphi$ is a reflexive and symmetric relation on

X, $\varphi(x,x)=+1$ for every $x\in X$ and $\varphi(x,y)=\varphi(y,x)$ for every $(x,y)\in D\varphi$. Each function φ defines three disjoint binary relations on X, denoted by

$$R^{+}, R^{-}, R^{0}$$

and defined as follows: for every $(x,y)\in D\varphi$

$$R^{+}(x,y) \text{ iff } \varphi(x,y)=+1,\quad R^{-}(x,y) \text{ iff } \varphi(x,y)=-1$$

and $R^{0}(x,y)$ for every $(x,y)\in X^2-D\varphi$. The relations R^{+}, R^{-}, R^{0} are called the alliance, conflict and neutrality relations, respectively.

By

$$\mu\colon X\longrightarrow\langle 0,+\infty)$$

we denote a function describing strengths of objects $x\in X$.

The triple $S=(X,\varphi,\mu)$ is called the situation (of X). The situation $S=(X,\varphi,\mu)$ is called the balanced situation iff there exists a function λ,

$$\lambda\colon X^2\longrightarrow\langle 0;+\infty)$$

called the balanced strategy in S, such that

$$(1)\quad \begin{cases} \lambda(x,y)=0 & \text{for every } (x,y)\in X^2-D\varphi \\ \lambda(x,y)>0 \Rightarrow \varphi(x,y)=-1 & \text{for every } (x,y)\in D\varphi \\ \sum\limits_{y\in X}\lambda(x,y)=\mu(x) & \text{for every } x\in X \\ \lambda(x,y)=\lambda(y,x) & \text{for every } x,y\in X. \end{cases}$$

The balanced strategy exists iff the system of linear algebraic equations with the unknowns $\lambda(x,y)$ corresponding to conditions (1) has a non-negative solution.

Without loss of generality we can assume that

$$\forall(x\in X)\ \ \mu(x)>0,$$

$$\forall(x \in X)\ \exists(y \in X)\quad \varphi(x,y) = -1.$$

1. Existence of a balanced strategy

Let $X = \{x_1,\ldots,x_n\}$. We shall use the following notation :

$N_n = \{1,\ldots,n\}$, $\mu_i = \mu(x_i)$ for $i \in N_n$, $C = \{(i,j) : \varphi(x_i,x_j) = -1\}$, $\lambda_{ij} = \lambda(x_i,x_j)$ for $(i,j) \in C$.

The situation $S = (X,\varphi,\mu)$ is balanced iff system of linear algebraic equations

$$\sum_{j:(i,j)\in C} \lambda_{ij} = \mu_i \qquad \text{for every } i \in N_n$$

$$\lambda_{ij} = \lambda_{ji} \qquad \text{for every } (i,j) \in C$$

with the unknowns λ_{ij} for $(i,j)\in C$, has a non-negative solution.

Now we define family $\mathfrak{I}$ of non-empty subsets of 2^{N_n} as follows:

$$I \in \mathfrak{I} \Leftrightarrow \forall(i,j \in I)\ (i,j) \notin C.$$

Moreover, for every $I \in \mathfrak{I}$ we define the set

$$I_C = \{i \in N_n : \exists(j \in I)\ (i,j) \in C\}.$$

Obviously, we have for any $i,j \in N_n$

$$\{i\} \in \mathfrak{I}\ ,\ (i,j) \notin C \wedge i \neq j \Rightarrow \{i,j\} \in \mathfrak{I}$$

and

$$I\in\mathfrak{I} \wedge K\subset I \Rightarrow K\in\mathfrak{I}\ ,\ I\in\mathfrak{I} \Rightarrow I\cap I_C = \emptyset.$$

Theorem 1. A balanced strategy exists in the situation $S = (X,\varphi,\mu)$ iff the strength μ fulfills the following condition

(2)

$$\sum_{i\in I} \mu_i \le \sum_{i\in I_C} \mu_i \qquad \text{for every } I\in\mathfrak{I}.$$

The proof of this theorem will be preceded by the following lemma.

Lemma 1. For any real numbers $x_1,\dots,x_k$ and $y_1,\dots,y_k$ such that $0<y_k\le y_{k-1}\le\dots\le y_1$ we have

$$\sum_{j=1}^{k} x_j y_j \ge 0 \quad \text{iff} \quad \sum_{j=1}^{r} x_j \ge 0 \quad \text{for every } r=1,\dots,k.$$

Proof. (⇒) Let $1\le r<k$. Let us $y_i = 1$ for $i=1,\dots,r$ and $y_i=\varepsilon$ $(0<\varepsilon<1)$ for $i=r+1,\dots,k$. By assumption $\sum_{j=1}^{r} x_j + \varepsilon\cdot\sum_{j=r+1}^{k} x_j \ge 0$ for any $0<\varepsilon<1$, therefore $\sum_{j=1}^{r} x_j \ge 0$. For $r=k$, it is sufficient to set y_i for all $i=1,\dots,k$. (⇐). The truth of the implication follows from the Abel's inequality ([5],p.34.).

Proof of Theorem 1. Let Y denote the set of all points $y=(y_1,\dots,y_n)$ such that

(3) $\qquad y_i+y_j\ge 0 \qquad$ for every $(i,j)\in C$.

In [2] it was shown that a balanced strategy exists in the situation S iff the function

$$f(y)=\sum_{i=1}^{n}\mu_i\cdot y_i$$

is non-negative in the set Y. For every $K\subset N_n$ we define the set Y_K as follows:

$$Y_K=\{(y_1,\dots,y_n)\in Y : \forall(i\in N_n)\ y_i<0 \Leftrightarrow i\in K\} \quad \text{for } K\neq\emptyset$$

and additionally

$$Y_\emptyset=\{(y_1,\dots,y_n)\in Y : \forall(i\in N_n)\,y_i\ge 0\}.$$

We have

$$Y=\bigcup_{K\subset N_n} Y_K \quad\text{and}\quad K\neq\emptyset \Rightarrow (Y_K\neq\emptyset \Leftrightarrow K\in\mathfrak{J}).$$

Obviously, for any strength μ the function $f(y)$ is non-negative in the set $Y_\emptyset$.

Let $K\in\mathfrak{I}$ and let card $K=k$. From (3) it follows that Y_K is the set of points $(y_1,\dots,y_n)$ satisfying conditions:

$$y_i < 0 \qquad \text{for every } i\in K$$

$$y_j \ge \max_{i\in K\cap\{j\}_C} |y_i| \qquad \text{for every } j\in K_C$$

$$y_j \ge 0 \qquad \text{for every } j\notin K\cup K_C.$$

Therefore the function $f(y)$ is non-negative in the set Y_K iff

$$(4)\qquad -\sum_{i\in K}\mu_i\cdot|y_i| + \sum_{j\in K_C}\mu_j\cdot\max_{i\in K\cap\{j\}_C}|y_i|\ge 0$$

for any negative value $y_i, i\in K$.

Let $y_i<0$ for $i\in K$ and let $(i_1,\dots,i_k)$ be the permutation of elements of the set K such that $|y_{i_k}|\le\dots\le|y_{i_1}|$. We define the sets

$$A_{i_1} = \{j\in K_C : (i_1,j)\in C\},$$

$$A_{i_2} = \{j\in K_C : (i_2,j)\in C \wedge j\notin A_{i_1}\},$$

$$\vdots$$

$$A_{i_k} = \{j\in K_C : (i_k,j)\in C\wedge j\notin \bigcup_{r=1}^{k-1} A_{i_r}\}.$$

Now the condition (4) can be written in the following form

$$\sum_{r=1}^{k} |y_{i_r}|\cdot\Big(\sum_{j\in A_{i_r}}\mu_j-\mu_{i_r}\Big)\ge 0.\ ^{1)}$$

Moreover, for every $s=1,\dots,k$

$$\{i_1,\dots,i_s\}\in\mathfrak{I},\ \{i_1,\dots,i_s\}_C=\bigcup_{r=1}^{s} A_{i_r}.$$

---

1) If the set of summation indexes is empty, then the corresponding sum is equal to 0.

From Lemma 1 we obtain, that the function $f(y)$ is non-negative in the set Y_K iff inequality in (2) is satisfied for every non-empty set $I \in 2^K$.

Considering all the sets $K \in \mathfrak{I}$ we obtain (2) as the necessary and sufficient condition of the existence of the balanced strategy in the situation S. The proof is complete.

Therefore, the situation $S=(X,\varphi,\mu)$ is the balanced situation iff the inequality in (2) is satisfied for every $I \in \mathfrak{I}$. It can be shown, that if φ satisfies some additional conditions, then the inequality in (2) does not have to be tested for some sets from $\mathfrak{I}$. Now we shall formulate two such conditions, which will be used in the future.

I. Let $\tilde{I}, I \in \mathfrak{I}$. If $\tilde{I} \subset I$ and $\tilde{I}_C = I_C$ then the inequality in (2) is satisfied for $\tilde{I}$, provided it is satisfied for I.

II. Let $N_n = A \cup B$, with $A, B \neq \emptyset$ and $A \cap B = \emptyset$. If for any $i \in A$ and for any $j \in B$ $(i,j) \in C$ then it is sufficient to test the inequality in (2) only for $I \in \mathfrak{I}$ such that $I \subset A$ or $I \subset B$. Indeed, for the remaining $I \in \mathfrak{I}$ the set I_C is the sum of the separable sets I'_C and I''_C, where $I' = I \cap A$, $I'' = I \cap B$.

2. Stable and unstable balanced situations

Let $\mathfrak{B}$ denote the family of all balanced situation for given X and φ.

Definition. The situation $(X,\varphi,\mu) \in \mathfrak{B}$ we called the stable balanced situation (with respect to μ) iff

$$\exists(\varepsilon>0)\forall(|\delta|<\varepsilon)\ \forall(x' \in X)\ (X,\varphi,\mu') \in \mathfrak{B}$$

where $\mu'(x') = \mu(x')+\delta$, $\mu'(x) = \mu(x)$ for every $x \neq x'$. Otherwise, S we called the unstable balanced situation (with respect to μ).

By Theorem 1 we obtain

Lemma 2. Let $S = (X,\varphi,\mu) \in \mathfrak{B}$.

a) S is an unstable balanced situation iff

$$\exists(I\in\mathfrak{I}) \quad \sum_{i\in I}\mu_i = \sum_{i\in I}\mu_i .$$

b) S is a stable balanced situation iff

$$\forall(I\in\mathfrak{I}) \quad \sum_{i\in I}\mu_i < \sum_{i\in I_C}\mu_i .$$

The situation $S=(X,\varphi,\mu)$ is called the simple situation iff there exists a non-empty set $X_1 \subsetneq X$ such that $X_1^2 \cup (X-X_1)^2 \subset R^+ \cup R^0$.

Let $S = (X,\varphi,\mu) \in \mathfrak{B}$ and let Λ denotes the set of all balanced strategies in S. We denote $C^* = \{(i,j) \in C : \forall(\lambda\in\Lambda)\lambda_{ij} = 0\}$ and $\varphi^* = \varphi|D\varphi^*$, where $D\varphi^* = D\varphi - \{(x_i,x_j) \in X^2 : (i,j) \in C^*\}$.

Theorem 2. Let $S=(X,\varphi,\mu)\in\mathfrak{B}$. If the situation S is an unstable balanced situation then there exists a non-empty set $X_1 \subseteq X$ such that

$(X_1, \varphi^*|X_1^2, \mu|X_1)$ is a simple situation

(5) $\forall(x\in X_1)\forall(y\in X-X_1)\ \varphi^*(x,y) \neq -1$, if $X_1 \neq X$.

Proof. From Lemma 2a there exists a set $I\in\mathfrak{I}$ such that $\sum_{i\in I}\mu_i = \sum_{i\in I_C}\mu_i$. For this set, we have

$$(i,j)\in C \wedge (i,j)\in(N_n-I)^2-(N_n-(I\cup I_c))^2 \Rightarrow (i,j)\in C^*.$$

Therefore the situation $(X_1, \varphi^*|X_1^2, \mu|X_1)$, where $X_1 = \{x_i \in X : i \in I \cup I_C\}$, is a simple situation. Moreover, if $X_1 \neq X$ then the function φ^* satisfies the constrain (5).

Remark. For any X and φ, the set of all points

$(\mu_1,\dots,\mu_n)$ such that (X,φ,μ) is a balanced situation is a convex cone defined by the inequality (2) and $\mu_i>0$ for every $i\in N_n$. The geometric interpretation of definitions introduced in this points is as follows: (X,φ,μ) is an unstable (stable) balanced situation iff $(\mu_i,\dots,\mu_n)$ is an edge (interior) point of this cone. Instead, in Theorem 2 the constrains have been formulated, to which the function φ is always subject, when (X,φ,μ) is an unstable balanced situation.

3. Special cases of a situations

The situation $S = (X,\varphi,\mu)$ is called a total conflict situation iff $D\varphi=X^2$ and $\varphi(x,y)=-1$ for every $x,y\in X$ such that $x\neq y$. For such a situation Theorem 1 was shown in [2]. In this case

$$C = N_n^2-\{(i,i):\ i\in N_n\},$$

$$\mathfrak{I}=\left\{\{1\},\{2\},\dots,\{n\}\right\},\quad \{i\}_C= N_n-\{i\}\quad \text{for } i\in N,$$

hence the constrain (2) has a form

(6) $$\max_{i\in N}\ \mu_i\le\omega$$

where $\omega=\frac{1}{2}\sum_{j=1}^{n}\mu_j$.

From (6) and Lemma 2 we obtain, that the total conflict situation $S=(X,\varphi,\mu)\in\mathfrak{B}$ is unstably balanced iff $\max_{i\in N_n}\mu_i=\omega$ and it is stably balanced iff $\max_{i\in N_n}\mu_i<\omega$.

In what follows we shall deal with the regular relations R^+,R^- (see[4]). In [4] it has been proved that the relations R^+,R^- are regular iff there exists non-empty sets $X_1,\bar{X}_1,\dots,X_k,\bar{X}_k$ $(1\le k\le n/2)$ generating the partition of X such that

$$R^{+} = \bigcup_{i=1}^{k} (X_i^2 \cup \overline{X}_i^2) , \quad R^{-} = \bigcup_{i=1}^{k} (X_i \times \overline{X}_i \cup \overline{X}_i \times X_i). \tag{7}$$

Let R^{+}, R^{-} will be a regular relations and let $X_1, \overline{X}_1, \ldots, X_k, \overline{X}_k$ will a partition of X satisfying (7). We shall denote $I_i = \{j: x_j \in X_i\}$, $\overline{I}_i = \{j: x_j \in \overline{X}_i\}$ for $i=1,\ldots,k$.

Theorem 3. A balanced strategy exists in the situation $S = (X, \varphi, \mu)$ iff

$$\sum_{j \in I_i} \mu_j = \sum_{j \in I_i} \mu_j \qquad \text{for every } i=1,\ldots,k. \tag{8}$$

Every balanced situation S is an unstable balanced situation.

Proof. From (6) it follows, that for $i=1,\ldots,k$

$$I_i, \overline{I}_i \in \Im \, , \ (I_i)_C = \overline{I}_i \ , \ (\overline{I}_i)_C = I_i .$$

Moreover, from the properties I, II it follows that it is enough to consider the inequality in (2) only for $I \in \{I_1, \overline{I}_1, \ldots, I_k, \overline{I}_k\}$. Then the obtained conditions take the form (8). From (8) and Lemma 2a it follows, that S is an unstable balanced situation.

REFERENCES

[1] Z. Pawlak : On conflicts, Intern. Journal of Man-Machine Studies, 21 (1984), 127-134.

[2] J. Wasowski : Existence of the balanced strategy in theory of conflicts, Acad. Polon. Sci. Math., 35(1987), 535-537.

[3] W. Zakowski: Investigation of the balanced situation in theory of conflicts, Bull. Acad. Polon. Sci. Math., 32 (1984), 379-382.

[4] W. Zakowski: On new characterization of regular configurations in theory of conflict situations, Demonstratio Math., 17(1984), 211-218.

[5] D.S. Mistrinović: Elementarne nierówności, PWN, Warszawa, 1972.

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SOME REMARKS ON SAFETY IN CONFLICT THEORY

Dedicated to the memory of Professor Edward Otto

1. Introduction

The notion of a conflict and a conflict situation was introduced by Z.Pawlak [Pa84]. Some basic results on this subject are established in [Za84a] and [Za84b]. The paper deals with some different approach to conflictness. As a starting point we assume that the primary cause of most conflicts lies in unequal distribution of strength in some group (of human beings, trading organizations, etc). Some members of the group can form coalition against the others and try to destroy them. But it can happen if two conditions are satisfied, namely:

(1) the coalition is stronger then the enemies

(2) an individual decide to participate in a given coalition only if he feels safe before and after the conflict.

To illustrate the notion of safety, let us consider the situation with three individuals x_1, x_2, x_3 representing the strength of 3, 2 and 2. There exists three possible coalitions $\{x_1, x_2\}$, $\{x_1, x_3\}$ and $\{x_2, x_3\}$, but only the third two can be called safe. In other cases after the first conflict (with proportional partition of a capture [Za84b]) the participant x_1 can and will attack his previous partner. Therefore x_2 (or x_3) does not participate in the coalition with x_1, so there will be no conflict.

The paper deals with the mathematical notion of a safe coalition. Some necessary and some sufficient conditions for coalitions to be safe are given.

2. Basic notions and definitions

Let $n \geq 2$ and $X = \{x_1, \ldots, x_n\}$ be a finite set. Elements x_i of X, $i = 1, \ldots, n$, will be called *objects*.

Let $\phi\colon X^2 -\!\square\!\!\longrightarrow \{-1,1\}$ be a partial function describing relations between object; for $x,y \in X$ we shall say that object x,y are:

- *in conflict* iff $\phi(x,y) = -1$
- *allied* iff $\phi(x,y) = 1$
- *neutral* iff $(x,y) \in X^2 - D\phi$,

where $D\phi$ denotes the domain of ϕ.

We assume also that ϕ is a reflexive and symmetric relation on X^2 and for any $x \in X$ $\phi(x,x) = 1$.

The *strength* of an object $x \in X$ is describing by a function μ, $\mu : X \longrightarrow \langle 0, +\infty)$.

The ordered triple $S = (X, \phi, \mu)$ is called a *situation* (of X). If S is a fixed situation, $S = (X, \phi, \mu)$, then we will write X_S, ϕ_S, μ_S, respectively.

Let S be a fixed situation satisfying the conditions:

(i) for any $x,y \in X$, $x \neq y$, if only $(x,y) \in D\phi$ then $\phi(x,y) = -1$.

(ii) $\mu(x) > 0$ for any $x \in X$.

Definition 2.1. The family of situations $\mathfrak{B}$ is called *the strength formation* with respect to the situation $S = (X,\phi,\mu)$ iff $\mathfrak{B}$ is the family of all situations (X,ϕ',μ) such that $\phi'|_{D\phi} = \phi$ (i.e. for any $(x,y)\in D\phi$ $\phi'(x,y) = \phi(x,y)$). The strength formation with respect to the situation $S = (X,\phi,\mu)$ will be denoted by $(X,\mu)_\phi$.■

Let Y be a proper subset of X.

Definition 2.2. The set Y of objects can arrange *an aggressor coalition* in the situation S iff

(2.2.1)
$$\sum_{x\in X} \mu(x) > \sum_{x\in X-Y} (x)$$

and

(2.2.2) there exists a function $\phi_Y \colon X^2 \longrightarrow \{-1,1\}$ such that

(i) $S_Y = (X,\phi_Y,\mu) \in (X,\mu)_\phi$

(ii) $$\phi_Y(x,y) = \begin{cases} 1 & (x,y)\in Y^2 \\ -1 & (x,y)\in [Y \times (X-Y) \cup (X-Y) \times Y]. \end{cases}$$■

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In that case the situation S_Y is called *a conflict situation* (cf.Za84b). It can be interpreted as follows: the elements of Y are able to form a coalition against the other elements of X and win the conflict. From (2.2.2) it follows that an aggressor coalition can be only formed by an Y such that enemies do not belong to it, i.e.

$$\text{card } [(Y^2 - \{(x,x)\ x\in X\}) \cap D\phi] = 0.$$

In the other words: objects x,y being enemies (i.e. $\phi(x,y) = -1$) can not belong simultaneously to the aggressor coalition Y.

Remark 2.1. Let $n\geq 3$. For any situation $S = (X,\phi,\mu)$ such that $D_\phi = \{(x,x)\colon x\in X\}$ there can be formed an aggressor coalition. Without the assumption about D_ϕ it may happen that the aggressor coalition can't be constituted in the given situation.■

Let us assume that after a conflict the strengths of defeated objects are equal to zero, so they do not exist any more. Therefore the conflict situation (X,ϕ_Y,μ) can be transformed (by realization of the conflict described by ϕ_Y) into the new situation $S' = (X',\phi',\mu')$ defined as follows:

(i) $X' = Y$

(ii) $D_{\phi'} = \{(x,x),\ x \in X\}$

(iii) $\sum_{x\in Y} \mu'(x) \geq \sum_{x\in Y} \mu(x) - \sum_{x\in X-Y} \mu(x)$

(iv) $\forall x,y \in Y\ \dfrac{\mu'(x)}{\mu(x)} = \dfrac{\mu'(y)}{\mu(y)}$.

We shall denote the above transformation by:

$$(X,\phi_Y,\mu) \xrightarrow[Y]{} (X',\phi',\mu').$$

From (i), (iii) and (iv) it is obvious that $\mu'(x) > 0$ for $x\in Y$ (but not necessary $\mu'(x) > \mu(x)$).

The sign "=" in (iii) (if happens) means that the winners do not get any capture (cf. [Za84b]).

Definition 2.3. A situation (X',ϕ',μ') is a *direct result* of the situation $S = (X,\phi,\mu)$ iff there exists an aggressor coalition Y in S such that

$$(X,\phi,\mu) \xrightarrow[Y]{} (X',\phi',\mu').$$

We shall denote it by $(X,\phi,\mu) \longrightarrow (X',\phi',\mu')$. The reflexive and transitive closure of "$\longrightarrow$" will be denoted by "$\xrightarrow{*}$".■

Definition 2.4. Let $(X,\mu)_\phi$ be a strength formation with respect to the situation S, Y a proper subset of

X. The existence of the aggressor coalition Y in S is *unprofitable for the object* x∈X iff there exist situations (X',ϕ',μ') and (X",ϕ",μ") such that

$$(X,\phi,\mu) \xrightarrow[Y]{} (X',\phi',\mu') \xrightarrow{*} (X'',\phi'',\mu'') \text{ and } x \notin X''.$$

It is *profitable for the object* x in the opposite case.■

Definition 2.5. A strength formation $(X,\mu)_\phi$ is *dangerous* iff there exists an aggressor coalition Y in $S=(X,\phi,\mu)$ profitable for every object x∈Y. In the opposite case a strength formation is called *safe*.■

3. Safety conditions

Remark 3.1. The strength formation $(X,\mu)_\phi$ is safe iff there doesn't exist any aggressor coalition Y in (X,ϕ,μ) or, for every aggressor coalition Y in (X,ϕ,μ) there exists an object x∈Y such that the existence of this coalition is unprofitable for x.■

Remark 3.2. Let $Z \subset X$ and $\mu(x) = \max_{z\in Z} \mu(z)$ for some x∈Z. If the strength formation $(Z,\mu_{|Z})_{\phi_{|Z^2}}$ is safe then $\mu(x) \le \sum_{z\in Z-\{x\}} \mu(z)$.■

Remark 3.3. Let $S=(X,\phi,\mu)$ and $S'=(X,\phi,\mu')$ be situations with $\mu'=\alpha\mu$ for some $\alpha > 0$. Then for every proper subset Y of X the elements of Y can form an aggressor coalition in S iff they can form it is S'.■

Lemma 3.1. Let $(X,\mu)_\phi$, $(X,\mu')_\phi$ be strength formations as in Remark 3.1. If the existance of the aggressor coalition Y is unprofitable for a given x in (X,ϕ,μ), then it is unprofitable for x in (X,ϕ,μ').

P r o o f. If the existence of the aggressor coalition Y is unprofitable for x, then there exists a sequence of situations $S_0 = (X_0,\phi_0,\mu_0)$, $S_1 = (X_1,\phi_1,\mu_1),\dots,$ $S_k = (X_k,\phi_k,\mu_k)$, $k\geq 1$, and the sequence of coalitions $Y_1,\dots,Y_k$, such that $X_0 = X$, $\phi_0 = \phi$, $\mu_0 = \mu$ and $\phi_{Y_1} = \phi_Y$ and

$$(X_{i-1},\phi_{i-1},\mu_{i-1}) \xrightarrow[Y_i]{} (X_i,\phi_i,\mu_i)$$

for $i = 1,2,\dots,k$. Moreover $x \notin X_k$.

Let us define the sequence of situations:

$$S'_0 = (X_o,\phi_o,\mu'_o),\ S'_1 = (X_1,\phi_1,\mu'_1),\dots,S'_k = (X_k,\phi_k,\mu'_k).$$

where $\mu'_0 = \mu'$, and $\mu'_i = \alpha\mu_i$ for $i = 1,\dots,k$.

For $i = 1,\dots,k$ the coalition Y_i can be formed in the situation $(X_{i-1},\phi_{i-1},\mu'_{i-1})$ and

$$(X_{i-1},\phi_{i-1},\mu'_{i-1}) \xrightarrow[Y_i]{} (X_i,\phi_i,\mu'_i)$$

Hence $(X,\phi,\mu') \xrightarrow[Y]{} (X_1,\phi_1,\mu'_1) \xrightarrow{*} (X_k,\phi_k,\mu'_k)$ and $x \notin X_k$.■

C o r o l l a r y 3.1. Let $(X,\mu)_\phi$, $(X,\mu')_\phi$ be strength formations with $\mu = \alpha\mu'$ for some $\alpha > 0$. Then: the strength formation $(X,\mu)_\phi$ is safe iff the strength formation $(X,\mu')_\phi$ is safe.■

C o r o l l a r y 3.2. If for situations $S = (X,\phi,\mu)$ and $S' = (X',\phi',\mu')$ the relation $(X,\phi,\mu) \xrightarrow{*} (X',\phi',\mu')$ holds and a strength formation $(X,\mu)_\phi$ is safe, then $X = X'$, $\phi = \phi'$ and $\mu = \mu'$.■

Lemma 3.2. A strength formation $(X,\mu)_\phi$ is dangerous iff there exists a safe strength formation $(Y,\mu_{|Y})_{\phi_{|Y^2}}$ for some proper subset Y of X and the elements of Y can form an aggressor coalition in (X,ϕ,μ).

Proof. " ⇒ " Let Y be a proper subset of X satisfying the following condition:

Y is the aggressor coalition in the situation (X,ϕ,μ) profitable for every $x \in Y$, i.e. if $x \in Y$ then $x \in X''$ for every μ', μ'', ϕ', ϕ'', X'' such that

$$(X,\phi,\mu) \xrightarrow[Y]{} (Y,\phi',\mu') \xrightarrow{*} (X'',\phi'',\mu'').$$

Let us suppose that the strength formation $(X,\mu')_{\phi'}$ is not safe. Therefore there would exist a proper subset Y_1 of Y such that the elements of Y_1 could form the aggressor coalition in (X,ϕ',μ') and $(X,\phi,\mu) \xrightarrow[Y]{} (Y,\phi',\mu') \xrightarrow[Y_1]{} (Y_1,\phi'',\mu'')$. But for $x \in Y-Y_1$ the Y coalition is unprofitable, a contradiction. Hence, by Corollary 3.2. the strength formation $(X,\mu')_{\phi'}$, is safe.

" ⇐ " There exists a situation (Y,ϕ',μ') such that $(X,\phi,\mu) \xrightarrow[Y]{} (Y,\phi',\mu')$. By Corollary 3.1. the strength formation $(X,\mu')_{\phi'}$ is safe and, by Corollary 3.2. for every strength formation $(Y_1,\mu'')_{\phi''}$ such that $(Y,\phi',\mu') \xrightarrow{*} (Y_1,\phi'',\mu'')$ there is $Y = Y_1$. This means that the Y coalition is profitable for every $x \in Y$, consequently the strength formation $(X,\mu)_\phi$ is dangerous.■

Corollary 3.3. A strength formation $(X,\mu)_\phi$ is safe iff for any proper subset Y of X exactly one of the following conditions holds:

(i) the elements can't form the aggressor coalition in the situation (X,ϕ,μ),

(ii) the strength formation $(Y,\mu_Y)_{\phi_{|Y^2}}$ is dangerous.■

In the sequel we assume that the situation S is such that $D_\phi = \{(x,x),\ x\in X\}$. For simplicity we shall write (X,μ) instead of $(X,\mu)_\phi$.

Theorem 3.1. Let $n\geq 3$, $X = \{x_1,x_2,\dots,x_n\}$ and let $\mu(x_1) \geq \mu(x_2) \geq \dots \geq \mu(x_n) > 0$. If one of the following conditions holds:

$$\text{(I)}\qquad \mu(x_1) = \mu(x_2) + \dots + \mu(x_n),$$

$$\text{(II)}\quad \begin{cases} \text{(II.1)} & \mu(x_1) < \mu(x_2) + \dots + \mu(x_n) \\ \text{(II.2)} & \mu(x_1) > \mu(x_2) + \dots + \mu(x_{n-1}) \\ \text{(II.3)} & \mu(x_2) > \mu(x_3) + \dots + \mu(x_n) \end{cases}$$

then the strength formation (X,μ) is safe.

Proof. Let us assume that the condition (I) does hold and let $Y = \{x_{i_1},\dots,x_{i_k}\}$ be a proper subset of X. Then the elements of Y can constitute the aggressor coalition in the situation (X,ϕ,μ) if and only if $k > 1$ and $x_1\in Y$. Let $x_1 = x_{i_1}$. But then $\mu(x_1) > \mu(x_{i_2}) + \dots + \mu(x_{i_k})$. By Remark 3.2. we obtain the strength formation $(Y,\mu_{|Y})$ is dangerous. Therefore, by Corollary 3.1. it follows that the strength formation (X,μ) is safe.

Assume that the conditions (II) hold and $Y = \{x_{i_1},\dots,x_{i_k}\}$, $i_1<\dots<i_k\leq n$. By (II.1) $k>1$. For $k = n-1$ either $x_{i_1} = x_1$ or

$x_{i_1} = x_2$. In both cases we get (by II.2 and II.3) $\mu(x_{i_1}) > \mu(x_{i_2}) + \dots + \mu(x_{i_k})$. Therefore the strength formation $(X,\mu_{|Y})$ is dangerous. Let $2 \le k \le n-2$. If $x_1 \notin Y$ and $x_2 \notin Y$ then the elements of Y can't form the aggressor coalition in (X,ϕ,μ). Providing that $x_1 \in Y$ or $x_2 \in Y$ we deduce, as previously, that the strength formation $(Y,\mu_{|Y})$ is dangerous. Hence, by Corollary 3.1. the strength formation (X,μ) is safe.■

It is interesting to observe, by Theorem 3.1. that the existence of two "superleaders" (here x_1 and x_2) assures the safety of the strength formation (X,μ).

Let (X,μ) be a strength formation. Let us denote by M the sum of strengths of all objects, $M = \mu(x_1) + \dots + \mu(x_n)$. Then the conditions (I) and (II) of theorem 3.1. can be rewritten as follows:

(I')
$$\mu(x_1) = \frac{M}{2},$$

(II')
$$\begin{cases} \frac{M}{2} - \frac{\mu(x_n)}{2} < \mu(x_1) < \frac{M}{2} \\ \frac{M}{2} - \frac{\mu(x_1)}{2} < \mu(x_2). \end{cases}$$

As an example, let us consider the existence of safe strength formations in n=2, n=3, n=4 and n=5 cases.

Example 3.1. Let n=2 and $X = \{x_1, x_2\}$. The strength formation (X,μ) is safe iff $\mu(x_1) = \mu(x_2)$.■

Example 3.2. Let n=3 and $X = \{x_1, x_2, x_3\}$ and a function μ fulfills the condition $\mu(x_1) \ge \mu(x_2) \ge \mu(x_3) > 0$. If the strength formation (X,μ) is safe, then by **Remark 3.2.**

$\mu(x_1) \leq \mu(x_2) + \mu(x_3)$. For $\mu(x_1) = \mu(x_2) + \mu(x_3)$ there exists no proper subset $Y \subset X$ such that the elements of Y can form the aggressor coalition in (X, ϕ, μ).

For $\mu(x_1) < \mu(x_2) + \mu(x_3)$ the strength formations $(\{x_1, x_2\}, \mu|\{x_1, x_2\})$ and $(\{x_2, x_3\}, \mu|\{x_2, x_3\})$ are dangerous, which leads to $\mu(x_1) > \mu(x_2)$ and $\mu(x_2) > \mu(x_3)$. Hence, one of the following conditions is satisfied:

(3.2.I) $\qquad \mu(x_1) = \mu(x_2) + \mu(x_3)$,

(3.2.II) $\qquad \begin{cases} \mu(x_1) < \mu(x_2) + \mu(x_3) \\ \mu(x_1) > \mu(x_2) \\ \mu(x_2) > \mu(x_3). \end{cases}$

But for n=3, by theorem 3.1., each of these conditions is also sufficient for the strength formation (X, μ) to be safe. Therefore:

a strength formation (X, μ) is safe iff one of the condition (3.2.I) or (3.2.II) is satisfied.

Let us denote - as previously - by M the sum $\mu(x_1) + \mu(x_2) + \mu(x_3)$ and let us write μ_i instead of $\mu(x_i)$ for $i = 1,2,3$.

From the point we shall omit the assumption about hierarchy of elements strength of X. Therefore the set D of all possible values of μ_1, μ_2, μ_3 with $\mu_1 + \mu_2 + \mu_3 = M$, within the safe strength formation (X, μ) can be described as follows:

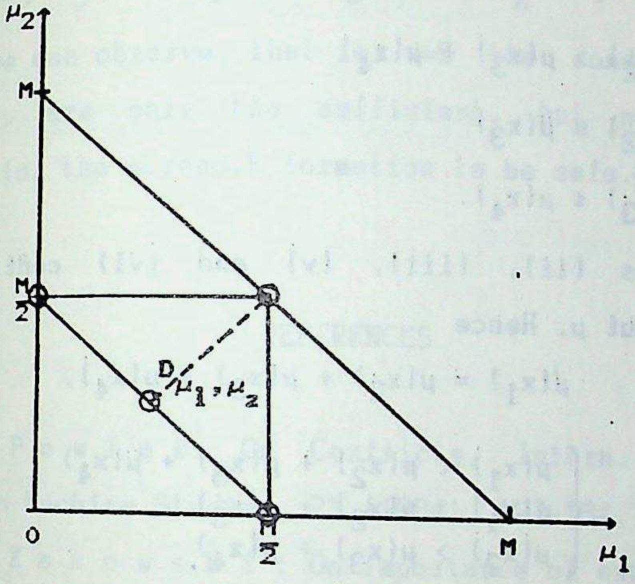
$$D = \{ (\mu_1, \mu_2, \mu_3) \in \Re^3_+ : (\mu_1, \mu_2) \in D_{\mu_1, \mu_2} \ \& \ \mu_3 = M - (\mu_1 + \mu_2) \},$$

where

$$D_{\mu_1, \mu_2} = \{ (\mu_1, \mu_2) \in \Re^2_+ : \mu_1 \neq \mu_2 \ \& \ 0 < \mu_1 \leq \frac{M}{2} \ \& \ \frac{M}{2} - \mu_1 \leq \mu_2 \leq \frac{M}{2} \} -$$

$\{(0,\frac{M}{2}),(\frac{M}{2},0)\}$.

The set D_{μ_1,μ_2} is presented by the figure below:



■

Example 3.3. Let n=4 and $X = \{x_1,x_2,x_3,x_4\}$ and a function μ is such that $\mu(x_1) > \mu(x_2) > \mu(x_3) > \mu(x_4) > 0$. If the strength formation (X,μ) is safe then

$$\mu(x_1) \le \mu(x_2) + \mu(x_3) + \mu(x_4).$$

For $\mu(x_1) = \mu(x_2) + \mu(x_3) + \mu(x_4)$ there exists no proper subset $Y \subset X$ such that elements of Y can form the aggressor coalition. Moreover, the strength formations $\{x_1,x_2,x_3\},\mu|\{x_1,x_2,x_3\})$ and $(\{x_2,x_3,x_4\},\mu|\{x_1,x_2,x_3\})$ are ngerous.

By Example 3.2. we can derive the following:

1) $\mu(x_1) > \mu(x_2) + \mu(x_3)$

(ii) $\mu(x_1) \le \mu(x_2)$

(iii) $\mu(x_2) \le \mu(x_3)$

and

(iv) $\mu(x_2) > \mu(x_3) + \mu(x_4)$

(v) $\mu(x_2) \le \mu(x_3)$

(vi) $\mu(x_3) \le \mu(x_4)$.

But conditions (ii), (iii), (v) and (vi) contradict the assumption about μ. Hence

$$(3.3.\mathrm{I}) \qquad \mu(x_1) = \mu(x_2) + \mu(x_3) + \mu(x_4),$$

$$(3.3.\mathrm{II}) \qquad \begin{cases} \mu(x_1) < \mu(x_2) + \mu(x_3) + \mu(x_4) \\ \mu(x_1) > \mu(x_2) + \mu(x_3) \\ \mu(x_2) > \mu(x_3) + \mu(x_4). \end{cases}$$

By Theorem 3.1. it follows that the above are also the sufficient conditions for safety of the strength formation (X,μ). Therefore we obtain:

The strength formation (X,μ) with $X = \{x_1,x_2,x_3,x_4\}$ and $\mu(x_1) > \mu(x_2) > \mu(x_3) > \mu(x_4) > 0$ is safe iff one of the conditions (3.3.I) or (3.3.II) holds.■

Example 3.4. Let n=5. Checking all possibilities out one can get the following result:

The strength formation (X,μ) with $X = \{x_1,x_2,x_3,x_4,x_5\}$ and $\mu(x_1) > \mu(x_2) > \mu(x_3) > \mu(x_4) > \mu(x_5) > 0$ is safe iff

$$\mu(x_1) = \mu(x_2) + \mu(x_3) + \mu(x_4) + \mu(x_5)$$

or

$$\begin{cases} \mu(x_1) < \mu(x_2) + \mu(x_3) + \mu(x_4) + \mu(x_5) \\ \mu(x_1) > \mu(x_2) + \mu(x_3) + \mu(x_4) \\ \mu(x_2) > \mu(x_3) + \mu(x_4) + \mu(x_5) \ \vee \ \mu(x_2) \le \mu(x_4) + \mu(x_5). \end{cases}$$

Therefore we can observe, that for n=5 the conditions of the Theorem 3.1. are only the sufficient, but not necessary, conditions for the strength formation to be safe.■

REFERENCES

[Pa84] Z. Pawlak : On Conflicts, Intern. Journal of Man-Machine Studies, 21 (1984), 127-134.

[Za84a] W. Zakowski : On importance of exploitation and help in the theory of conflicts, Bull. Pol. Acad. Sci, Math., 9-10, vol. 32 (1984) 535-537.

[Za84b] W. Zakowski : On profitable conflicts, Bull. Pol. Acad. Sci, Math., 7-8, vol. 32 (1984) 535-537.37.

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SOME REMARKS ON PERPENDICULAR PROJECTION

Dedicated to the memory
of Professor Edward Otto

Introduction

The paper [1] deals with projections in projective and affine space, which are considered as lattices. In order to complete this analysis it is necessary to present projection methods in Euclidean space. This paper describes just properties of perpendicular projection. The most important methods of projection used in geometry are those which enable us to reconstruct a manifold from its projections and traces [1]. Such perpendicular projection systems will be considered in this paper. The lattice theory can be very convenient for such an analysis. Our terminology and notations follow [1].

Basic definitions and notations

By $\langle\mathfrak{L},\cup,\cap,\Theta,1\rangle$ we denote a lattice, where $\mathfrak{L}$ is a set, $\cap$ are operations $\mathfrak{L}^2 \longrightarrow \mathfrak{L}$, Θ is the zero and 1 is the unit of a lattice. By an n-dimensional metric lattice, or projective space, we shall mean a structure $\langle\mathfrak{L},\cup,\cap,\Theta,1,\dim\rangle$, where $\langle\mathfrak{L},\cup,\cap,\Theta,1\rangle$ is a lattice and $\dim:\mathfrak{L} \longrightarrow \{-1,0,1,\dots,n\}$ is a function of dimension. By an n-dimensional affine space we shall mean the structure $\langle \mathcal{L}, \cup, \cap, \theta, 1, \hbar, \dim \rangle$, where $\langle \mathcal{L}, \cup, \cap, \theta, 1, \dim \rangle$ in n-dimensional projective space, $\hbar \in \mathcal{L}$ and $\dim(\hbar) = n-1$. The element $\hbar$ will be called an (n-1)- hyperplane at infinity. The set $\mathcal{I} = \{x \in \mathcal{L} : x \subset \hbar\}$ will be known as the set of manifolds at infinity. $\mathcal{P} = \{x \in \mathcal{L} : \sim x \subset \hbar\}$ will be called the set of proper manifolds.

By an orthocomplemented lattice we shall mean the structure $\langle \mathcal{L}, \cup, \cap, \theta, 1, ' \rangle$ where $\langle \mathcal{L}, \cup, \cap, \theta, 1 \rangle$ is a lattice and $' : \mathcal{L} \rightarrow \mathcal{L}$ is a function such that for every $a, b \in \mathcal{L}$: if $a \subset b$, then $b' \subset a'$,

$(a')' = a$,

$a \cup a' = 1$,

$a \cap a' = \theta$.

By an n-dimensional Euclidean space we shall mean the structure $\langle \mathcal{L}, \cup, \cap, \theta, 1, \hbar, \dim, ' \rangle$ where $\langle \mathcal{L}, \cup, \cap, \theta, 1, \hbar, \dim \rangle$ is n-dimensional affine space, $\langle \mathcal{I}, \cup, \cap, \theta, 1, \hbar, ' \rangle$ is an orthocomplemented lattice, and $\mathcal{I}$ is the set of manifolds at infinity.

From the above definition it is seen that function $'$ is described on $\mathcal{I}$.

In affine space two proper manifolds a and b are called parallel (notation $a \| b$) if $\theta \neq a \cap \hbar \subset b \cap \hbar$.

In Euclidean space two proper manifolds a and b are called perpendicular (notation: $a \perp b$) if $\theta \neq b \cap \hbar \subset (a \cap \hbar)' \neq \hbar$.

Theorem 1. Observe that in Euclidean space if a is a manifold at infinity, then $\dim(a') + \dim(a) = n-2$.

Proof. We have consequently

$\dim(a' \cup a) + \dim(a' \cap a) = \dim(a') + \dim(a)$,

$$\dim(\hbar) + \dim(\Theta) = \dim(a') + \dim(a),$$

$$n-2 = \dim(a') + \dim(a). \blacksquare$$

Perpendicular projection

By a projection system we understand a pair $F = (s, r)$ of manifolds s (called the centre) and r (called the projection manifold). The manifold $(a \cup s) \cap r$ will be called the projection of the element a. We denote this by a^F. The manifold $a \cap r$ will be called the trace of the element a (denoted by a_F).

Definition 2. In Euclidean space a projection with respect to a projection system $F = (s, r)$ will be called a perpendicular projection if F is a parallel projection system and $s' = r \cap \hbar$.

Theorem 3. Observe that in Euclidean space each perpendicular projection is a regular projection.

Proof is based on Theorem 1.

Theorem 4. Observe that in Euclidean space if s is a manifold at infinity, r is a proper manifold, $s' \subset r \cap \hbar$, then $F = (s, r)$ is semiregular projection system.

Proof is based on properties of functions dim and $'$.

Theorem 5. If $F = (s, r)$ is a perpendicular projection system and $a \parallel r$, then $a^F \perp b^F$ implies $a \perp b$.

Proof. Assume $a^F \perp b^F$. We have

$a^F \cap \hbar \subset (b^F \cap \hbar)'$ (by Definition $\perp$),

(1) $a \cap r \cap \hbar \subset (b^F \cap \hbar)' \cap s'$ (since $s' = r \cap \hbar$ and by properties of operations $\cap, \cup$),

$s \cup (b^F \cap \hbar) = s \cup (b \cap \hbar)$ (using the modular law, and by Theorem 2 in [1]),

$s' \cap (b^F \cap \hbar)' = s' \cap (b \cap \hbar)'$ (by properties of function '),

(2) $a \cap r \cap \hbar \subset (b \cap \hbar)' \cap s'$ (by (1)),

$a \cap r \cap \hbar \subset (b \cap \hbar)'$ (by properties of operation $\cap$),

$\theta \neq a \cap \hbar \subset (b \cap \hbar)' \neq \hbar$ (since $a \parallel r$, by (2), and by properties of functions dim and ').

It follows $a \perp b$.■

Theorem 6. If $F = (s,r)$ is a perpendicular projection system, $a \parallel r$ and $b \cap s = \theta$, then $a \perp b$ if and only if $a^F \perp b^F$.

Proof. Assume $a \perp b$. We get consequently:

(1) $a \cap r \cap \hbar \subset (b \cap \hbar)' \cap r \cap \hbar$ (since $a \perp b$ and by properties of operation $\cap$),

(2) $a \cap \hbar = a \cap \hbar \cap r \neq \theta$ (since $a \parallel r$),

$a \cap \hbar \subset (b \cap \hbar)' \cap r \cap \hbar$ (by (1) and (2)),

$a \cap \hbar \subset (b \cap \hbar)' \cap s'$ (since $s' = r \cap \hbar$),

$a \cap \hbar \subset ((b \cap \hbar) \cup s)'$ (by properties of function '),

$a \cap \hbar \subset ((b \cup s) \cap \hbar)'$ (using the modular law),

$(a \cap \hbar) \cup s \subset ((b \cup s) \cap \hbar)' \cup s$ (by properties of operation $\cup$),

$(a \cap \hbar) \cup s \subset ((b \cup s) \cap \hbar)' \cup (r \cap \hbar)'$ (since $s = (r \cap \hbar)'$),

$(a \cap \hbar) \cup s \subset ((b \cup s) \cap \hbar \cap r)'$ (by properties of function '),

$(a \cup s) \cap \hbar \subset (b^F \cap \hbar)'$ (by definition b^F),

$(a \cap \hbar) \cup s \subset (b^F \cap \hbar)'$ (using the modular law),

(3) $a^F \cap \hbar \subset (b^F \cap \hbar)'$ (by properties of operation on $\cap$ and definition a^F),

(4) $\hbar \neq (b^F \cap \hbar)'$, $a^F \cap \hbar \neq \theta$ (by assumption $b \cap s = \theta$ and properties of functions dim and ').

Combining (3) and (4) we get $a^F \perp b^F$.

By the last relation and Theorem 5 the proof is completed.■

Reconstruction of manifolds

Basing on Theorem 6 in [1] we can formulate

Theorem 7. Let $q > 0$. Assume $t_1, t_2, \dots, t_u$ are independent manifolds in a projective space, $s_1, s_2, \dots, s_p$ with $p = \binom{u}{q}$ the different manifolds

$$s_i = \bigcup_{j \in J_i} t_j \ , \quad J_i \subseteq \{1,2,\dots,u\}, \quad |J_i| = q.$$

Then every manifold a with $0 \le \dim(a) \le \frac{u-2}{q} - 1$ can be reconstructed from its semiregular projections $a^1, a^2, \dots, a^p$ with respect to the projection systems $F_i = (s_i, r_i)$.

Manifolds $a_1, a_2, \dots, a_p$ in an n-dimensional projective space are said to be $\cap$-independent if

$$\dim\left(\bigcap_{i=1}^{p} a_i\right) = \sum_{i=1}^{p} \dim(a_i) + n - n \cdot p.$$

If we are unable to reconstruct a manifold from its projections, then we have to reconstruct it from its traces.

Assume that in a projective space we have p projection systems $F_i = (s_i, r_i)$, $i = 1,2,\dots,p$. If the p traces a_i

$(i = 1,2,\ldots,p)$ of the manifold a (with respect to F_i) are known, then we only know that the manifolds a_i $(i = 1,2,\ldots,p)$ are included in the manifold a. We say that the manifold a is reconstructed from its traces $a_1,a_2,\ldots,a_p$ whenever

$$a = \bigcup_{i=1}^{p} a_i .$$

Let us here notice, that the condition describing the reconstruction from projections in semiregular projection systems is dual to the condition describing the reconstruction from traces in arbitrary projection systems.

Using Theorem 7 and the principle of duality we get

Theorem 8. Let $q > 0$. Assume $t_1,t_2,\ldots,t_u$ are n-independent manifolds in an n-dimensional projective space, $r_1,r_2,\ldots,r_p$ with $p = \binom{u}{q}$ the different manifolds

$$r_i = \bigcup_{j\in J_i} t_j \ , \quad J_i \subseteq \{1,2,\ldots,u\}, \quad |J_i| = q.$$

and $s_1,s_2,\ldots,s_p$ arbitrary manifolds.

Then every manifold a with $n > \dim(a) \geq n - \frac{u-2}{q}$ can be reconstructed from its traces $a_1,a_2,\ldots,a_p$ with respect to the projection systems $F_i = (s_i,r_i)$.

Theorem 9. Let $q > 0$. Assume $t_1,t_2,\ldots,t_u$ are n-independent manifolds in an n-dimensional affine space, $\bigcap_{i=1}^{u} t_i$ is a proper manifold, $r_1,r_2,\ldots,r_p$ with $p = \binom{u}{q}$ the different manifolds

$$r_i = \bigcap_{j\in J_i} t_j\ ,\quad J_i \subseteq \{1,2,\dots,u\},\quad |J_i| = q.$$

Then every manifold at infinity a with $n-1 > \dim(a) \ge n-1-\frac{u-2}{q}$ can be reconstructed from its traces $a_1, a_2, \dots, a_p$ with respect to the projection systems $f_i = (s_i, r_i)$.

P r o o f. Since $\bigcap_{i=1}^{u} t_i$ is a proper manifold, $t_1 \cap \hbar$, $t_2 \cap \hbar$, $\dots, t_u \cap \hbar$ are n-independent in the space $\langle \mathcal{J}, \cup, \cap, \theta, \hbar, \dim \rangle$ where $\mathcal{J}$ is the set of manifolds at infinity of $\langle \mathcal{L}, \cup, \cap, \theta, 1, \hbar, \dim \rangle$. Then we can use Theorem 8 for (n-1)-dimensional projective space $\langle \mathcal{J}, \cup, \cap, \theta, \hbar, \dim \rangle$.■

In an Euclidean space we are interested in reconstruction of perpendicular manifolds.

T h e o r e m 10. Let $q > 0$. Assume $t_1, t_2, \dots, t_u$ are n-independent manifolds in an n-dimensional Euclidean space, $\bigcap_{i=1}^{u} t_i$ is a proper manifold, $r_1, r_2, \dots, r_p$ with $p = \binom{u}{q}$ the different manifolds

$$r_i = \bigcap_{j\in J_i} t_j\ ,\quad J_i \subseteq \{1,2,\dots,u\},\quad |J_i| = q,$$

$f_i = (s_i, r_i)$ $(i=1,2,\dots,p)$ perpendicular projection systems. If $> \dim(a) \ge n-\frac{u-2}{q}$, then

$a_i \perp b^i$ for every $i \in \{1,2,\dots,p\}$ implies $a \perp b$.

If a_i is a proper manifold, $b \cap s_i = \theta$, $a_i \cap \hbar \ne \theta$ for every $i \in \{1,2,\dots,p\}$, then $a \perp b$ implies $a_i \perp b^i$ for every

$i \in \{1,2,\dots,p\}$.

Proof. Assume $a_i \perp b^i$. We have

(1) $\theta \neq b^i \cap \hbar \subset (a_i \cap \hbar)' \neq \hbar$ (by assumption),

(2) $a_i \cap \hbar \subset r_i \cap \hbar$ (since $a_i \subset r_i$),

(3) $\theta \neq a_i \cap \hbar \subset r_i \cap \hbar$ (by (1) and (2)),

(4) $\theta \neq b \cap \hbar \subset (a_i \cap \hbar)' \neq \hbar$ (since $(a_i)^i = a_i$ and by (1), (2) and (3)),

(5) $b \cap \hbar \subset \bigcap_{i=1}^{p} (a_i \cap \hbar)' = \left(\bigcup_{i=1}^{p} (a_i \cap \hbar) \right)'$ (by (4)),

(6) $n-1 > \dim(a \cap \hbar) \geq n - \frac{u-2}{q} - 1$ (by assumption),

(7) $a \cap \hbar$ is reconstructed from $(a \cap \hbar)_1, (a \cap \hbar)_2, \dots, (a \cap \hbar)_p$ (by (6) and Theorem 9),

(8) $\theta \neq b \cap \hbar \subset (a \cap \hbar)'$ (by (5) and (7)),

$a \perp b$ (since $a \cap \hbar \supset a \cap r_i \cap \hbar \neq \theta$ and by (8)).

Assume $a \perp b$. We get consequently

$a \cap \hbar \subset (b \cap \hbar)'$ (by assumptions),

$$\left.\begin{aligned} & a \cap r_i \cap \hbar \subset (b \cap \hbar)' \cap r_i \\ & a \cap r_i \cap \hbar \subset (b \cap \hbar)' \cap (r_i \cap \hbar) \end{aligned}\right\} \text{(by properties of operation } \cap\text{),}$$

$a \cap r_i \cap \hbar \subset (b \cap \hbar)' \cap (s_i)'$ (since $s_i = (r_i \cap \hbar)'$),

$a \cap r_i \cap \hbar \subset ((b \cap \hbar) \cup s_i)'$ (by properties of function $'$),

$a \cap r_i \cap \hbar \subset ((b \cup s_i) \cap \hbar)'$ (using the modular law),

(9) $(a \cap r_i) \cap \hbar \subset ((b \cup s_i) \cap \hbar)' \cup s_i$ (by properties of $\cup$),

(10) $a_i \cap \hbar \subset (b^i \cap \hbar)'$ (by (9) and definitions a_i, b^i),

$a_i \perp b^i$ (by assumptions and (10)).■

Final remarks

In this point some properties of incidence relation and perspective projection will be described. It completes the results presented in the paper [1].

Suppose that in a projective space we have p projection systems $F_i = (s_i, r_i)$, $i=1,2,\dots,p$.

If a,b can be reconstructed from their projections $a^1,\dots,a^p$ $b^1,\dots,b^p$, respectively, then $a \subset b$ if and only if $a^i \subset b^i$ for every $i \in \{1,2,\dots,p\}$.

If a, b can be reconstructed from their traces $a_1,\dots,a_p$ $b_1,\dots,b_p$, respectively, then $a \subset b$ if and only if $a_i \subset b_i$ for every $i \in \{1,2,\dots,p\}$.

Perspective projection is used on paintings and drawings. In this method the elements of the space are shown by the use of projections and traces.

For every a the manifold $((a \cap \hbar) \cup s) \cap r$ (confluence trace) will be denoted by $a^F_\hbar$. It presents point or line on drawings where in our imagination parallel manifolds confluent.

In an affine space a projection with respect to a projection system $F = (s,r)$ will be called a perspective projection if F is regular, r a proper manifold, and $s \cap \hbar = \theta$.

The following theorems hold:

In an affine space for any perspective projection system $F = (s,r)$

(1) $a = a_F \cup ((a^F_\hbar \cup s) \cap \hbar)$ whenever a_F is a proper manifold (reconstruction),

(2) if $a_F \cap \hbar \neq \theta$, a_F is a proper manifold, and $\sim a \subset r$, then $a_F \parallel a^F_\hbar$ (trace and confluence trace are parallel),

(3) for proper manifolds a, b $a \parallel b$ if and only if $\theta \neq a^F_\hbar \subset b^F_\hbar$,

(4) $a \subset b$ if and only if $a^F_\hbar \subset b^F_\hbar$ and $a_F \subset b_F$.

(incidence).

The proofs of these theorems are left to the reader.

REFERENCES

[1] A. Służalec: Projection systems described in lattice theory, Geometriae Dedicata, 18 (1985) 34-41.

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CONTRIBUTIONS TO THE THEORY OF PARAANALYTIC FUNCTIONS

Dedicated to the memory
of Professor Edward Otto

We provide and discuss in this paper some examples of C^∞ functions which are interesting from the view-point of theory of D-analytic and D-paraanalytic functions in the case of $D = \frac{d}{dt}$.

1. Paraanalytic functions

The notion of paraanalytic function was introduced by D. Przeworska-Rolewicz in [2]. We refer to Przeworska-Rolewicz [3,4] and von Trotha [5] for the general setting of the theory of right invertible operators and basic results concerning D-R spaces. We only recall in this paper some background constructions and definitions leading to the notion of paraanalycity.

A linear operator $D : X \supset D_1 \longrightarrow X$ defined on a domain D_1 in a linear space X (over $\mathbb{R}$ or $\mathbb{C}$) is called a right invertible operator if and only if there exists a linear operator $R : X \longrightarrow X$, defined on the whole X, such that $DR = I$. For any $k \in N$, we denote by D_k the domain of D^k and $D_0 := \bigcap_{k \in N} D_k$. A pair (D,R) consisting of a given right invertible operator D and an operator R which is one of its right inverses determines uniquely the so called initial operator F, defined as follows:

(1.1) $F : X \longrightarrow X$ is a linear operator defined on X such that $F^2 = F$, $FX = \ker D$ and $FR = 0$.

On the domain D_1 of D, the initial operator for (D,R) satisfies

$$F|_{D_1} = I - RD . \tag{1.2}$$

The following equality is crucial for further considerations

$$I|_{D_n} = \sum_{k=0}^{n-1} R^k F D^k + R^n D^n . \tag{1.3}$$

This is generalized Taylor's formula. It can easily be verified if one substitutes F from formula (1.2). Both terms in the right hand side of formula (1.3) are projections, since $(R^nD^n)^2 = R^nD^n$. Formula (1.3) implies that

$$\{ x \in D_\infty \mid R^nD^nx = x \quad \forall\, n \in N\} = \{ x \in D_\infty \mid FD^kx = 0 \quad \forall\, k \in N \cup \{0\}\}. \tag{1.4}$$

The above subspace will be denoted by $Q_R(D)$ and its elements are called D-R singular elements.

Suppose now that X is a metric linear space. The subspace $A_R(D) \subset X$ of D-R analytic elements in X is defined as follows.

$$A_R(D) := \{ x \in D_\infty \mid \lim_{n \to \infty} R^nD^nx = 0 \} . \tag{1.5}$$

Clearly,

$$A_R(D) = \{ x \in D_\infty \mid \sum_{k=0}^{\infty} R^kFD^kx \text{ is convergent to } x\},$$

by virtue of generalized Taylor's formula (1.3).

The subspace $PA_R(D) \subset X$ of D-R paraanalytic elements is defined by

(1.6) $$PA_R(D) := \{ x\in D_\infty \mid \lim_{n\to\infty} R^n F D^n x = 0 \} .$$

Formulae (1.4-6) imply the following inclusions:

(1.7) $$A_R(D) \subset A_R(D) + Q_R(D) \subset PA_R(D) \subset D_\infty \subset X .$$

Let us denote by $\mathcal{R}(D)$ the set of all right inverses of an operator D. D. Przeworska-Rolewicz defines in [2] the sets A(D) and PA(D) of D-analytic and D-paraanalytic elements, respectively, as follows:

(1.8) $$A(D) := \bigcup_{R\in\mathcal{R}(D)} A_R(D) , \quad PA(D) := \bigcup_{R\in\mathcal{R}(D)} PA_R(D) .$$

2. Analytic functions and $\frac{d}{dt}$ - analytic functions

In this section $X = C(\langle -1,1\rangle)$ is the Banach space of continuous real functions on $\langle -1,1\rangle$ with usual norm $|x| = \max_{|t|\leqslant 1} |x(t)|$ and $D = \frac{d}{dt}$ is defined on $D_1 = C^1(\langle -1,1\rangle)$. Then $D_k = C^k(\langle -1,1\rangle)$ and $D_\infty = C^\infty(\langle -1,1\rangle)$. Let $R\in\mathcal{R}(D)$ be chosen as follows:

(2.1) $$(Rx)(t) := \int_0^t x(s)ds .$$

Then

$$\mathcal{R}(D) = \{ R_1 : X \longrightarrow X \mid \exists \varphi\in L(X,R) \ \forall x\in X \ R_1 x = Rx + \varphi(x)\cdot \mathbf{1} \},$$

where $\mathbf{1}$ is the constant function identically equal to 1 on $\langle -1,1\rangle$ and $L(X,R)$ is the space of all, not necessarily continuous, linear functionals on X. The initial operator corresponding to D and R chosen has the form

(2.2) $$Fx = x(0)\cdot\mathbf{1} .$$

Taking formulae (1.4) and (2.2) into account, it is easy to see that the space $Q_R(D)$ of D-R singular elements consists of all

C^∞ functions whose all derivatives vanish at 0.

$$(2.3)\qquad Q_R(D) = \{ x \in C^\infty(\langle -1,1\rangle) \mid x^{(n)}(0) = 0 \quad \forall\, n \in N\cup\{0\} \}.$$

It is also not difficult to calculate that

$$(2.4)\qquad (R^kFD^kx)(t) = \frac{t^k}{k!}\,x^{(k)}(0)$$

and

$$(R^nD^nx)(t) = x(t) - \sum_{k=0}^{n-1}\frac{t^k}{k!}\,x^{(k)}(0)$$

for every $x\in D_\infty$, $k\in N\cup\{0\}$ and $n\in N$.

For a different right inverse operator R_a,

$$(R_ax)(t) := \int_a^t x(s)ds,$$

where $a\in\langle -1,1\rangle$, we obtain $F_ax = x(a)\cdot 1$ and

$$(2.5)\qquad (R_a^kF_aD^kx)(t) = \frac{1}{k!}\,x^{(k)}(a)(t-a)^k.$$

We show now an example illustrating the fact that subspaces $PA_R(D)$ and $PA_{R_a}(D)$ are essentially different.

Proposition 2.6. Let X,D,R and R_a, $0<|a|\le 1$, be as above in this section. Then the functions

$$x_1(t) = \arctan t,\quad x_2(t) = \arctan 2t,\quad t\in\langle -1,1\rangle,$$

have the following properties:

(2.6.1) $x_1\in A_R(D)$ and $x_1\notin PA_{R_a}(D)$ $\forall\, a\in\langle -1,0)\cup(0,1\rangle$,

(2.6.2) $x_2\notin PA_R(D)$ and $x_2\notin PA_{R_a}(D)$ $\forall\, a\in\langle -1,0)\cup(0,1\rangle$.

Proof. By virtue of (2.4) we obtain $R^{2k}FD^{2k}x_1 = 0$ and $(R^{2k+1}FD^{2k+1}x_1)(t) = \frac{(-1)^k}{2k+1}\,t^{2k+1}$. Since for $t\in\langle -1,1\rangle$

$$\left|\sum_{k=l}^{\infty}\frac{(-1)^k}{2k+1}\,t^{2k+1}\right| \le \frac{|t|^{2l+1}}{2l+1} \le \frac{1}{2l+1},$$

we see that $\sum_{n=0}^{\infty} R^n F D^n x_1$ is uniformly convergent on $\langle -1,1\rangle$, that is, $x_1 \in A_R(D)$. Now, for x_2, we have

$$x_2(t) = \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} 2^{2k+1} t^{2k+1} \quad \text{for } |t| < \frac{1}{2}$$

and, by virtue of formula (2.4),

$$(R^{2k+1} F D^{2k+1} x_2)(t) = \frac{(-1)^k}{2k+1} 2^{2k+1} t^{2k+1}.$$

The sequence $\left\{\frac{(-1)^k}{2k+1} 2^{2k+1} t^{2k+1}\right\}_{k\in N}$ is unbounded for $|t| > \frac{1}{2}$, therefore the sequence $\{R^n F D^n x_2\}_{n\in N}$ is divergent in X. This means that $x_2 \notin PA_R(D)$.

To prove the rest, we expand x_1 and x_2 into Taylor series around $a \in \langle -1,1\rangle$. By means of standard techniques of elementary complex analysis, one gets

$$x_1'(t) = \frac{i/2}{t+i} - \frac{i/2}{t-i} = \sum_{n=0}^{\infty} \frac{(-1)^n \mathrm{Im}[(a+i)^{n+1}]}{(a^2+1)^{n+1}} (t-a)^n.$$

Thus, integrating both sides, one obtains

(2.6.3) $$\arctan t = \arctan a + \sum_{n=1}^{\infty} \frac{(-1)^{n-1} \mathrm{Im}[(a+i)^n]}{n(a^2+1)^n} (t-a)^n.$$

The above formula is valid for $t \in \left(a - \sqrt{1+a^2}, a + \sqrt{1+a^2}\right)$, so it allows to calculate $x_1^{(k)}(a)$. By (2.5) and (2.6.3) we see that for any $n \in N$ and $t \in \langle -1,1\rangle$

(2.6.4) $$(R_a^n F_a D^n x_1)(t) = \frac{(-1)^{n-1}(t-a)^n}{n(a^2+1)^{n/2}} \sin\left[n \arccos \frac{a}{\sqrt{1+a^2}}\right]$$

or, equivalently for $a \neq 0$,

$$(R_a^nF_aD^nx_1)(t) = \frac{(-1)^{n-1}(t-a)^n}{n(a^2+1)^{n/2}} \sum_{k=0}^{[(n-1)/2]} \binom{n}{2k+1}(-1)^k a^{n-2k-1}.$$

Since $\alpha = \arccos\frac{a}{\sqrt{1+a^2}} \in (0,\pi)$ for every $a\in R$, the sequence $\{\sin n\alpha\}_{n\in N}$ infinitely many times takes values greater than, say, $\frac{1}{2}$. The sequence $\left\{\frac{(-1)^n}{n}\left(\frac{t-a}{\sqrt{1+a^2}}\right)^n\right\}_{n\in N}$ is bounded if and only if $t\in\langle a-\sqrt{1+a^2},\, a+\sqrt{1+a^2}\rangle$. Thus, the sequence $\{(R_a^nF_aD^nx_1)(t)\}_{n\in N}$ described by formula (2.6.4) is unbounded for $t\in(a+\sqrt{1+a^2},1\rangle$ if $a<0$ and for $\langle -1,a-\sqrt{1+a^2})$ in the case of $a>0$. This implies that $x_1\notin PA_{R_a}(D)$ for $a\in\langle -1,0)\cup(0,1\rangle$. Since $x_2^{(n)}(a) = 2^n\frac{d^n}{dt^n}\Big|_{2a}\arctan t$ and formula (2.6.3) is valid on a suitable neighbourhood of any $a\in R$, one obtains applying (2.5)

$$(R_a^nF_aD^nx_2)(t) = \frac{2^n(-1)^{n-1}(t-a)^n}{n(4a^2+1)^{n/2}}\sin\left[n\arccos\frac{2a}{\sqrt{4a^2+1}}\right].$$

By the same motivation as above, the sequence $\{(R_a^nF_aD^nx_2)(t)\}_{n\in N}$ is bounded if and only if $|2t-2a| \le \sqrt{1+4a^2}$, that is, for $t\in\langle a-\sqrt{a^2+1/4},\, a+\sqrt{a^2+1/4}\rangle \cap \langle -1,1\rangle$. Hence, this sequence is unbounded, in particular, for $t\in\langle -1,a-\sqrt{a^2+1/4})$ if $a>0$ and for $t\in(a+\sqrt{a^2+1/4},1\rangle$ in the case of $a<0$. If suffices to conclude that $x_2\notin PA_{R_a}(D)$.■

Let us point out that $x_2(t) = \arctan 2t$ is analytic in the classical sense and is not D-analytic and, moreover, it is not D-paraanalytic. The proof of fact that $x_2 \notin PA_{R'}(D)$ for every $R' \in \mathcal{R}(D)$ goes through hard and boring calculus and this justifies the decision of omitting it here. We suppose that the idea given in proof of Proposition 2.6 for $R, R_a \in \mathcal{R}(D)$ is satisfactory.

Of course, all D-R analytic functions in $C(\langle -1,1\rangle)$ are analytic in the classical meaning. Concluding these remarks, we can say that the condition of $\frac{d}{dt}$ -analycity in $C(\langle -1,1\rangle)$ is stronger than the classical conditions for an analytic function.

3. Local properties of functions and paraanalycity

As D. Przeworska - Rolewicz pointed out, the function

$$x(t) = \begin{cases} 0 & \text{for } t=0 \\ \exp(-1/t^2) & \text{for } 0<|t|\le 1 \end{cases}$$

is a D-R paraanalytic element of X, $x \in Q_R(D)$, where X, R, D, are as in Section 2, and $x \notin A_R(D)$. We will show now an example of function which is analytic on a neighbourhood of $0 \in \langle -1,1\rangle$, D-R paraanalytic, but is not analytic on $\langle -1,1\rangle$. Then, in particular, it is not D-analytic.

Example 3.1. Let $\eta \in C^\infty(\langle -1,1\rangle)$ be defined as follows

$$\eta(t) = \begin{cases} 0 & \text{if } 3/4 \le |t| \le 1 \\ 1 & \text{if } |t| \le 1/2 \\ \left(1 - \exp\frac{-1}{|t|-1}\right) \exp\frac{2-4|t|}{3-4|t|} & 1/2 < |t| < 3/4. \end{cases}$$

Then $x_3(t) := \frac{2\eta(t)}{2-t}$ has the properties:

(3.1.1) $x_3 \in D_\infty = C^\infty(\langle -1,1\rangle)$, (3.1.2) x_3 is analytic on $\langle -1,1\rangle - \{-\frac{3}{4},-\frac{1}{2},\frac{1}{2},\frac{3}{4}\}$ and is not analytic on $\langle -1,1\rangle$,

(3.1.3) $(R^nFD^nx_3)(t) = \frac{1}{n!}\, x_3^{(n)}(0)\cdot t^n = \frac{t^n}{2^n}$,

since $x_3(t) = \frac{1}{1-t/2}$ for $|t|<1/2$.

We see that $x_3 \in PA_R(D)$. Obviously $x_3 \notin A(D)$, but $\sum_{n=0}^{\infty} R^nFD^nx_3$ is convergent and $\sum_{n=0}^{\infty} (R^nFD^nx_3)(t) = \frac{2}{2-t}$. It is easy to see that $x_3 \in A_R(D)+Q_R(D)$, because

$$x_3(t) = \frac{2}{2-t} + \left(x_3(t) - \frac{2}{2-t}\right).$$

Our aim now is to complete the proof that each of inclusions mentioned in (1.7) is not an equality. We have still to show that $D_\infty \neq X$ and $PA_R(D) \neq A_R(D) + Q_R(D)$. The first fact is easy, for instance, defining $f(t) := \sqrt{1+t}$ for $t\in\langle -1,1\rangle$, we see that $f\in X-D_\infty$. Note that $\sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(0)\cdot t^n$ is convergent on $(-1,1\rangle$ and $f(t) = \sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(0)\cdot t^n$ for every $t\in(-1,1\rangle$.

The following example shows that $PA_R(D) \neq A_R(D) + Q_R(D)$.

Example 3.2. Let $\eta \in C^\infty(\langle -1,1\rangle)$ be the function defined in Example 3.1. Recall that $\eta|_{\langle -1/2,1/2\rangle} = 1$ and $\eta(t)$ vanishes for $|t|\geq\frac{3}{4}$. The function

$$x_4(t) := \begin{cases} 0 & \text{for } t=1 \\ \eta(t)\cdot[t+(1-t)\ln(1-t)] & \text{for } t\in\langle -1,1) \end{cases}$$

is obviously an element of D_∞. To show that $x_4 \in PA_R(D)$, let us note that for $|t|<1/2$

$$x_4(t) = t + (1-t)\ln(1-t) = \sum_{n=2}^{\infty} \frac{t^n}{n(n-1)}$$

and $x_4|_{(-1/2,1/2)}$ is analytic. By virtue of formula (2.4), we see that $(R^nFD^nx_4)(t) = \frac{t^n}{n(n-1)}$ for $t \in \langle -1,1\rangle$ and hence $x_4 \in PA_R(D)$. Suppose now that $x_4 = \tilde{x}_4 + \tilde{\tilde{x}}_4$, where $\tilde{x}_4 \in A_R(D)$ and $\tilde{\tilde{x}}_4 \in Q_R(D)$. Then equality (2.3) implies that $\tilde{\tilde{x}}_4^{(n)}(0) = 0$ for all $n \in N \cup \{0\}$ and $\tilde{x}_4^{(n)}(0) = x_4^{(n)}(0)$. Applying formula (2.4), we get

$$(R^nFD^n\,\tilde{x}_4)(t) = \begin{cases} 0 & \text{if } n=0 \text{ or } n=1 \\ \frac{t^n}{n(n-1)} & \text{for } n \geq 2. \end{cases}$$

Therefore

$$\tilde{x}_4(t) = \sum_{n=2}^{\infty} \frac{t^n}{n(n-1)} = \begin{cases} 1 & \text{if } t=1 \\ t + (1-t)\ln(1-t) & \text{for } t \in \langle -1,1). \end{cases}$$

Of course $\tilde{x}_4 \notin D_\infty$, because this function is not differentiable in 1. This contradiction means that $x_4 \notin A_R(D) + Q_R(D)$.

However, any D-R paraanalytic function admits a local decomposition.

Proposition 3.3. Let $(Y, \|\ \|_Y)$ be a Banach pace and $X_\varepsilon := C(\langle -\varepsilon, \varepsilon\rangle, Y)$, $\varepsilon > 0$, let be the Banach space of -valued continuous functions with the standard norm $\|x\|_\varepsilon = \max_{|t| \leq \varepsilon} \|x(t)\|_Y$. Define $D = \frac{d}{dt}$ with a domain $_1 = C^1(\langle -\varepsilon, \varepsilon\rangle, Y)$ and $(Rx)(t) = \int_0^t x(s)ds$. Then any element $\in PA_R(D)$ splits on $(-\varepsilon, \varepsilon)$ into a sum of an analytic function a smooth "singular" function, that is, there exist an alytic Y-valued function $\tilde{x} : (-\varepsilon, \varepsilon) \longrightarrow Y$ and

$\tilde{\tilde{x}}\in C^{\infty}((-\varepsilon,\varepsilon),Y)$ such that $x(t)=\tilde{x}(t)+\tilde{\tilde{x}}(t)$ for every $t\in(-\varepsilon,\varepsilon)$ and $\tilde{\tilde{x}}(0)=0=\tilde{\tilde{x}}^{(n)}(0)$ for every $n\in N$.

Proof. From (1.6) and (2.4) it is easy to deduce that $x\in PA_R(D)$ if and only if $\lim\limits_{n\to\infty}\frac{1}{n!}\|x^{(n)}(0)\|_Y\,\varepsilon^n=0$. Thus the sequence $\left\{\frac{1}{n!}\varepsilon^n x^{(n)}(0)\right\}_{n\in N}$ is bounded in Y. Therefore, the function $\psi(s):=\sum\limits_{n=0}\frac{s^n}{n!}\varepsilon^n x^{(n)}(0)$ is an analytic function on $(-1,1)$ and $\tilde{x}(t)=\psi(t/\varepsilon)$ is an analytic function on $(-\varepsilon,\varepsilon)$. Define $\tilde{\tilde{x}}(t):=x(t)-\tilde{x}(t)$ for $t\in(-\varepsilon,\varepsilon)$. It is clear that $\tilde{\tilde{x}}\in C^{\infty}((-\varepsilon,\varepsilon),Y)$ vanishes at 0 with all its derivatives and $x(t)=\tilde{x}(t)+\tilde{\tilde{x}}(t)$ for all $t\in(-\varepsilon,\varepsilon)$.■

Remark. A similar result holds if we substitute in Proposition 3.3 the interval $\langle b,c\rangle$ instead of $\langle-\varepsilon,\varepsilon\rangle$ and the operator $(R_a x)(t)=\int\limits_a^t x(s)ds$, $a\in\langle b,c\rangle$, instead of R. Then any Y-valued $D-R_a$ paraanalytic function on $\langle b,c\rangle$ can be decomposed on (b,c) into a sum of an analytic function and a smooth function vanishing at a with all its derivatives. The proof of this fact is analogical; it uses equality (2.5) instead of (2.4).

Consider now the function arctan 2t on different domains as an element of spaces $X_\varepsilon=C(\langle-\varepsilon,\varepsilon\rangle)$ and let D, R be defined as previously (the same letters will denote operators in different spaces X_ε, $\varepsilon>0$). Using arguments and formulae derived in the proof of Proposition 2.6, one can check that arctan 2t is not a D-paraanalytic element of X_ε for

$\varepsilon\in(1/2,\infty)$ but it is a D-R analytic element of X_ε for $0<\varepsilon\le\frac{1}{2}$. However, in general, reducing a domain of a smooth function, one cannot obtain a D-R analytic (or even D-R paraanalytic) function on $\langle-\varepsilon,\varepsilon\rangle$. This can be viewed by means of Mandelbrojt's function (see [1])

$$x(t)=\sum_{n=1}^{\infty}e^{-\sqrt{n!}}\cos(n!t) \tag{3.4}$$

considered on the interval $\langle-\varepsilon,\varepsilon\rangle$. It is not difficult to show that $x\in C^{\infty}(\langle-\varepsilon,\varepsilon\rangle)=D_\infty\subset X_\varepsilon=C(\langle-\varepsilon,\varepsilon\rangle)$ for any $\varepsilon>0$. We have

$$x^{(2k)}(0)=(-1)^k\sum_{n=1}^{\infty}e^{-\sqrt{n!}}(n!)^{2k}. \tag{3.5}$$

Lemma 3.6. For any $\varepsilon>0$

$$\lim_{m\to\infty}\frac{\varepsilon^m}{m!}\sum_{n=1}^{\infty}e^{-\sqrt{n!}}(n!)^m=\infty.$$

Proof. Consider the function $f_m(s):=e^{-\sqrt{\Gamma(s)}}(\Gamma(s))^m$ for $s\in\langle 2,\infty)$. Note that $f_m(n+1)=e^{-\sqrt{n!}}(n!)^m$. Since

$$f_m'(s)=e^{-\sqrt{\Gamma(s)}}(\Gamma(s))^{m-1}\left[-\frac{1}{2}\sqrt{\Gamma(s)}+m\right]\Gamma'(s),$$

we see that. $f_m'(s)=0$ if $\Gamma(s)=4m^2$, that is, if $s=\Gamma^{-1}(4m^2)$, where Γ^{-1} denotes the inverse function of $\Gamma|_{\langle 2,\infty)}$. It is easy to see now that f_m has maximum on $\langle 2,\infty)$ at $\Gamma^{-1}(4m^2)$. Thus, taking the maximal integer number $[\Gamma^{-1}(4m^2)]$ which does not exceed $\Gamma^{-1}(4m^2)$, we obtain the lower bound of our series.

$$\sum_{n=1}^{\infty}e^{-\sqrt{n!}}(n!)^m>(\Gamma([\Gamma^{-1}(4m^2)]))^m\exp-(\Gamma([\Gamma^{-1}(4m^2)]))^{\frac{1}{2}}. \tag{3.6.1}$$

This is evident, because the right hand side of inequality

(3.6.1) equals to $e^{-\sqrt{k!}}(k!)^m$ for $k := [\Gamma^{-1}(4m^2)]-1$. Since $\Gamma|_{\langle 2,\infty)}$ and Γ^{-1} increase monotonically, we have

$$e^{-\sqrt{k!}}(k!)^m > (\Gamma(\Gamma^{-1}(4m^2)-1))^m \exp-(\Gamma(\Gamma^{-1}(4m^2)))^{\frac{1}{2}} =$$

$$= \left(\frac{\Gamma(\Gamma^{-1}(4m^2))}{\Gamma^{-1}(4m^2)-1}\right)^m e^{-2m} = \left(\frac{2m}{e}\right)^{2m}(\Gamma^{-1}(4m^2)-1)^{-m}.$$

We use in the first equality above the well known formula $\Gamma(z) = (z-1)\Gamma(z-1)$. From the above estimate and by means of inequality (3.6.1), we obtain the following estimate.

$$(3.6.2) \qquad \frac{\varepsilon^m}{m!}\sum_{n=1}^{\infty} e^{-\sqrt{n!}}(n!)^m > \frac{1}{m!}\left(\frac{4\varepsilon m}{e^2}\right)^m\left(\frac{m}{\Gamma^{-1}(4m^2)-1}\right)^m.$$

To simplify this inequality, we first prove that

$$\Gamma^{-1}(s) < 2 + 2\ln s \quad \text{for} \quad s \geq 2.$$

Since $\Gamma^{-1}(2) = 3$ and Γ' monotonically increases, we have

$$\Gamma^{-1\prime}(s) = \frac{1}{\Gamma'(\Gamma^{-1}(s))} < \frac{1}{\Gamma(\Gamma^{-1}(s))-\Gamma(\Gamma^{-1}(s)-1)} =$$

$$= \frac{1}{\Gamma(\Gamma^{-1}(s))}\cdot\frac{1}{1-(\Gamma^{-1}(s)-1)^{-1}} < \frac{2}{\Gamma(\Gamma^{-1}(s))} = \frac{2}{s}.$$

Therefore

$$\Gamma^{-1}(s) = \Gamma^{-1}(2) + \int_2^s \Gamma^{-1\prime}(t)\,dt < 3 + \int_2^s \frac{2}{t}\,dt =$$

$$= 3 - 2\ln 2 + 2\ln s < 2 + 2\ln s.$$

This inequality and estimate (3.6.2) both imply that

$$(3.6.3) \qquad \frac{\varepsilon^m}{m!}\sum_{n=1}^{\infty} e^{-\sqrt{n!}}(n!)^m > \frac{1}{m!}\left(\frac{4\varepsilon m}{e^2}\right)^m\left(\frac{m}{4\ln(2m)+1}\right)^m.$$

Thus, it is sufficient to prove that

$$(3.6.4)\qquad \lim_{m\to\infty} \frac{m}{4\ln(2m)+1}\cdot\frac{4\varepsilon m}{e^2\sqrt[m]{m!}} = \infty.$$

It is well known for positive sequences $\{a_m\}_{m\in N}$ that if the limit $\lim_{m\to\infty}\frac{a_{m+1}}{a_m}$ exists, then the sequence $\{\sqrt[m]{a_m}\}_{m\in N}$ is convergent and $\lim_{m\to\infty}\sqrt[m]{a_m} = \lim_{m\to\infty}\frac{a_{m+1}}{a_m}$. Taking $a_m = \frac{1}{m!}\left(\frac{4\varepsilon m}{e^2}\right)^m$, we obtain $\frac{a_{m+1}}{a_m} = \frac{4\varepsilon}{e^2}\left(\frac{m+1}{m}\right)^m$. Thus

$$\lim_{m\to\infty}\frac{4\varepsilon m}{e^2\sqrt[m]{m!}} = \lim_{m\to\infty}\sqrt[m]{a_m} = \lim_{m\to\infty}\frac{a_{m+1}}{a_m} = \frac{4\varepsilon}{e} \neq 0.$$

Therefore, regarding equality $\lim_{m\to\infty}\frac{m}{4\ln(2m)+1} = \infty$, we conclude that (3.6.4) holds and Lemma follows.■

This lemma allows us to formulate some conclusions concerning the function x defined by formula (3.4). By virtue of Lemma 3.6 and equality (3.5), we see that

$$\lim_{k\to\infty}\|R^{2k}FD^{2k}x\|_{X_\varepsilon} = \lim_{k\to\infty}\sup_{|t|\le\varepsilon}\left|\frac{t^{2k}}{(2k)!}x^{(2k)}(0)\right| = \infty .$$

This means that for every $\varepsilon>0$, $x \notin PA_R(D) \subset X_\varepsilon = C(\langle-\varepsilon,\varepsilon\rangle)$.

4. Paraanalytic functions on open intervals

Let X be the space of continuous Y-valued functions on $(-1,1)$, $X := C((-1,1),Y)$, endowed with a metric d inducing almost uniform convergence, say

$$d(x_1,x_2) := \sum_{k=1}^{\infty} 2^{-k}d_k(x_1,x_2)(1+d_k(x_1,x_2))^{-1},$$

where

$$d_k(x_1,x_2) := \sup_{|t|<k/(k+1)} \|x_1(t) - x_2(t)\|_Y .$$

The operator $D = \frac{d}{dt}$ is defined on $D_1 = C^1((-1,1),Y)$ and R, F are defined by formulae (2.1) and (2.2), respectively. It is clear that formulae (2.3-5) are still valid in this case.

Theorem 4.1. In the space $X = C((-1,1),Y)$

$$PA_R(D) = A_R(D) + Q_R(D).$$

Proof. If $x\ PA_R(D)$ then $\lim_{n\to\infty} d(R^nFD^nx,0) = 0$ and this implies that

$$(4.1.1) \qquad \lim_{n\to\infty} \frac{1}{n!}a^n\|x^{(n)}(0)\|_Y = 0 \quad \text{for every } a\in(0,1).$$

This implies in turn that the function $x_1(t) := \sum_{n=0}^{\infty} \frac{t^n}{n!} x^{(n)}(0)$ is well defined and analytic on $(-1,1)$. Note that $x_1^{(n)}(0) = x^{(n)}(0)$. Thus, regarding (4.1.1), we have

$$(4.1.2) \qquad d_k\Big(x_1, \sum_{n=0}^{m} R^nFD^nx_1\Big) =$$

$$= \sup_{|t|\le k/(k+1)} \Big\|\sum_{n=m+1}^{\infty} \frac{t^n}{n!} x^{(n)}(0)\Big\|_Y \le \sum_{n=m+1}^{\infty} \Big(\frac{k}{k+1}\Big)^n \frac{1}{n!}\Big\|x^{(n)}(0)\Big\|_Y .$$

Let us denote the right hand side of inequality (4.1.2) by $f_m(k)$. Using (4.1.1) it is not difficult to show that

$$(4.1.3) \qquad \lim_{m\to\infty} f_m(k) = 0 \quad \text{for each } k\in N.$$

For every $\varepsilon>0$ one can choose $k_\varepsilon\ge2$ such that $\sum_{k=k_\varepsilon} 2^{-k} < \frac{\varepsilon}{2}$.

By virtue of 4.1.3, there exists m_ε such that

$$(4.1.4) \qquad \forall\, k \le k_\varepsilon\ \forall\, m > m_\varepsilon \quad f_m(k) < \frac{\varepsilon}{2} .$$

Thus, for each $\varepsilon > 0$ and for every $m > m_\varepsilon$, we obtain from estimates (4.1.2) and (4.1.4)

$$d\Big(x_1, \sum_{n=0}^{m} R^nFD^nx_1\Big) \le \sum_{k=1}^{\infty} 2^{-k}f_m(k)(1+f_m(k))^{-1} \le$$

$$\le \sum_{k=1}^{k_\varepsilon} 2^{-k}\cdot\frac{\varepsilon}{2} + \sum_{k=k_\varepsilon+1}^{\infty} 2^{-k} < \frac{\varepsilon}{2}+\frac{\varepsilon}{2} = \varepsilon .$$

Hence $x_1 \in A_R(D)$. For $x_2 = x - x_1$ we have $x_2 \in D_\infty$ and $x_2^{(n)}(0) = x^{(n)}(0) - x_1^{(n)}(0) = 0$ for every $n \in N \cup \{0\}$, that is, $x_2 \in Q_R(D)$. The proof is complete.■

Remarks 4.2.

(4.2.1) Both subspaces $A_R(D)$ and $Q_R(D)$ are closed and $A_R(D) \cap Q_R(D) = \{0\}$, that is, $PA_R(D)$ splits into a direct sum of these subspaces.

(4.2.2) The decomposition $PA_R(D) = A_R(D) + Q_R(D)$ remains valid if we substitute any open and connected subset $I \subseteq R$ instead of $(-1,1)$ in Theorem 4.1 and if we take any operator R_a instead of R, where $(R_ax)(t) = \int_a^t x(s)ds$, $a \in I$. This can be proved by means of the same method as in proof of Theorem 4.1 with use of a suitable metric.

(4.2.3) There exist functions in $D_\infty \subset X$ which are Y-valued analytic functions on $(-1,1)$ but are not $\frac{d}{dt}$-paraanalytic. The function $(-1,1) \ni t \longmapsto (\arctan 2t)y \in Y$, $y \in Y$, $y \neq 0$, is an example of such function.

Let us consider now smooth Y-valued functions on $(-1,1)$ which are analytic on $(-1,0) \cup (0,1)$. The question arises, if it is possible to express such a function as a sum of an analytic function and an element of $Q_R(D)$. The answer is "strongly negative", that is, in general it is not possible even locally (on a sufficiently small neighbourhood of 0) and, moreover, if a local decomposition exists, it may happen that a decomposition defined on the entire interval $(-1,1)$ does not exist. We will illustrate this pathology on two examples in the case of $Y = R$. One can easily generalize them for the case of any Banach space by multiplying functions in examples by a non-zero element of Banach space.

Example 4.3. Let $x_o(t) := \sum_{k=1}^{\infty} \frac{1}{k!} g_k(t)$, where $g_k(t) := (1+k^2t^2)^{-1}$. Since for any $n\in N$

$$(4.3.1) \quad g_k^{(n)}(t) = \frac{(-1)^n n! k^n}{\sqrt{(1+k^2t^2)^{n+1}}} \cdot \sin\left[(n+1)\arccos \frac{kt}{\sqrt{1+k^2t^2}} \right],$$

then for every $t\in R$ and for any $n \in N\cup\{0\}$, $k\in N$

$$|g_k^{(n)}(t)| \le n!k^n.$$

Therefore, the series $\sum_{k=1}^{\infty} \frac{1}{k!} g_k(t)$ and $\sum_{k=1}^{\infty} \frac{1}{k!} g_k^{(n)}(t)$ are uniformly convergent on R. Hence x_o is a C^∞ function. Consider $g_k(z) = (k^2z^2 + 1)^{-1}$ as a function of complex variable on a domain $\overline{G}_\varepsilon := \{ z\in C \mid |\mathrm{Re}\, z| \ge \varepsilon \}$, $\varepsilon>0$. For $z\in\overline{G}_\varepsilon$ we have $|g_k(z)| = |kz+i|^{-1}|kz-i|^{-1} \le k^{-2}\varepsilon^{-2}$, because $\inf_{z\in\overline{G}_\varepsilon} |kz+i| = \inf_{z\in\overline{G}_\varepsilon} |kz-i| = k\varepsilon$. Therefore, by virtue of Weierstrass theorem, we obtain that $\sum_{k=1} \frac{1}{k!} g_k(z)$ is a holomorphic function on two half-planes $G_\varepsilon = \{ z\in C \mid |\mathrm{Re}\, z| > \varepsilon \}$. Since $\varepsilon>0$ is arbitrary, we conclude that x_o is analytic on $R - \{0\}$.

Suppose now that

$$x_o(t) = \tilde{x}_o(t) + \tilde{\tilde{x}}_o(t)\ ,\ t\in(-\varepsilon,\varepsilon), \tag{4.3.2}$$

where $\tilde{x}_o$ is an analytic function on $(-\varepsilon,\varepsilon)$ and $\tilde{\tilde{x}}_o$ is a C^∞ function on $(-\varepsilon,\varepsilon)$ such that $\tilde{\tilde{x}}_o^{(n)}(0) = 0$ for every $n\in N\cup\{0\}$. Then, for every $t\in(-\varepsilon,\varepsilon)$,

$$\tilde{x}_o(t) = \sum_{n=0}^{\infty} \frac{t^n}{n!}\, x_o^{(n)}(0). \tag{4.3.3}$$

Applying formula (4.3.1), we obtain

$$\left| \frac{t^{2m}}{(2m)!}\, x_o^{(2m)}(0) \right| = |t|^{2m} \sum_{k=1}^{\infty} \frac{k^{2m}}{k!}\ .$$

Therefore, for $t = \frac{\varepsilon}{2}$,

$$\left(\frac{\varepsilon}{2}\right)^{2m} \left| \frac{x_o^{(2m)}(0)}{(2m)!} \right| > \sum_{k=[2/\varepsilon]+1}^{\infty} \frac{(k\varepsilon/2)^{2m}}{k!} > \sum_{k=[2/\varepsilon]+1}^{\infty} \frac{1}{k!} = c_\varepsilon\ .$$

This means that (4.3.3) is divergent if $t = \frac{\varepsilon}{2}$, since the constant $c_\varepsilon > 0$ does not depend on m. Hence, the decomposition (4.3.2) is impossible for every $\varepsilon > 0$.

Example 4.4. We define $\varphi \in C^\infty(R)$ as follows

$$\varphi(t) = \begin{cases} -\exp(-t^{-2}) & \text{for } t>0 \\ 0 & \text{if } t=0 \\ \exp(-t^{-2}) & \text{for } t<0. \end{cases}$$

The function φ is analytic on $R-\{0\}$ and

$$0 > \varphi'(t) \ge \varphi'(\sqrt{2/3}) = -\frac{3\sqrt{3}}{\sqrt{2}}\, e^{-3/2} > -1\ .$$

We rotate now the graph of φ around origin by angle $\frac{\pi}{4}$ in contrary-clockwise direction and we thus obtain a graph of monotonically increasing function Φ. The function Φ may be expressed implicitly by the formula

$$\Phi(t) = t + \sqrt{2}\, \varphi\left(\frac{t}{\sqrt{2}} + \frac{1}{\sqrt{2}}\, \Phi(t)\right). \tag{4.4.1}$$

Clearly, $\Phi \in C^{\infty}(R)$ and Φ is analytic on $R-\{0\}$. Using simple computer calculation, we get $|\Phi(t)| < 0.46$ for every $t\in\langle -1,1\rangle$. Therefore, the function f,

$$f(t) := (1 - 2\Phi(t))^{-1}, \quad t\in(-1,1), \tag{4.4.2}$$

is a C^{∞} function on $(-1,1)$ which is analytic on $(-1,0)\cup(0,1)$. Since $\Phi'(0) = 1$ and $\Phi(0) = \Phi^{(k)}(0) = 0$ for every $k\geq 2$, we can calculate

$$f^{(n)}(0) = n!2^{n}. \tag{4.4.3}$$

We explain how we calculate $f^{(n)}(0)$ later on. Before that, let us analyze the possibility of splitting $f = f_1 + f_2$, where f_1 is an analytic function and $f_2^{(n)}(0) = 0$ for every $n \in N \cup \{0\}$. Reasoning as previously, we obtain

$$f_1(t) = \sum_{n=0}^{\infty} \frac{t^n}{n!} f^{(n)}(0) = \sum_{n=0}^{\infty} 2^n t^n = \frac{1}{1-2t}$$

on a neighbourhood of 0. The function f_1 can not be analytically extended on $(-1,1)$. Concluding, f has no the required decomposition on $(-1,1)$, notwithstanding it can be decomposed on $(-1,1/2)$.

It remains to prove formula (4.4.3).

Lemma 4.5. Let $g \in C^{\infty}(a,b)$, $\Phi \in C^{\infty}(a',b')$ and suppose that $a < \Phi(t) < b$ for every $t\in(a',b')$. Then for every $n\geq 2$ there exist functions $h_{2n},\dots,h_{nn}\in C^{\infty}(a',b')$ such that

$$(g\circ\Phi)^{(n)} = (g^{(n)}\circ \Phi)(\Phi')^{n} + \sum_{k=2}^{n} h_{kn}\Phi^{(k)}. \tag{4.5.1}$$

Proof. For $n=2$, we have

$$(g\circ\Phi)^{(2)} = (g^{(2)}\circ \Phi)(\Phi')^{2} + (g'\circ \Phi)\Phi^{(2)}.$$

Therefore, formula (4.5.1) folds for n=2 with $h_{22} = g'\circ\Phi$. Differentiating both sides of (4.5.1), we obtain

$$(g\circ\Phi)^{(n+1)} = (g^{(n+1)}\circ\Phi)(\Phi')^{n+1} + n(g^{(n)}\circ\Phi)(\Phi')^{n-1}\Phi^{(2)} + \sum_{k=2}^{n}\left[h'_{kn}\Phi^{(k)} + h_{kn}\Phi^{(k+1)}\right] =$$

$$= (g^{(n+1)}\circ\Phi)(\Phi')^{n+1} + \Phi^{(2)}\left[n(g^{(n)}\circ\Phi)(\Phi')^{n-1} + h'_{2n}\right] + \sum_{k=3}^{n}(h_{k-1,n} + h'_{kn})\Phi^{(k)} + h_{nn}\Phi^{(n+1)}.$$

The sum $\sum_{k=3}^{n}$ disappears in case of n=2. If we define $h_{n+1,n+1} := h_{nn}$, $h_{k,n+1} := h_{k-1,n} + h'_{kn}$ for $3\le k\le n$, and $h_{2,n+1} := n(g^{(n)}\circ\Phi)(\Phi')^{n-1} + h'_{2n}$, we obtain, by virtue of Induction Principle, the conclusion of proof.■

To calculate formula (4.4.3), it suffices to notice that $f = g\circ\Phi$, where $g(t) = \frac{1}{1-2t}$ and then use the equalities $\Phi(0) = 0 = \Phi^{(k)}(0)$ for $k\ge 2$, $\Phi'(0) = 1$. By Lemma 4.5

$$f^{(n)}(0) = g^{(n)}(0)(\Phi'(0))^{n} = g^{(n)}(0) = 2^{n}n!.$$

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REFERENCES

[1] S. Mandelbrojt : Analytic Functions and Classes of Infinitely Differentiable Functions, The Rice Institute Pamflet, 29, 2, Houston, Texas, 1942.

[2] D. Przeworska-Rolewicz: Smooth solutions of linear equations with scalar coefficients in a right invertible operator, to appear.

[3] D. Przeworska-Rolewicz: Shifts and Periodicity for Right Invertible Operators, Research Notes in Mathematics, 47, Pitman, London, 1980.

[4] D. Przeworska-Rolewicz: Algebraic Analysis, PWN, Warszawa - Dordrecht, 1987.

[5] H. von Trotha: Structure Properties of D-R Spaces, Dissertationes Math. 184, Warszawa, 1981.

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PROPERTY (C) AND INTERPOLATION FORMULAE INDUCED BY RIGHT INVERTIBLE OPERATORS

Dedicated to the memory
of Professor Edward Otto

In the present paper we shall solve the following problem: To find a D-polynomial, i.e. a polynomial induced by a right invertible operator D in a linear space X which admits the given values for given initial operators. When $D = \frac{d}{dt}$, we have a classical interpolation problem. An unified approach to various interpolation problems in Hilbert spaces by applications of properties of right invertible operators has been given by M. Tasche [4].

1. Preliminaries

Let X be a linear space over a field $\mathcal{F}$ of scalars (where $\mathcal{F} = \mathbb{R}$ or $\mathcal{F} = \mathbb{C}$). Denote by $L(X)$ the set of all linear operators with domains and ranges in X, by $L_o(X)$ the set $\{A \in L(X) : \operatorname{dom} A = X\}$ and by $R(X)$ the set of all right invertible operators belonging to $L(X)$. For a $D \in R(X)$ we write

$$\mathcal{R}_D = \{R \in L_o(X) : DR = I\},$$

$$\mathcal{F}_D = \{F \in L_o(X) : F^2 = F,\ FX = \ker D \text{ and } \exists_{R \in \mathcal{R}_D}\ FR = 0\}.$$

We admit here and in the sequel that $\ker D \neq \{0\}$, i.e. operators D under consideration are right invertible but not invertible. Any operator $F \in \mathcal{F}_D$ is said to be an initial operator for D corresponding to R. Right inverses and initial operators have the following properties:

(i) An operator $F \in L_o(X)$ is initial for D if and only if there exists an $R \in \mathcal{R}_D$ such that $F = I - RD$ on dom D.

(ii) Any projection F_1 onto $\ker D$ is an initial operator for D corresponding to a right inverse $R_1 = R - F_1R$ and R_1 does not depend on the choice of an $R \in \mathcal{R}_D$. Right inverses (respectively: initial operators), which commute, are equal.

Any element $z \in \ker D$ is said to be a constant. Initial operators preserve constants: $Fz = z$ for all $z \in \ker D$, $F \in \mathcal{F}_D$. Elements of the form R^kz for $z \in \ker D$, $k \in \mathbb{N}$, $R \in \mathcal{R}_D$ are called D-monomials. We have

$$\ker D^n = \{ \sum_{k=0}^{n-1} R^k z_k : z_o, \ldots, z_{n-1} \in \ker D\} \quad \text{for } n \in \mathbb{N}.$$

Let $A(\mathbb{R}) = \mathbb{R}_+$ or $\mathbb{R}$. Let $D \in R(X)$ and F be an initial operator for D corresponding to an $R \in \mathcal{R}_D$. A family $\{S_h\}_{h \in A(\mathbb{R})} \subset L_o(X)$ is said to be a family of R-shifts if $S_o = I$ and

$$S_h R^k F = \sum_{j=0}^{k} \frac{h^{k-j}}{(k-j)!} R^j F \quad \text{for all } h \in A(\mathbb{R}),\ k \in \mathbb{N}_o = \mathbb{N} \cup \{0\}.$$

R-shifts preserve constants $S_hz = z$ for all $z \in \ker D$, $h \in A(\mathbb{R})$. The family of R-shifts is a semigroup if $A(\mathbb{R}) = \mathbb{R}_+$ and a group if $A(\mathbb{R}) = \mathbb{R}$ (with respect to superposition as a structure operations). R-shifts are uniquely determined on the set

$$P(R) = \text{lin}\,\{R^k z : z \in \ker D,\ k \in \mathbb{N}_0\}$$

(this set does not depend on the choice of $R \in \mathcal{R}_D$).

The listed properties and other informations about right invertible operators, as well as examples and applications, can be found in the author's book [3].

2. Property c(R)

Definition 2.1. Let $D \in R(X)$. An initial operator F_0 for D has the property c(R) for an $R \in \mathcal{R}_D$ if there exist scalars c_k such that

$$F_0 R^k z = \frac{c_k}{k!}\, z \quad \text{for all} \quad z \in \ker D,\ k \in \mathbb{N}, \tag{2.1}$$

and $c_k = 0$ for all $k \in \mathbb{N}$ if $F_0 = F$, where F is an initial operator for D corresponding to R. We shall write: $F_0 \in c(R)$. A set $\mathcal{F}_D^0 \subset \mathcal{F}_D$ has the property (c) if for every $F_0 \in \mathcal{F}_D^0$ there exists an $R \in \mathcal{R}_D$ such that $F_0 \in c(R)$. We admit $c_0 = 1$, since $F_0 z = z$ for all $z \in \ker D$.

The property (c) for two initial operators has been introduced by the author [2] (cf. also [3], p.299).

Theorem 2.1. Let $D \in R(X)$. The set $\mathcal{F}_D$ of all initial operators has the property (c) if and only if $\dim \ker D = 1$.

Proof. Sufficiency. By our assumption we have $\ker D = \{\mu e : \mu \in \mathcal{F}\}$ for an element e. Let F and $F_1 \neq F$ be arbitrarily fixed initial operators corresponding to right inverses R and R_1, respectively. By our assumption there exist scalars c_k such that $F_1 R^k e = c_k e$ for $k \in \mathbb{N}$. This implies that for an arbitrary $z \in \ker D$, $k \in \mathbb{N}$, we have

$$F_1R^kz = F_1R^k\mu e = \mu F_1R^ke = \mu c_ke = c_kz.$$

The arbitrariness of $F, F_1 \in \mathcal{F}_D$ implies that the set has the property (c).

Necessity.[1] Let $D \in R(X)$ and let $\dim \ker D > 1$. In order to prove that not all initial operators for D have the property c(R) it is enough to show that there exist $z \in \ker D$, $R \in \mathcal{R}_D$, $F^\wedge \in \mathcal{F}_D$ such that $F^\wedge Rz \neq cz$ for all scalars c (where $F^\wedge \neq F$, which is an initial operator corresponding to R). To do this, let us fix an $e \in \ker D$, $e \neq 0$, and an $R \in \mathcal{R}_D$. Write $X_e = \mathrm{lin}\,\{R^ke : k \in \mathbb{N}_0\}$. Denote by $X_e^\perp$ the direct complement of X_e in X and by Z_e the direct complement of lin{e} in ker D, i.e.

$$X = X_e \oplus X_e^\perp, \quad \ker D = \mathrm{lin}\{e\} \oplus Z_e$$

(we admit here and in the sequel the axiom of choice). By definition, $RX_e \subset X_e$ and $DX_e \subset X_e$.

Define

$$D_e = \begin{cases} D & \text{on } X_e \\ I & \text{on } X_e^\perp \end{cases}, \quad D_e^\perp = \begin{cases} I & \text{on } X_e \\ D & \text{on } X_e^\perp \end{cases},$$

$$R_e = \begin{cases} R & \text{on } X_e \\ I & \text{on } X_e^\perp \end{cases}, \quad R_e^\perp = \begin{cases} I & \text{on } X_e \\ R & \text{on } X_e^\perp \end{cases}.$$

Then $D = D_eD^\perp = D_e^\perp D_e$ and $R = R_eR_e^\perp = R_e^\perp R_e$. Moreover, $D_e \in R(X_e)$, $R_e \in \mathcal{R}_{D_e}$, $D_e^\perp \in R(X_e^\perp)$, $R_e^\perp \in \mathcal{R}_{D_e^\perp}$. Let $F \in \mathcal{F}_D$ correspond to R.

---

[1] The idea of this proof was suggested by S. Rolewicz.

Then

$$F = \begin{cases} F_e & \text{on } X_e \\ F_e^\perp & \text{on } X_e^\perp \end{cases}, \text{ where } \begin{cases} F_e \in \mathcal{F}_{D_e} \text{ and } F_eR_e = 0,\ F_e = 0 \text{ on } X_e^\perp, \\ F_e^\perp \in \mathcal{F}_{D_e^\perp} \text{ and } F_e^\perp R_e^\perp = 0,\ F_e^\perp = 0 \text{ on } X_e. \end{cases}$$

Let $\tilde{F} \in \mathcal{F}_D$, $\tilde{F} \neq F$. In a similar way we construct operators $\tilde{F}_e$ and $\tilde{F}_e^\perp$. Write $F_e^\wedge = (1-d)F_e + d\tilde{F}_e$, where $d \in \mathbb{R}$ is a parameter which can be chosen arbitrarily small. Then $F_e^\wedge \in \mathcal{F}_{D_e}$ (as a combination of initial operators with the sum of coefficients equal 1). Since dim ker $D_e = 1$, the first part of our theorem implies that there exists a scalar $c \neq 0$ such that $\tilde{F}_e R_e e = ce$, i.e. $\tilde{F}_e$ has the property $c(R_e)$. If not all initial operators for $D_e^\perp$ have the property $c(R_e^\perp)$, then our theorem is already proved for the space $X_e^\perp$ satisfies our requirement. Suppose then that there exists a scalar c' such that $\tilde{F}_e^\perp R_e^\perp z_e = c'z_e$ for all $z_e \in Z_e$. Let $z = e + z_e$ and let $d = \frac{c'}{2c}$. Define $F^\wedge = F_e^\wedge$ on X_e and $F^\wedge = \tilde{F}_e^\perp$ on $X_e^\perp$. Then $F^\wedge \in \mathcal{F}_D$ and (since $F_eR_e = 0$) we find

$$F^\wedge Rz = F_e^\wedge R_e e + \tilde{F}_e^\perp R_e^\perp z_e = [\,(1-d)F_e + d\tilde{F}_e]R_e e + \tilde{F}_e^\perp R_e^\perp z_e =$$

$$= (1-d)F_eR_e e + d\tilde{F}_e R_e e + \tilde{F}_e^\perp R_e^\perp z_e = d\tilde{F}_e R_e e + \tilde{F}_e^\perp R_e^\perp z_e = dce + c'z_e =$$

$$= \frac{c'}{2c}\, ce + c'ze = \frac{c}{2}\, e + z_e,$$

The arbitrariness of $z_e \in Z_e$ implies that $F^\wedge$ has not the property $c(R)$.

Proposition 2.1. Suppose that $D \in R(X)$ and dim ker $D = 1$, i.e. ker $D = \{\mu e. : \mu \in \mathcal{F}\}$ for an $e \neq 0$. Denote by X' the space of all linear functionals over X. Then every initial operator is of the form

$$F_\varphi x = \varphi(x)\cdot e \quad \text{for} \quad x \in X, \text{ where } \varphi \in X' \text{ and } \varphi(e) = 1,$$

and

$$c_k = c_k(\varphi) = k!\ \ \varphi(R^k e) \quad \text{for} \quad k \in \mathbb{N},\ \varphi \in X',\ \varphi(e) = 1,\ R \in \mathcal{R}_D.$$

P r o o f. Let $\varphi \in X'$ be arbitrarily fixed. Then the operator F_φ defined by $F_\varphi x = \varphi(x)e$ for $x \in X$ maps X onto ker D and we have $F_\varphi^2 x = \varphi(x)\cdot\varphi(e)\cdot e = \varphi(e)F_\varphi x$. Hence F_φ is a projection onto ker D if and only if $\varphi(e) = 1$. If φ has this property, then for arbitrary $k \in \mathbb{N}$ and $\mu \in \mathcal{F}$ we have $z = \mu\cdot e \in \ker D$ and

$$F_\varphi R^k z = F_\varphi R^k(\mu\cdot e) = \mu F_\varphi R^k e = \mu\varphi(R^k e)\cdot e = \frac{k!\,\varphi(R^k e)}{k!}\,z = \frac{c_k(\varphi)}{k!}\,z.$$

P r o p o s i t i o n 2.2. Suppose that X is a complete linear metric space, $D \in R(X)$, F is an initial operator for D corresponding to an $R \in \mathcal{R}_D$, the set $P(R) = \operatorname{lin}\{R^k z : z \in \ker D, k \in \mathbb{N}_0\}$ is dense in X and $F_1 \neq F$ is an arbitrarily fixed continuous initial operator having the property c(R). Then not all numbers c_k, $k \in \mathbb{N}_0$, vanish simultaneously.

P r o o f. Suppose that $F_1 R^k z = 0$ for all $k \in \mathbb{N}$, $z \in \ker D$, i.e. $c_k = 0$ for $k \in \mathbb{N}$. By our assumptions, $\overline{P(R)} = X$ and F_1 is continuous, hence $F_1 x = 0$ for all $x \in X$. This contradicts our assumption that F_1 is an initial operator, i.e. a projection onto ker D.

Clearly, if the system $\{F_0,\ldots,F_{n-1}\} \subset \mathcal{F}_D$ has the property c(R) with constants d_{ik}, i.e.

$$F_i R^k z = \frac{d_{ik}}{k!}\,z \quad \text{for } i = 0,1,\ldots,N-1;\ k \in \mathbb{N}_0,$$

and $F_0,\ldots,F_{N-1}$ are linearly dependent, then

$V_N = \det(d_{ik})_{i,k} = 0,1,\ldots,N-1 = 0$. Indeed, if

$F_m = \sum_{j=0, j\neq m}^{N-1} c_j F_j$, where c_j are scalars not vanishing simultaneously, for an m, then

$$F_m R^k z = \sum_{j=0, j\neq m}^{N-1} c_j F_j R^k z = \sum_{j=0, j\neq m}^{N-1} c_j \frac{d_{jk}}{k!} z.$$

Hence

$$d_{mk} = \sum_{j=0, j\neq m}^{N-1} c_j d_{jk}, \quad k = 0,1,\ldots,N-1,$$

i.e. the m-th row of V_N is a linear combination of other rows.

The following question arises: is

$V_N = \det(d_{ik})_{i,k=0,1,\ldots N-1} \neq 0$ for any system $\{F_o,\ldots,F_{N-1}\}$ of linearly independent initial operators having the property c(R) ?

Proposition 2.3. Suppose that $D \in R(X)$, F is an initial operator for D corresponding to a right inverse R and $\{S_h\}_{h \in A(\mathbb{R})}$ is a family of R-shifts (where $A(\mathbb{R}) = \mathbb{R}$ or $\mathbb{R}_+$), write $F_h = FS_h$ for $h \in A(\mathbb{R})$. By definition, F_h are initial operators for D corresponding to right inverses $R_h = R-F_hR$ (where we admit $R_o = R$). Then $F_h \in c(R)$ for all $h \in A(\mathbb{R})$ and

$$c_k(h) = h^k, \quad h \in A(\mathbb{R}),\ k \in \mathbb{N}.$$

Proof. Let $z \in \ker D$, $h \in A(\mathbb{R})$ and $k \in \mathbb{N}$ be arbitrarily fixed. Then, by our assumptions,

$$F_h R^k z = FS_h R^k z = F \sum_{j=0}^{k} \frac{h^{k-j}}{(k-j)!} R^j z =$$

$$\sum_{j=0}^{k} \frac{h^{k-j}}{(k-j)!} FR^j z = \frac{h^k}{k!} Fz = \frac{h^k}{k!} z$$

for $FR = 0$. Thus we have $c_k(h) = h^k$ and $c_k(h) \neq$ for $h \neq 0$.

Corollary 2.1. Suppose that all assumptions of Proposition 2.3 are satisfied. Then $F_h \in c(R_{h_o})$ for all $h \in A(\mathbb{R})$ and an arbitrarily fixed $h_o \in A(\mathbb{R})$ and

$$(2.3) \qquad F_h R_{h_o}^k z = \frac{(h-h_o)^k}{k!} z \qquad \text{for } z \in \ker D,\ k \in \mathbb{N}.$$

Proof. Formula (5.2.19) in [3] implies that

$$(2.4) \qquad R_h^k z = S_{-h} R^k z \quad \text{for } z \in \ker D,\ h \in A(\mathbb{R}),\ k \in \mathbb{N}.$$

Let $h_o \in A(\mathbb{R})$ be arbitrarily fixed. Then for all $h \in \mathbb{R}$ Formula (2.4) implies

$$F_h R_{h_o}^k z = FS_h S_{-h_o} R^k z = FS_{h-h_o} R^k z = F_{h-h_o} R^k z = \frac{(h-h_o)^k}{k!} z.$$

We therefore conclude that $F_h \in c(R_{h_o})$ and that $c_k(h) = (h-h_o)^k$ for $h \in \mathbb{R}$, $k \in \mathbb{N}$.

Corollary 2.2. Suppose that all assumptions of Proposition 2.3 are satisfied. Then the following initial operators belong to $c(R)$ for all $h \in A(\mathbb{R})$:

(i) $F' = (1-d)F + dF_n$ (where $d \in A(\mathbb{R})$ is a parameter) with $c_k(h) = dh^k$,

(ii) $F'' = F + F_h D$ with $c_k(h) = kh^{k-1}$,

(iii) $F''' = F_h + FD$ with $c_k(h) = h^k + \delta_{1k}$,

(iv) $F^{(iv)} = \frac{1}{h} F_h R$ with $c_k(h) = \frac{h^k}{k+1}$,

(v) $F^{(v)} = (1-d)F + dF^{(iv)}$ with $c_k(h) = d\frac{h^k}{k+1}$,

(vi) $F^{(vi)} = \sum_{j=1}^{m} \alpha_j F_{h_j}$, where $\sum_{j=1}^{m} \alpha_j = 1$, $\alpha_j, h_j \in A(\mathbb{R})$, $h_j \neq h_k$, for $j \neq k$, $j = 1,\dots,m$, with $c_k(h) = \sum_{j=1}^{m} \alpha_j h_j^k$.

Observe that for all initial operators $F', F'', F''', F^{(iv)}, F^{(v)}$ we have $c_k(h) \neq 0$ if $h \neq 0$.

Proof. All listed operators are projections onto $\ker D$, hence initial operators (cf. [3]). For arbitrary $z \in \ker D$ and $k \in \mathbb{N}$, $h \in \mathbb{R}$, Proposition 2.1 implies

$$F'R^k z = [(1-d)F + dF_h]R^k z = (1-d)FR^k z + dF_h R^k z = d\,\frac{h^k}{k!}\,z;$$

$$F''R^k z = (F + F_h D)R^k z = FR^k z + F_h R^{k-1} z = \frac{h^{k-1}}{(k-1)!}\,z = \frac{1}{k!}\,kh^{k-1} z\ ;$$

$$F'''R^k z = (F_h + FD)R^k z = F_h R^k z + FR^{k-1} z =$$

$$\frac{h^k}{k!}\,z + \delta_{1k} z = \frac{1}{k!}\,(h^k + \delta_{1k} k!)z = \frac{1}{k!}\,(h^k + \delta_{1k})z;$$

$$F^{(iv)}R^k z = \frac{1}{h}\,F_n R R^k z = \frac{1}{h}\,F_n R^{k+1} z = \frac{1}{h}\,\frac{h^{k+1}}{(k+1)!}\,z = \frac{1}{k!}\,\frac{h^k}{k+1}\,z;$$

$$F^{(v)}R^k z = [(1-d)F + dF^{(iv)}]\,R^k z =$$

$$(1-d)FR^k z + dF^{(iv)}R^k z = \frac{1}{k!}\,d\,\frac{h^k}{k+1}\,z;$$

$$F^{(vi)}R^k z = \sum_{j=1}^{m} \alpha_j F_{h_j} R^k z = \sum_{j=1}^{m} \alpha_j \frac{h_j^k}{k!}\,z = \frac{1}{k!}\,\sum_{j=1}^{m} \alpha_j h_j^k z.$$

Example 2.1. Let $X = C(\mathbb{R})$, $D = \frac{d}{dt}$, $R = \int_0^t$ and $(Fx)(t) = x(0)$. Then R-shifts are defined as usual shift operators

$$(S_h x)(t) = x(t+h) \quad \text{for} \quad t,h \in \mathbb{R},\ x \in X.$$

Thus we have

$$(F_h x)(t) = x(h), \quad R_h = \int_h^t \quad \text{and} \quad F_h R^k c = c\,\frac{h^k}{k!}.$$

for $c,h,t \in \mathbb{R}$, $x \in X$, $k \in \mathbb{N}$. We therefore conclude that $F_h \in c(R)$ and that $c_k(h) = h^k$. By Corollary 2.2, the initial operators defined by

$$(F'x)(t) = (1-d)x(0) + dx(h)\ , \quad (F''x)(t) = x(0) + x'(h)\ ,$$

$$(F'''x)(t) = x(h) + x'(0)\ , \quad (F^{(iv)}x)(t) = \frac{1}{h}\int_0^h x(s)ds,$$

$$(F^{(v)}x)(t) = (1-d)x(0) + \frac{d}{h}\int_0^h x(s)ds\ , \quad (F^{(vi)}x)(t) = \sum_{j=1}^{m} \alpha_j x(h_j),$$

also belong to $c(R)$. Recall that in this case we have $\dim \ker D = 1$. Hence, by Theorem 2.1, all initial operators have the property (c).

Example 2.2. Let $X = C[1,\infty]$, $D = t\frac{d}{dt}$,

$(Rx)(t) = \int_1^t s^{-1}x(s)ds$, $(Fx)(t) = x(1)$. Here we have

$$(S_h x)(t) = x(e^h t),\ (F_h x)(t) = x(e^h),\ R^k c = \frac{c(\ln t)^k}{k!}$$

for $h,t \in \mathbb{R}$, $x \in X$, where c is an arbitrary constant (cf. [3]). Since $\dim \ker D = 1$, we conclude that all initial operators have the property (c).

Example 2.3. Suppose that $X = (s)$ is the space of all sequences $x = \{x_n\}$ (where either $x_n \in \mathbb{R}$ or $x_n \in \mathbb{C}$ for $n \in \mathbb{N}$). Consider the operator of forward shift $D\{x_n\} = \{x_{n+1}\}$. Then $D \in R(X)$ and $\dim \ker D = 1$. Write $R\{x_n\} = \{x_{n-1}\}$, where we admit x_{n-k} if $n \le k$ for all $x \in X$, $F\{x_n\} = x_1\{\delta_{1n}\}$. Then F is an initial operator for D corresponding to $R \in \mathcal{R}_D$. Since

$\dim \ker D = 1$, we conclude that all initial operators have the property (c). To calculate numbers c_k observe that every initial operator can be written in the form $F_f\{x_n\} = f(x)\{\delta_{1n}\}$, where f is a linear functional over X, i.e. $f = \sum_{j=1}^{m} \alpha_j x_j$, and $\alpha_1, \ldots, \alpha_m$ are scalars such that $\sum_{j=1}^{m} \alpha_j = 1$. Let $z \in \ker D$ be arbitrary, i.e. $z = c\{\delta_{1n}\}$. We have for $k \in \mathbb{N}$:

$$R^k z = cR^k\{\delta_{1n}\} = c\{\delta_{1,n-k}\} = c\{\delta_{n,k+1}\},$$

$$F_f R^k z = f(R^k z)\{\delta_{1n}\} = c\{\delta_{1n}\} \sum_{j=1}^{m} \alpha_j \delta_{j,k+1}.$$

Hence

$$c_k(f) = \sum_{j=1}^{m} \alpha_j \delta_{j,k+1} = \begin{cases} k!\alpha_k & \text{if } k \le m-1, \\ 0 & \text{otherwise,} \end{cases}$$

and not all $c_k(f)$ vanish simultaneously, if $f \neq 0$, since not all α_k vanish simultaneously, by the assumption that $\sum_{j=1}^{m} \alpha_j = 1$.

Example 2.4. Let X, D, R, S_h be defined as in Proposition 2.3. Let $P(t) = \prod_{j=1}^{n} (t-h_j)^{r_j}$ and $r(n) = (\prod_{j=1}^{n} r_j!)^{-1}$, where we admit $h_i \neq h_j$ for $i \neq j$.

Corollary 2.1 implies

$$\left[\prod_{j=1}^{n} F_h R_{h_j}^{r_j} \right] z = r(n)P(h)z \text{ for all } z \in \ker D \text{ and } h \in A(\mathbb{R}). \tag{2.5}$$

In order to prove Formula (2.5) we apply step by step formula (2.3).

3. Interpolation formulae

Suppose that $D \in R(X)$. The interpolation problem (L) is to find a D-polynomial of degree $N-1$, i.e. an element $u = \sum_{k=0}^{N-1} R^k z_k$, where $R \in \mathcal{R}_D, z_0, \dots, z_{N-1} \in \ker D$, which admits for given initial operators $F_0, \dots, F_{N-1}$ the given values

$$F_i u = u_i, \quad i = 0, 1, \dots, N-1,$$

where $u_i \in \ker D$ ($F_i \neq F_j$ for $i \neq j$).

Theorem 3.1. Suppose that $D \in R(X)$, $F_0, \dots, F_{N-1} \in c(R)$ for an $R \in \mathcal{R}_D$, i.e.

$$(3.2) \qquad F_i R^k z = \frac{d_{ik}}{k!} z \quad \text{for} \quad i = 0, 1, \dots, N-1; \; k \in \mathbb{N}$$

(we admit $d_{i0} = 1$ for $i = 0, 1, \dots, N-1$, since $F_i z = z$ for all $z \in \ker D$). If

$$(3.3) \qquad V_N = \det(d_{ik})_{i,k=0,1,\dots,N-1} \neq 0,$$

then the interpolation problem (L) has a unique solution for every $u_0, \dots, u_{N-1} \in \ker D$ of the form

$$(3.4) \qquad u = w_N(u_0, \dots, u_{N-1}),$$

where

$$(3.5) \qquad w_N(u_0, \dots, u_{N-1}) = \sum_{j=0}^{N-1} V_{Nj}(R) u_j,$$

$$V_{Nj}(t) = \frac{1}{V_N} \sum_{k=0}^{N-1} (-1)^{k+j} k! V_{Njk} t^k, \quad j = 0, 1, \dots, N-1,$$

and V_{Njk} is the minor determinant obtained by cancelling in V_N the k-th column and the j-th row, $j, k = 0, 1, \dots, N-1$.

Proof. We are looking for a D-polynomial $u = \sum_{k=0}^{N-1} R^k z_k$, satisfying conditions (3.1), where $z_0,\dots,z_{N-1} \in \ker D$ are to be determined. Property c(R) implies that for $i = 0,1,\dots,N-1$ we have

$$u_i = F_i u = F_i \sum_{k=0}^{N-1} R^k z_k = \sum_{k=0}^{N-1} F_i R^k z_k = \sum_{k=0}^{N-1} \frac{d_{ik}}{k!} z_k = \sum_{k=0}^{N-1} d_{ik} w_k,$$

where $w_k = \frac{z_k}{k!}$, $k = 0,1,\dots,N-1$.

Therefore we have obtained a system of N algebraic equations

$$\sum_{k=0}^{N-1} d_{ik} w_k = u_i \ , \ i = 0,1,\dots,N-1 \tag{3.7}$$

with N unknowns $w_0,\dots,w_{N-1}$.

The determinant V_N of this system is, by our assumption, different from zero. Hence the system (3.7) has a unique solution

$$w_k = \frac{1}{V_N} \sum_{j=0}^{N-1} (-1)^{k+j} V_{Njk} u_j \ , \quad k = 0,1,\dots,W-1,$$

where V_{Njk} is the minor determinant obtained by cancelling in V_N the k-th column and the j-th row, $j,k = 0,1,\dots,N-1$. Since $z_k = k! w_k$ for $k = 0,1,\dots,N-1$, we conclude that the interpolation problem (L) has a unique solution, for every $u_0,\dots,u_{N-1} \in \ker D$, of the form

$$u = \sum_{k=0}^{N-1} R^k z_k = \sum_{k=0}^{N-1} k! R^k w_k = \sum_{k=0}^{N-1} k! R^k \frac{1}{V_N} \sum_{j=0}^{N-1} (-1)^{k+j} V_{Njk} u_j =$$

$$= \sum_{j=0}^{N-1} \frac{1}{V_N} \Big(\sum_{k=0}^{N-1} (-1)^{k+j} V_{Njk} R^k \Big) u_j = \sum_{j=0}^{N-1} V_{Nj}(R) u_j = W_N(u_0,\dots,u_{N-1}).$$

Observe that the mapping $W_N : (\ker D)^N \longrightarrow \operatorname{dom} D^N$, which appears in formula (3.4), is a sum of N linear operators such that the j-th operator acts on the j-th variable and

$$W_N(0,0,\dots,0) = 0.$$

This, in particular, means that the interpolation problem (L) with zero conditions $u_i = 0$ for $i = 0,1,\dots,N-1$ has only a trivial solution. By definition, we have

$$(3.8) \qquad u_i = F_i W_N(u_o,\dots,u_{N-1}) \quad \text{for} \quad i = 0,1,\dots,N-1,$$

where $u_o,\dots,u_{N-1} \in \ker D$.

Corollary 3.1. Suppose that $D \in R(X)$, F is an initial operator for D corresponding to an $R \in \mathcal{R}_D$, $\{S_h\}_{h \in A(\mathbb{R})}$ is a family of R-shifts and $F_i = F_{h_i} = FS_{h_i}$, where $h_i \neq h_j$ for $i \neq j$, $i,j = 0,1,\dots,N-1$. Then the interpolation problem (L) has a unique solution for every $u_o,\dots,u_{N-1} \in \ker D$, of the form (3.4), where $V_N = V_N(h_i^k) \neq 0$.

Proof. By our assumptions and Proposition 2.2, initial operators $F_o,\dots,F_{N-1} \in c(R)$ and $d_{ik} = h_i^k$, $i,k = 0,1,\dots,N-1$. The determinant $V_N = V_N(h_i^k)$ is the Vandermonde determinant of N different numbers $h_o,\dots,h_{N-1}$, hence is different from zero. This and Theorem 3.1 together imply our Corollary.

Example 3.1. Let X, D, R, S_h, F_h, $h \in \mathbb{R}$, be defined as in Example 2.1. Formulae (3.4)-(3.6) give the well-known Lagrange interpolation formula

$$(3.9) \qquad u(t) = \sum_{j=0}^{N-1} \Big(\prod_{\substack{k=0 \\ k \neq j}}^{N-1} \frac{t-h_k}{h_j-h_k} \Big) u_j \ ,$$

where $u_j = u(h_j)$, $j = 0,1,\dots,N-1$.

Indeed, we can verify that

$$V_{Nj}(R) = \prod_{\substack{k=0\\k\neq j}}^{N-1} \frac{t-h_k}{h_j-h_k}, \quad j = 0,1,\dots,N-1 \tag{3.10}$$

(provided that $h_j \neq h_k$ for $j\neq k$).

Example 3.2. Suppose that X,D,R,S_h,F_h, $h \in \mathbb{R}_+$, are defined as in Example 2.2. Then

$$V_{Nj}(R)u_j = \frac{1}{V_N}\left[\sum_{k=0}^{N-1} (-1)^{k+1} V_{Nkj}(\ln t)^k\right]u_j, \quad j = 0,1,\dots,N-1.$$

In a similar manner we can obtain a generalization of the Hermite interpolation formula, i.e. a formula for interpolation with multiple knots. Namely, the interpolation problem (H) is to find a D-polynomial u of degree $N-1$ which admits for $n \leq N$ given different initial operators $F_0,\dots,F_{n-1}$ the given values together with the "derivatives" D^ku up to the order r_j, i.e. for a solution of the following problem

$$F_jD^ku = u_{jk}, \quad k = 0,1,\dots,r_j-1; \; j = 1,\dots,n; \tag{3.11}$$

$r_1 + \dots + r_n = N$, where $u_{jk} \in \ker D$ are given, $u = \sum_{k=0}^{N-1} R^k z_k$ for an $R \in \mathcal{R}_D$ and $z_0,\dots,z_{N-1}$ are to be determined.

Theorem 3.2. Suppose that all assumptions of Theorem 3.1 are satisfied. Denote by H_j the matrix of dimension $r_j \times N$

$$H_j = \left(\frac{d_{j,m-k}}{(m-k)!}\right)_{\substack{m=k,\dots,N-1\\k=0,1,\dots,N-1}}, \quad j=1,\dots n, \tag{3.12}$$

and by H the square matrix of dimension N

$$H = \begin{pmatrix} H_1 \\ \vdots \\ H_n \end{pmatrix}. \tag{3.13}$$

If det H ≠ 0, then the interpolation problem (H) has a unique solution, for every $u_{jk} \in \ker D$, $k = 0,1,\dots,r_j-1$; $j = 1,\dots,n$, of the form

$$(3.14)\qquad u = W_N^H(u_0,\dots,u_{N-1}),$$

where

$$(3.15)\qquad W_N^H(u_0,\dots,u_{N-1}) = \sum_{j=0}^{N-1} V_{Nj}^H(R)u_j\ ,$$

$$(3.16)\qquad V_{Nj}^H(t) = \frac{1}{\det H}\sum_{k=0}^{N-1}(-1)^{k+j}V_{Njk}^H\, t^k, \quad j = 0,1,\dots,N-1.$$

V_{Njk} is the minor determinant obtained by cancelling in V_N the k-th column and the j-th row, $j,k = 0,1,\dots,N-1$, and elements $u_0,\dots,u_{N-1} \in \ker D$ are defined as follows

$$u_\nu = u_{1\nu} \quad \text{if} \quad \nu = 0,1,\dots,r_1-1,$$

$$(3.17)\qquad \begin{cases} u_{r_1+\nu} = u_{2\nu} & \text{if}\quad \nu = 0,1,\dots,r_2-1,\\ u_{r_1+r_2+\nu} = u_{3\nu} & \text{if}\quad \nu = 0,1,\dots,r_3-1,\\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots & \\ u_{r_1+r_2+\dots+r_{n-1}+\nu} = u_{n\nu} & \text{if}\quad \nu = 0,1,\dots,r_n-1\ . \end{cases}$$

P r o o f. By our assumptions, we have for $k=0,1,\dots,r_j-1$ and $j = 1,\dots,n$

$$u_{jk} = F_jD^ku = F_jD^k\sum_{m=0}^{N-1}R^mz_m = F_j\sum_{m=0}^{N-1}D^kR^mz_m =$$

$$= F_j\Big(\sum_{m=0}^{N-k}D^{k-m}z_m + \sum_{m=k+1}^{N-1}R^{m-k}z_m\Big) = F_j\Big(z_m + \sum_{m=k+1}^{N-1}R^{m-k}z_m\Big) =$$

$$= F_j\sum_{m=k}^{N-1}R^{m-k}z_m = \sum_{m=k}^{N-1}F_jR^{m-k}z_m = \sum_{m=k}^{N-1}\frac{d_{j,m-k}}{(m-k)!}\,z_m.$$

We obtain a system of N equations with N unknowns

$$\sum_{m=k}^{N-1} \frac{d_{j,m-k}}{(m-k)!} z_m = u_{jk}, \quad k = 0,1,\dots,r_j-1; \; j = 1,\dots,n.$$

Let us fix j. Then the matrix of coefficients is the matrix H_j defined by formula (3.12). Since $\det H \neq 0$, applying the denotation (3.17) we obtain a unique solution of the problem in question in a similar way as in the proof of Theorem 3.1.

NOTE. The matrices H_j defined by formulae (3.12) have the following property rank: $H_j = r_j$, $j = 1,2,\dots,n$.

Indeed,

$$\det H_j = \det \left(\frac{d_{j,m-k}}{(m-k)!} \right) = d_{j_o}^{r_j} = 1.$$

Moreover,

$$\sum_{j=1}^{N} \operatorname{rank} H_j = \sum_{j=1}^{n} r_j = N,$$

where N is the dimension of the square matrix H. If $\operatorname{rank} H = N = \sum_{j=1}^{N} \operatorname{rank} H_j$, then, of course, $\det H \neq 0$. In general, we do not know if H has this property. Thus we have to assume that $\det H \neq 0$. However, $\det H \neq 0$ if $d_{ik} = \frac{h_i^k}{k!}$ (cf. Example 3.1) as the following example shows.

Example 3.3. Consider the classical case of the problem (H), i.e. X, D, R, S_h, F_h, $h \in \mathbb{R}$, defined as in Example 2.1. Then the Hermite interpolation problem has a unique solution for every u_{jk}, $k = 0,1,\dots,r_{j-1}$; $j = 1,\dots,n$, hence $\det H \neq 0$ (cf. Hermite [1], also the author [3], p.38).

Consider now another problem of interpolation. Namely, the interpolation problem (Nt) is to find a D-polynomial u of degree N-1 which satisfies for N given initial operators $F_o,\dots,F_{N-1}$ the following conditions

$$(3.18)\qquad F_mD^mu = u_m\ ,\quad m = 0,1,\dots,N-1,$$

where $u_o,\dots,u_{N-1}\in \ker D$ are given (we assume that $F_m \neq F_j$ for $m\neq j$).

Theorem 3.3. Suppose that all assumptions of Theorem 3.1 are satisfied. Then the interpolation problem (Nt) has a unique solution for every $u_o,\dots,u_{N-1}\in \ker D$ which is of the form

$$(3.19)\qquad u = \sum_{m=0}^{N-1} V_{Nm}^{Nt}(R)u_m\ ,$$

where

$$(3.20)\qquad V_{Nm}^{Nt} = \sum_{k=0}^{N-1}(-1)^{m+k}V_{Nmk}^{Nt}\ t^k,\quad m = 0,1,\dots,N-1$$

and V_{Nmk}^{Nt} is the minor determinant obtained by cancelling the k-th column and the m-th row, $k,m = 0,1,\dots,N-1$ in the determinant

$$(3.21)\qquad V_N^{Nt} = \det\left(\frac{d_{m,k-m}}{(k-m)!}\right)_{\substack{k=m,\dots,N-1\\ m=0,1,\dots,N-1}}\ .$$

Proof. Let $u = \sum_{k=0}^{N-1}R^kz_k$, where $z_o,\dots,z_{N-1}\in \ker D$ are to be determined. Conditions (3.18) and the property c(R) together imply that , for $m = 0,1,\dots,N-1$, we have

$$u_m = F_mD^mu = F_mD^m\sum_{k=0}^{N-1}R^kz_k = F_m\left(\sum_{k=0}^{m}D^mR^kz_k + \sum_{k=m+}^{N-1}{}^{m}R^kz_k\right) =$$

$$= F_m\left(\sum_{k=0}^{m}D^{m-k}z_k + \sum_{k=m+1}^{N-1}R^{k-m}z_k\right) = F_m\left(z_m + \sum_{k=m+1}^{N-1}R^{k-m}z_k\right) =$$

$$= F_m \sum_{k=m}^{N-1} R^{k-m} z_k = \sum_{k=m}^{N-1} F_m R^{k-m} z_k = \sum_{k=m}^{N-1} \frac{d_{m,k-m}}{(k-m)!} z_k .$$

We therefore have obtained a system of N equations

$$\sum_{k=m}^{N-1} \frac{d_{m,k-m}}{(k-m)!} z_k = u_m , \quad m = 0,1,\ldots,N-1, \tag{3.22}$$

with N unknown. The determinant of this system is the determinant V_N^{Nt} defined by formula (3.12). Since the matrix of this determinant has only zeros below the principal diagonal, we have $d_{oo} d_{1o} \ldots d_{N-1,o} = 1$. We conclude that the system (3.22) has a unique solution for every $u_o, \ldots, u_{N-1} \in \ker D$ and that it is of the form (3.19)-(3.21).

Note. We should point out that the uniqueness of the solution of the interpolation problem (Nt) is an immediate consequence of the Taylor-Gontcharov formula for right invertible operators (cf. [3]). However, a solution obtained by means of this Formula is much more complicated for calculations. Indeed, the interpolation problem (Nt) can be interpreted as a mixed boundary value problem

$$D^N u = 0, \quad F_m D^m u = u_m \in \ker D, \quad m = 0,1,\ldots,n-1. \tag{3.23}$$

The Taylor-Gontcharov formula for $u \in \operatorname{dom} D^N$ implies

$$u = F_o u + \sum_{m=1}^{N-1} R_o \ldots R_{m-1} F_m D^m u + R^N D^N u = u_o + \sum_{m=1}^{N-1} R_o \ldots R_{m-1} u_m ,$$

where F_m corresponds to a right inverse R_m , $m = 0,1,\ldots,N-1$. Thus we have a unique u satisfying conditions (3.23), but instead of powers of one right inverse R we have superpositions of m right inverses $R_o, \ldots, R_{m-1}$.

$m = 0,1,\dots,N-1$.

Example 3.4. Suppose that X, D, R, S_h, F_h, $h \in \mathbb{R}$, are defined as in Example 3.1. Then formulae (3.19)-(3.21) determine the well-know Newton interpolation formula

$$(3.24)\quad u(t) = u_o + u_1(t-h_o) + \dots + u_{N-1}(t-h_o)\dots(t-h_{N-2}).$$

Now, we shall consider the general interpolation problem (shortly: (GI)). Given n finite sets I_j of positive integers. Denote by p_j the power of the set I_j: $I_j = p_j$ for $j=1,\dots,n$. Let $N = \sum_{j=1}^{n} p_j$. We are looking for a D-polynomial u of degree N-1 satisfying, for n given different initial operators $F_o,\dots,F_{n-1}$, the conditions

$$(3.25)\qquad F_jD^ku = u_{jk}\ ,\quad k \in I_j;\quad j = 1,\dots,n,$$

where $u_{jk} \in \ker D$ are given, $u = \sum_{k=0}^{N-1} R^kz_k$ for an $R \in \mathcal{R}_D$ and $z_o,\dots,z_{N-1} \in \ker D$ are to be determined.

If $I_j = \{0\}$ for $j = 1,\dots,n$, then we obtain conditions (3.1) with N=n, i.e. the interpolation problem (L).

If $I_j = \{j\}$ for $j = 1,\dots,n$, then we obtain conditions (3.18) with N=n, i.e. the interpolation problem (Nt).

If $I_j = \{0,\dots,p_j-1\}$ for $j = 1,\dots,n$, then we obtain the conditions (3.11) with $r_j = p_j$, $N = \sum_{j=1}^{n} p_j$, i.e. the interpolation problem (H).

Theorem 3.4. Suppose that all assumptions of Theorem 3.1 are satisfied. If

$$\det V_N^G = \det \begin{pmatrix} V_{N,1}^G \\ \cdots\cdots \\ V_{N,n}^G \end{pmatrix} \neq 0, \tag{3.26}$$

where

$$V_{N,j}^G = \left(\frac{d_{j,m-k}}{(m-k)!} \right)_{\substack{k \in I_j \\ m=k,\dots,N-1}} ; \quad j = 1,\dots,n \tag{3.27}$$

then the general interpolation problem (GI) has a unique solution (which can be constructed in a similar way as in Theorem 3.2) for every $u_{jk} \in \ker D$, $k \in I_j$; $j = 1,\dots,n$.

The proof is going on similar lines as that of Theorem 3.2. Observe that we have rank $V_{N,j}^G = p_j$ for $j = 1,\dots,n$, hence $\sum_{j=1}^{n}$ rank $V_{Nj}^G = N$ and N is the dimension of the square matrix V_N^G. However, we do not know, is rank $V_N^G = N$? Thus we have to assume $\det V_N^G \neq 0$.

The following question arises. Suppose that X, D, R, S_h, F_h, $h \in \mathbb{R}$, are defined as in Example 2.1. Is $\det V_N^G \neq 0$? Observe that for the Lagrange, Hermite and Newton interpolations problems the corresponding determinants are different from zero (cf. Examples 3.1, 3.3, 3.4) and the problem in question is a generalization of these three problems. Moreover, we have here

$$V_{N,j}^G = V_{N,j}^G(h_j), \text{ where } V_{N,j}^G(t) = \left(\frac{t^{m-k}}{(m-k)!} \right)_{\substack{k \in I_j \\ m=k,\dots,N-1}}$$

$j = 1,\dots,n$, and $V_{N,j}^G$ are defined by formulae (3.27).

REFERENCES

[1] Ch. Hermite: Sur la formule d'interpolation de Lagrange, J.Reine Angew. Math. 84 (1878), 70-79.

[2] D. Przeworska-Rolewicz: Concerning boundary value problems for equations with right invertible operators, Demonstratio Math. 7(1974), 365-380.

[3] D. Przeworska-Rolewicz: Algebraic Analysis. PWN-Polish Scientific Publishers and D.Reidel, Warszawa-Dordrecht, 1988.

[4] M. Tasche: A Unified Approach to Interpolation Methods, Journ. of Integral Eqs. 4 (1982), 55-75.

[5] V. L. Gontcharov: Theory of Interpolation and Approximation of Functions (in Russian), Moscow, 1954.

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ON CONGRUENCE CONVEX BCK-ALGEBRAS

Dedicated to the memory
of Professor Edward Otto

It is proved that if in a BCK-algebra A every chain is well-ordered by the relation (3) then every congruence relation on A is convex. Also a strictly increasing sequence of varieties of BCK-algebras is selected with the variety of commutative BCK-algebras at the bottom.

1. Introduction

Convex congruences have been successfully used in developing the theory of partially ordered groups (see e.g. L.Fuchs [2] and W.M.Kopytov [4]).

Definition. Let A be any partially ordered algebra (abbreviated: p.o.algebra); i.e. A is an algebra, A is a p.o.set, every algebraic operation f on A fulfills a monotony law. Then a congruence θ on A is said to be convex iff

(1) $a \le b \le c$ and $a \sim c(\theta)$ imply $a \sim b(\theta)$

for all a,b,c in A.

If A is a p.o. group, then the monotony law

(2) $a \le b$ implies $ca \le cb$ and $ac \le bc$

for all a,b,c, in A holds (G.Birkhoff [1]).

The following simple facts, concerning p.o. groups are known, or easy to be proved.

Theorem 1.1. Let θ be a congruence on a p.o. group G. Then θ is convex if and only if the set $H = \{x \in G : x \sim e(\theta)\}$ is a convex and normal subgroup of G, where e denotes the neutral element of the group G.

Theorem 1.2. A normal subgroup H of a p.o. group G is the kernel of an order-preserving homomorphism (abbreviated: o-homomorphism) if and only if H is convex.

Theorem 1.3 (second isomorphism theorem). If φ is an o-epimorphism of G onto $\overline{G}$ and $\overline{H}$ is a normal and convex subgroup of $\overline{G}$ then the counter-image $\varphi^{-1}(\overline{H}) = H$ is a normal and convex subgroup of G and the factor groups G/H and $\overline{G}/\overline{H}$ are o-isomorphic.

Each BCK-algebra can also be considered as a p.o. algebra. Let us recall to mind that an algebra of type (2,0) is said to be a BCK-algebra whenever the following axioms are fulfilled

BCK1 $(xy \cdot xz) \cdot zy = 0$

BCK2 $(x \cdot xy)y = 0$

BCK3 $xx = 0$

BCK4 $0x = 0$

BCK5 $xy = 0$ and $yx = 0$ imply $x = y$.

Dot has only been used here to avoid too many brackets; for example BCK1 reads $((xy)(xz))(zy)=0$. The ordering relation is given by the equivalence

(3) $x \leq y$ iff $xy = 0$,

and the binary operation is of type $(\uparrow,\downarrow)$ with respect to this order, which means (see L.Fuchs [2]) that

(4) $a\leq b$ implies both $ac\leq bc$ and $ca\geq cb$

for all a,b,c in the considered BCK-algebra.

Theorem 1.4. Let θ be a congruence relation on a BCK-algebra A. Then the following conditions are equivalent

(i) θ is convex

(ii) $xy\sim 0(\theta)$ and $yx\sim 0(\theta)$ imply $x\sim y(\theta)$

(iii) A/θ is a BCK-algebra again

(see H. Yutani [6], and T. Traczyk and W. Zarębski [5]).

Lemma 1.5. A congruence relation θ on a BCK-algebra A is convex if and only if, for every a,b,b',c in A, $a\leq b$ and $b\sim b'(\theta)$ and $b'\leq c$ and $a\sim c(\theta)$ together imply $a\sim b(\theta)$.

Proof. Evidently, by putting $b = b'$, (5) implies convexity. Now let us suppose convexity, and let $a\leq b\sim b'\leq c$ and $a\sim c(\theta)$. Then $ba\sim b'a\leq ca\sim 0(\theta)$. Hence $b'a\sim 0(\theta)$, by convexity. Hence $ba\sim 0(\theta)$. Since $ab\sim 0(\theta)$ trivially, we obtain $a\sim b(\theta)$, by Theorem 1.4.

In the case of BCK-algebras both isomorphism theorems hold, and so they do in apparently stronger version, as follows.

Theorem 1.6. Let B be a subalgebra of a BCK-algebra A. Let θ be a congruence on A such that for every $a \in A$ there is $b\in B$ with $a\sim b(\theta)$. Let $\theta_B = \theta\cap B^2$. Then θ_B is convex if and only if θ is convex and the factor algebras A/θ and B/θ_B are isomorphic.

Theorem 1.7. Let φ be an epimorphism of a BCK-algebra A onto a BCK-algebra B. Let θ be a congruence relation on B and let

$$\bar{\theta} = \{(x,y)\in A^2 : \varphi(x)\sim\varphi(y) \quad (\theta)\}.$$

Then $\bar{\theta}$ is convex if and only if θ is convex and the factor algebras $A/\bar{\theta}$ and B/θ are isomorphic.

Lemma 1.5 provides us with a straightforward proof of Theorem 1.6.

2. BCK-algebras with each congruence convex

It follows from Theorem 1.4 that if V is a variety of BCK-algebras, then each congruence relation on every A in V is convex. J.K.Kabziński and A.Wroński proved, [3], that there does not exist a variety of BCK-algebras containing all finite ones. Therefore of certain importance is the following result obtained by H.Yutani [6].

Theorem 2.1. Every finite BCK-algebra has each congruence convex.

We shall prove the following generalization of this result.

Theorem 2.2. If every strongly decreasing sequence in a BCK-algebra A is finite, then every congruence relation θ on A is convex. So it is, in particular, if every chain in A is well ordered by the relation $\leq$ defined by (3).

Proof. Let $x_0 \leq y_0 \leq z_0$ and $x_0 \sim z_0(\theta)$. In the case $x_0 = 0$, we get $y_0x_0 \sim y_0z_0(\theta)$ and $y_0z_0 = 0$, i.e. $y_0 \sim 0(\theta)$, as claimed. So, we only have to consider the case $x_0 \neq 0$. This time, let

(i) $x_k = y_{k-1} \cdot y_{k-1}x_{k-1}$, $y_k = x_{k-1}$, $z_k = y_{k-1}$ for $k=1,2,\ldots$

Evidently, for $k = 0,1,\ldots$ we have

(ii) $x_k \geq x_{k+1}$

and

(iii) $x_k \leq y_k \leq z_k$.

If $x_k \sim z_k(\theta)$ for certain k, then $y_k x_k \sim y_k z_k(\theta)$ and $y_k z_k = 0$ whence $x_{k+1} \sim y_k(\theta)$, i.e. $x_{k+1} \sim z_{k+1}(\theta)$, by (i). Therefore, by induction argument,

(iv) $x_k \sim z_k(\theta)$ for every positive integer k.

Now, by our assumption, (ii) implies that there exists a positive integer n such that $x_n = x_{n+1}$. Then, by (i) and (iv),

(v) $x_{n+1} \sim y_{n+1}(\theta)$ and $y_{n+1} \sim z_{n+1}(\theta)$

and, again by (i) and (iv), and by (v),

(vi) $x_n \sim y_n(\theta)$ and $y_n \sim z_n(\theta)$.

This procedure can be iterated, therefore after finite number of backwards steps we arrive at $x_0 \sim y_0(\theta)$ and $y_0 \sim z_0(\theta)$, what completes the proof of the Theorem.

3. n-commutative BCK-algebras

The above proof of Theorem 2.2 inspired the following definition. A BCK-algebra A is said to be n-commutative iff, for all x_0, x_1 in A such that $x_1 \leq x_0$ and

(5) $x_k = x_{k-2} \cdot x_{k-2} x_{k-1}$ for $k=2,3,\ldots$

holds

(6) $x_n = x_{n+1}$.

Theorem 3.1. The class V_n of all n-commutative BCK-algebras is a variety.

Proof. It follows from Theorem 2.2 that every algebra in V_n has each congruence convex. Then, by Theorem 1.4, the homomorphic image of an algebra in V_n again is in V_n. Since any subalgebra of an algebra A in V_n is in V_n and direct products of algebras from V_n are in V_n as well, it follows by the Birkhoff theorem that V_n is a variety.

Its equational basis easily derives from BCK1 - BCK5 by exchanging quasi-equation BCK5 with the following equation

BCK5' $x_n(x,y) = x_{n+1}(x,y)$

where $x_0(x,y) = x$, $x_1(x,y) = y \cdot yx$ and

$x_k(x,y) = x_{k-2}(x,y) \cdot x_{k-2}(x,y)x_{k-1}(x,y)$ for $k=2,3,\ldots$

Evidently, BCK5' holds in V_n, by this definition.

Conversely, if $xy = 0 = yx$ then $x_{2k-1}(x,y) = y$ and $x_{2k}(x,y) = x$ for $k=1,2,\ldots$ Therefore BCK5' implies $x=y$, in this case, and BCK5 follows.

Of course, the sequence $V_1, V_2, \ldots$ of classes is increasing. Now, we shall show that it is strongly increasing.

Theorem 3.2. For every positive integer n, $V_n \neq V_{n+1}$.

Proof. We shall, in fact prove, that for $n \geq 2$, $V_{n-1} \neq V_n$. Let A be an algebra of type (2,0) defined on a strictly decreasing sequence $x_0 > x_1 > \ldots > x_n > 0$ with the binary operation satisfying the following set of postulates:

(i) $0 = 0 \cdot 0 = 0x_i = x_ix_j$ for $j \leq i$ and $i=0,\ldots,n$

(ii) $x_ix_j = x_n$ for $j=i+1$ and $i=0,\ldots,n-1$

(iii) $x_ix_j = x_{n+i+2-j}$ for $i+1 < j \leq n$ and $i=0,\ldots,n-2$

(iv) $x_i0 = x_j$ for every $i=0,\ldots,n$.

By this definition, if we additionally denote

(v) $x_{n+1} = x_{n-1} \cdot x_{n-1}x_n$

then (5) and (6) hold, although $x_{n-1} \neq x_n$. Therefore, in order to prove that $V_{n-1} \neq V_n$, it ought to be shown that A is a BCK-algebra and belongs to V_n.

Since, by (i) - (iv), $xx=0$ and $0x=0$, and $xy=0$ iff $x\leq y$, it follows that BCK3, BCK4 and BCK5 hold. In order to prove BCK1 and BCK2 let us consider BCK2, first. Since it holds for either $x=0$ or $y=0$, by (i) and (iv), we only have to show that

(7) $x_i \cdot x_ix_j \leq x_j$ for all $i,j=0,\dots,n$

This inequality holds if $j\leq i$, evidently by (i) and (iv). If $j = i+1$ then

$$x_i\cdot x_ix_j = x_ix_n = \begin{cases} x_n & \text{for } j = n \\ x_{n+i+2-n} = x_{i+2} < x_j \end{cases}$$

by (ii) and (iii), and (7) follows.

In the last case, i.e. when $i+1<j\leq n$, also $2+i\leq n+i+2-j<n+1$ and this implies that $x_i \cdot x_ix_j = x_ix_{n+i+2-j} = x_j$, by (iii), which completes the proof of BCK2.

For to prove BCK1, let us firstly observe that, by (i) and (iv) and by BCK2, BCK1 holds if either $x=0$ or $x\leq y$ or $y = 0$. Since, by (i) and (ii) and (iii), we have $xy\leq x$ always in A, it follows that BCK1 holds if $z=0$, as well. So, it remains to show that

(8) $x_ix_j \cdot x_ix_k \leq x_kx_j$

for all i, j, k with $j>i$. (8) holds, trivially, for $k=i$. If $k<i<j$, then

$$x_ix_j \cdot x_ix_k = x_ix_j = x_{n+i+2-j} \leq x_{n+k+2-j} = x_kx_j$$

and (8) follows.

Now let $j>i$ and $k>i$. If $k\geq j\geq i+1$ then, by (ii) or (iii), $x_ix_j \le x_ix_k$ and the inequality (8) follows in this case (because its left-hand side equals zero). If $j>k=i+1$ then

$$x_ix_j\cdot x_ix_k = x_{n+i+2-j}x_n = \begin{cases} x_n \le x_kx_j & \text{for } j=i+3 \\ x_{n+i+4-j} \le x_{n+i+3-j} = x_kx_j & \text{for } j>i+3 \end{cases}$$

and the inequality (8) follows in both above cases.

In the last possible case, i.e. if $j>k>i+1$, we use (ii) and (iii) as follows

$$x_ix_j\cdot x_ix_k = \begin{cases} x_n \le x_kx_j & \text{for } j=k+1 \\ x_{n+i+2-j}x_{n+i+2-k} = x_{n+k+2-j} = x_kx_j & \text{for } j>k+1 \end{cases}$$

and again (8) follows. So we have just proved that A is, in fact, a BCK-algebra. Its non-zero members form an n-element chain, which fulfills (5) and, also, (6) in virtue of the denotation (v). Since we cannot find any chain in A longer than that one and satisfying (5), this algebra belongs to the variety V_n. Evidently, it does not belong to the variety V_{n-1}, and this remark completes the proof of Theorem 3.2.

4. Final remarks and problems

For arbitrary x,y in a given BCK-algebra A, two sequences satisfying (5) can be built up

$$x_0 = x,\quad x_1 = y\cdot yx,\ \ldots,\quad x_k = x_{k-2}\cdot x_{k-2}x_{k-1},\ \ldots$$

$$y_0 = y,\quad y_1 = x\cdot xy,\ \ldots,\quad y_k = y_{k-2}\cdot y_{k-2}y_{k-1},\ \ldots$$

By certain general properties of BCK-algebras, the following inequalities hold

(9) $x_0 \ge y_1 \ge x_2 \ge y_3$,

(10) $y_0 \ge x_1 \ge y_2 \ge x_3$.

In fact, $x_0 \ge y_1$, trivially.

Now, $x_2y_1 = x(x(y\cdot yx))\cdot(x\cdot xy) = x(x\cdot xy)\cdot x(y\cdot yx) = xy\cdot x(y\cdot yx) = 0$ and $y_2x_1 = 0$, by exchanging x with y in the above line. Similarly,

$$y_3x_2 = (y_1\cdot y_1y_2)\cdot x_2 = ((x\cdot xy)\cdot x(x(y\cdot yx)))\cdot y_1y_2 = (x(y\cdot yx)\cdot xy)\cdot y_1y_2 =$$
$$= ((x\cdot xy)(y\cdot yx))\cdot y_1y_2 = y_1x_1\cdot y_1y_2 = 0$$

because $x_1 \geq y_2$. Again, by exchanging x with y in the above two lines, we get $x_3y_2 = 0$, as well. Besides (2), two other general rules have been used here: $xy\cdot z = xz\cdot y$ and $x(x\cdot xy) = xy$.

Proposition 4.1. The class V_1 coincides with the variety of all commutative BCK-algebras.

Proof. Since $x_1 = x_2$ (and $y_1 = y_2$) in V_1, (9) implies $y_1 \geq x_1$ and (10) implies $x_1 \geq y_1$. Hence $x_1 = y_1$ which means commutativity.

Conversely, every commutative BCK-algebra belongs to V_1.

Proposition 4.2. The variety V_2 is characterized by the equation

(11) $x_2 = y_2$.

Proof. A similar proof to the above one is left to the reader.

Problem 1. Prove or disprove that for any BCK-algebra A the sequences of inequalities (9) and (10) can be prolonged.

Problem 2. Prove or disprove that the variety V_n is characterized by the equation $x_n = y_n$.

Remark. Besides the theory of p.o. groups, this paper may be considered as further motivation for studying convex congruences in Universal Algebra setting.

REFERENCES

[1] G. Birkhoff : Lattice theory. Third edition, Providence, R.I. 1967.

[2] L. Fuchs : Partially ordered algebraic systems, Pergamon Press, New York, 1963.

[3] J. K. Kabziński, A. Wroński : There is no largest variety of BCK-algebras, Math. Japon. 29 (1984), 545-549.

[4] W. M. Kopytow : Lattice ordered groups (Russian), Moskwa, 1984.

[5] T. Traczyk, W. Zarębski : Convex congruences on BCK-algebras, Demonstratio Math., 18 (1985), 319-323.

[6] H. Yutani : Quasi-commutative BCK-algebras and congruence relations, Math.Seminar Notes, 5 (1977), 469-480.

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EXISTENCE AND STABILITY OF GLOBAL CLASSICAL SOLUTIONS FOR SEMILINEAR PARABOLIC EQUATIONS OF ORDER 2M

Dedicated to the memory
of Professor Edward Otto

1. Introduction

Let Ω be a bounded domain in the n-dimensional Euclidean space R^n, $n\geq 2$ with C^∞- boundary Γ. Let $H^r(\Omega)$, $r\in N$ and $H^r_o(\Omega)$ be Sobolev spaces with the norm $\|\cdot\|_{H^r}$. Let A_o be an uniformly strongly elliptic operator of order 2m, $m\geq 1$ of the form

$$A_o = \sum_{|p|=|q|=0}^{m} (-1)^{|p|} D^p a_{pq}(x) D^q \text{ , } x\in\Omega$$

functions $a_{pq}:\Omega \longrightarrow R$ and p, q are multiindices $p=(p_1,\dots,p_n)$, $q=(q_1,\dots,q_n)$, $|p| = p_1+\dots+p_n$. Then for any $v\in H^m_o(\Omega)$ the operator A_o satisfies Gårding's inequality $B(v,v)\geq \gamma_o\|v\|^2_m - \lambda_o\|v\|^2_o$ with constants $\gamma_o\geq 0$, $\lambda_o\geq 0$ and the bilinear form $B(u,v)$, corresponding to operator A_o, is given by the formula

$$B(u,v) = \sum_{|p|=|q|=0}^{m} \int_\Omega a_{pq} D^q u D^p v \, dx \, .$$

If $A = A_o + a(x)I$, $(a:\Omega\longrightarrow R,\ a(x)\geq\lambda_o$ for $x\in\Omega$, I-the identity operator), then the bilinear form $B(u,v)$ corresponding to operator A satisfies inequality

(1) $$B(v,v) \geq \gamma_0 \|v\|_m^2 .$$

In this paper existence and stability of global bounded classical solutions of the following problem have been investigated:

(2) $u_t + Au = F(t,x,u,Du,\dots,D^m u), \quad (t,x)\in R^+\times \Omega, \ R^+ = \langle 0,\infty)$,

(3) $u(x,0) = \phi(x), \quad x\in\Omega$,

(4) $D^\beta u|_\Gamma = 0$ in the sense of traces for $|\beta| \leq m-1$,

$(t,x)\in R^+\times\Gamma$,

$F : R^+\times\Omega\times R^N \longrightarrow R, \qquad \phi : \Omega \longrightarrow R$ given functions,

$N = n + \frac{(n+m)!}{n!m!}$.

The existence of solutions for the problem similar (2)-(4) has been investigated by many authors. For a linear equation with $F = f(t,x)$ G. Fichera [3] has proved the existence of solutions in C^∞-class. J. Kačur [4] has investigated the existence of solutions of the nonlinear equation with $a_{pq} = a_{pq}(x,t,u)$. He has used Rother's method. A. Pazy [9] using semigroups method has proved existence of solutions with $F = f(t,x,u,Du,\dots,D^{2m-1}u)$ and $a_{pq} = a_{pq}(x)$. J. Lions [8] applying compactness method has searched solutions of the nonlinear equation with term $|u|^\rho u$, $\rho > 0$. The existence of global solutions has been investigated, among others, by A. Haroux [5] B. Kawohl [6] M. Slemrod [11]. First they have proved local existence for $t\in\langle 0,\rho\rangle$, $\rho > 0$, using the Banach or Schauder theorems. Next, they have obtained a-priori estimates and used them to prove global existence for any $t\in R^+$. A. Haroux [5] has too proved existence of global solutions for dissipative equation $\frac{\partial u}{\partial t} - \Delta u + \beta(u) \ni 0$, where β is a maximal monotone graph in R^2. He has used theory of monotone operators.

Our method has been based on M. Yamaguchi's ideas [12] and consists of two parts. At first the existence of the time global solution for a linear problem with $F = F(t,x)$ will be proven and the Fourier method will be used. In the general case the Picard iteration method will be applied.

In order to solve our problem we will define the subspace $K^r(\Omega)$ of the space $H^r(\Omega)$, which is connected with the boundary condition.

Any function which belongs to the subspace $K^r(\Omega)$ satisfies the compatibility conditions and can be expanded into the Fourier series convergent in the spaces $K^r(\Omega)$ to this function. Let $r = sm$, $s \in N$. We define $K^r(\Omega)$ in the following way:

$$K^{sm}(\Omega) = \left\{ u \in H^{sm}(\Omega),\ A^{\alpha}u \in H_o^m(\Omega),\ 0 \le \alpha \le \left[\frac{s-1}{2}\right],\ s \ge 1 \right\},$$

where $A^{\alpha} = \underbrace{A(A(\ldots))}_{\alpha\text{-times}}$.

The subspace $K^{sm}(\Omega)$ is a Banach space with the same norm $\|\cdot\|_{H^{sm}}$.

Let be given Banach spaces $C^i(R^+, K^{(s-i)m}(\Omega))$ $i = 0,1$ with the norms $\sup_{t\in R^+} \sum_{k=0}^{i} \left\| \frac{d^k u(t,\cdot)}{dt^k} \right\|_{H^{(s-k)m}}$.

Let be given spaces $F^{sm}(\Omega) := C^o(R^+, K^{sm}(\Omega))$ with the norm $\|u\|_{F^{sm}} = \sup_{t\in R^+} \|u(t,\cdot)\|_{H^{sm}}$ and $G^{sm}(\Omega) = \bigcap_{i=0}^{1} C^i(R^+, K^{(s-2i)m}(\Omega))$ with the norms $\|u\|_{G^{sm}} = \sup_{t\in R^+} \sum_{k=0}^{1} \left\| \frac{d^k u(t,\cdot)}{dt^k} \right\|_{H^{(s-2k)m}}$.

As it was mentioned above the linear problem would have been solved by the Fourier method. Therefore we have considered the eigenvalue problem $A\phi = \mu\phi$ in Ω, $D^{\beta}u|_{\Gamma} = 0$ for $|\beta| \le m-1$. This problem has been investigated, among others, by J. Berezanski [2], G. Fichera [3], K. Roktorys [10]. We recall some of the results established in [2]. Assume that:

A1. Domain Ω has C^{∞}-boundary Γ. The functions a_{pq}, a are C^{∞}-class and $a_{pq} = a_{qp}$ for any multiindices p,q.

A2. The operator A satisfies the condition (1).

Then:

(i) There exists a sequence of the eigenvalues $0 < \lambda_1^2 \le \lambda_2^2 \le \lambda_3^2 \le \dots$ which has only one accumulation point at ∞.

(ii) The sequence $\{\phi_k\}$ of the corresponding eigenfunctions can be taken as a complete and orthonormal system in $L^2(\Omega)$. Because $a_{pq} \in C^{\infty}(\Omega)$ and Γ is C^{∞}-class, then each ϕ_k belongs to $K^{\infty}(\Omega)$.

(iii) Any $u \in K^{sm}(\Omega)$, $s \ge 1$ is expanded into the Fourier series $u = \sum_{k=1}^{\infty} u_k\phi_k$ and this series converges to $u \in K^{sm}(\Omega)$.

We introduce now in the space $K^{sm}(\Omega)$ a new norm

$$|u|_{K^{sm}} := \Big(\sum_{k=1}^{\infty} \lambda_k^{2s} u_k^2 \Big)^{\frac{1}{2}}, \text{ where } u = \sum_{k=1}^{\infty} u_k\phi_k .$$

This norm is equivalent to the norm $\|\cdot\|_{H^{sm}}$ because

(5) $\delta_1\|u\|_{H^{sm}} \le |u|_{K^{sm}} \le \delta_2\|u\|_{H^{sm}}$, positive constants δ_1, δ_2 depending on Ω.

First, the solution u of the problem (2) - (4) is searched in the space $G^{sm}(\Omega)$. Next, due to Sobolev's Imbedding Theory (see [1]), for $s = \left[\frac{n}{2m}\right] + 3$ u is a classical solution. It means that u has first continuous derivatives with respect t, continuous partial derivatives $D^{\gamma}u$, $|\gamma| \le 2m$ and u and its derivatives satisfy equation (2), initial (3) and boundary conditions (4) in the classical sense.

2. The linear case with F = F(t,x)

Let us assume that :

A3. $F = F(t,x)$, $F \in C^{o}(R^{+}, K^{(s-1)m}(\Omega))$,

$\sup_{t \in R^{+}} |F(t,\cdot)|_{K^{(s-1)m}} = H.$

A4. $\phi \in K^{sm}(\Omega)$, $s \ge 2$,

and consider equation

(2')
$$u_t + Au = F(t,x) \ , \quad (t,x) \in R^{+} \times \Omega.$$

Theorem 1. If the assumptions A1.- A4. are hold, then the problem (2'), (3), (4) has a unique global solution $u \in G^{sm}(\Omega)$, satisfying

(6)
$$\|u\|_{F^{sm}} \le C_o\left(\|\phi\|_{H^{sm}} + \frac{1}{|\lambda_1|} \sup_{t \in R^{+}} \|F(t,\cdot)\|_{H^{(s-1)m}}\right),$$

where C_o is a positive constant.

Proof. We set up a solution in a formal Fourier series

(7)
$$u(t,\cdot) = \sum_{k=1}^{\infty} u_k(t)\phi_k \ , \quad u_k(t) = (u(t,\cdot),\phi_k(\cdot))_o \ ,$$

where $(\cdot,\cdot)_o$ - the inner product in the space $L^2(\Omega)$ and show that this formal series converges in $F^{sm}(\Omega)$.

From the assumptions A3., A4., functions ϕ and F can be expanded into the Fourier series:

(8) $\phi = \sum_{k=1}^{\infty} p_k \phi_k$ in the space $K^{sm}(\Omega)$

and $|\phi|_{K^{sm}} = \sum_{k=1}^{\infty} \lambda_k^{2s} p_k^2$, $p_k = (\phi, \phi_k)_o$.

(9) $F(t,\cdot) = \sum_{k=1}^{\infty} f_k(t)\phi_k$ in the space $K^{(s-1)m}(\Omega)$ for $t \in R^+$

and $|F(t,\cdot)|^2_{K^{(s-1)m}} = \sum_{k=1}^{\infty} \lambda_k^{2(s-1)} f_k^2(t)$, $f_k(t) = (F(t,\cdot),\phi_k)_o$

Substituting (8), (9) into (2') and taking the inner product $(\cdot,\cdot)_o$ with ϕ_k , due to the (ii) property of eigenfunctions, we obtain an infinite ordinary differential system of equations

$$(10) \qquad \dot{u}_k(t) + \lambda_k^2 u_k(t) = f_k(t)$$

with the initial data

$$(11) \qquad u_k(0) = p_k , \quad f = 1,2,\ldots .$$

The solution u_k of the above system has a form

$$(12) \qquad u_k(t) = p_k \exp(-\lambda_k^2 t) + \int_0^t \exp(-\lambda_k^2(t-\tau)) f_k(\tau)\, d\tau ,$$

$$t \in R^+,\ k=1,2,\ldots$$

The formal solution of the problem (2'), (3), (4) is given by (7). We want to show now that this series converges to $u \in F^{sm}(\Omega)$. Using the inequality $(a + b)^2 \leq 2(a^2 + b^2)$ and the Schwarz inequality we obtain

$$(13) \qquad u_k^2(t) \leq 2\Big(p_k^2 \exp(-2\lambda_k^2 t) + \frac{1}{\lambda_k^2} \int_0^t \exp(-\lambda_k^2(t-\tau)) f_k^2(\tau)\, d\tau\Big).$$

Thus

(14)
$$\sum_{k=l}^{r} u_k^2(t)\lambda_k^{2s} \le$$

$$\le 2\Big(\exp(-2\lambda_1^2 t)\sum_{k=l}^{r} p_k^2\,\lambda_k^{2s} + \int_0^t \sum_{k=l}^{r} \lambda_k^{2(s-1)} f_k^2(\tau)\exp(-\lambda_1^2(t-\tau))\,d\tau\Big).$$

By A3., A4., the right-hand side of (14) tends to zero uniformly in $t \in R^+$ as l, r ⟶ ∞. This means that the series $\sum_{k=1}^{\infty} \lambda_k^{2s} u_k^2(t)$ converges uniformly in $t \in R^+$.

Due to (8), (9) it follows from (14) that

(15)
$$|u(t,\cdot)|^2_{K^{sm}} \le$$

$$\le 2\Big(\exp(-2\lambda_1^2 t)\,|\phi|^2_{K^{sm}} + \int_0^t |F(\tau,\cdot)|^2_{K^{(s-1)m}}\exp(-\lambda_1^2(t-\tau))\,d\tau\Big).$$

Using the inequality (5) we have desired estimate

(16)
$$\|u(t,\cdot)\|^2_{H^{sm}} \le$$

$$\le C_0^2\Big(\exp(-2\lambda_1^2 t)\,\|\phi\|^2_{H^{sm}} + \int_0^t |F(\tau,\cdot)|^2_{H^{(s-1)m}}\exp(-\lambda_1^2(t-\tau))\,d\tau\Big)$$

with $C_o = \dfrac{\sqrt{2}\,\delta_2}{\delta_1}$. From (16) we obtain the inequality (6).

In order to show that $u \in C^1(R^+, K^{(s-2)m}(\Omega))$ it is enough to prove that $\sum_{k=l}^{r} \lambda_k^{2(s-2)}\dot{u}_k^2(t)$ converges uniformly to zero as l, r → ∞.

From the equation (10) after some calculations we have

$$\sum_{k=l}^{r} \lambda_k^{2(s-2)}\dot{u}_k^2(t) \le 2\Big(\sum_{k=l}^{r} \lambda_k^{2(s-2)} f_k^2(t) + \sum_{k=l}^{r} \lambda_k^{2s} u_k^2\Big).$$

By the assumptions A3., A4. the right - hand side of the previous inequality tends to zero uniformly for $t \in R^+$ as l,

$r \longrightarrow \infty$ and this implies that $u \in C^{o}(R^{+}, K^{(s-2)m}(\Omega))$. Thus

$$|\dot{u}(t,\cdot)|^{2}_{K^{(s-2)m}} \le 2\Big(|F(t,\cdot)|^{2}_{K^{(s-2)m}} + |u(t,\cdot)|^{2}_{K^{sm}}\Big).$$

Having $|F(t,\cdot)|_{K^{(s-2)m}} \le \tilde{C}_1 |F(t,\cdot)|_{K^{(s-1)m}}$ for some positive constant $\tilde{C}_1$, we get

$$\|\dot{u}(t,\cdot)\|^{2}_{H^{(s-2)m}} \le \tilde{C}_2\Big(\|F(t,\cdot)\|^{2}_{H^{(s-1)m}} + \|u(t,\cdot)\|^{2}_{H^{sm}}\Big)$$

$$\text{with } \tilde{C}_2 = C_o^2 \max(C_1^2, 1).$$

The uniqueness of the solution $u \in G^{sm}(\Omega)$ follows directly from (6) ■

For $s = \left[\frac{n}{2m}\right] + 3$ u is a classical solution i.e. $u \in C^{1}(R^{+}\times\bar{\Omega})$ and $D^{p}D^{q}u \in C^{o}(R^{+}\times\bar{\Omega})$ for $|p| \le m$ and $|q| \le m$.

3. The nonlinear case

Due to generalized Moser's Theorem (see [12], Appendix) the following Lemma holds:

Lemma. Let $f(t,x,a)$, $a = (a_o, a_1, a_2, \ldots, a_{N-1}) \in R^{N}$ be defined on $R^{+}\times\Omega\times R^{N}$. Let f be of C^{ms+1}- class in (x,a) and $D^{k}D^{l}_{a}f(t,x,a)$, $0 \le k+l \le ms+1$ be continuous in $t \in R^{+}$. Let $B(Q)$ be any bounded domain in R^{N} of the form

$$B(Q) = \{\, a : a \in R^{N},\ |a_i| \le Q\ ,\ \ i = 0,1,\ldots,N-1 \,\},$$

Q - is any positive real number.

Set $h_Q(t) := \max\limits_{0\le k+l\le sm+1}\ \sup\limits_{x\in\bar{\Omega}, a\in B(Q)} |D^{k}D^{l}_{a}f(t,x,a)|$.

This function is well defined because $f \in C^{sm+1}(R^{+}\times\bar{\Omega}\times R^{N})$.

Denote by

$$F_Q = \{\, u : u \in F^{sm}(\Omega),\ |D^{\alpha}u(t,x)| \le Q \quad \text{for } |\alpha| \le m \,\}.$$

If the above assumptions are satisfied then the following assertions hold :

(1) For any function $u \in F_Q$ it holds

$$\|f(t,\cdot,u(t,\cdot),Du(t,\cdot),\dots,D^m u(t,\cdot))\|_{H^{(s-1)m}} \le$$

$$\le C_1 h_Q(t)\Big(\|u(t,\cdot)\|_{H^{sm}} + 1\Big),$$

where C_1 is a constant depending on f, Q, N,

(2) For any functions u_1, $u_2 \in F_Q$ satisfying the conditions $\|u_1\|_{H^{sm}} \le M$, $\|u_2\|_{H^{sm}} \le M$, with some positive constant M it holds

$$\|f(t,\cdot,u_1(t,\cdot),Du_1(t,\cdot),\dots,D^m u_1(t,\cdot)) -$$

$$- f(t,\cdot,u_2(t,\cdot),Du_2(t,\cdot),\dots,D^m u_2(t,\cdot))\|_{H^{(s-1)m}} \le$$

$$\le C_2 h_Q(t)\|u_1(t,\cdot) - u_2(t,\cdot)\|_{H^{sm}},$$

where C_2 is a constant dependent on f, Q, M and N.

Assume that :

$\bar{A}1$ g satisfies the assumptions in Lemma.

$\bar{A}2$ For any $u \in F^{sm}$ $g(t,\cdot,u(t,\cdot),Du(t,\cdot),\dots,D^m u(t,\cdot))$ belongs to $C^0(R^+,K^{(s-1)m}(\Omega))$.

$\bar{A}3$ There exists a positive constant $\tilde{H}$ such that for any $Q > 0$ $\sup_{t\in R^+} h_Q(t) = \tilde{H}$.

Now let us consider the equation

(2*) $$u_t + Au = \varepsilon g(t,\cdot,u,Du,\dots,D^m u)\ , \quad \varepsilon > 0,$$ with conditions (3), (4)

Theorem 2. If the assumptions A1, A2, A4, $\overline{A1}$ - $\overline{A3}$ are satisfied, then for any $M > C_oL$ (C_o is the constant from Theorem 1) and $L = \|\phi\|_{H^{sm}}$, there exists a positive constant

$$\varepsilon_o = \min\left(\frac{|\lambda_1|(M-C_oL)}{C_oC_1\tilde{H}(M+1)} , \frac{\theta|\lambda_1|}{C_oC_2\tilde{H}} \right), \quad \theta\in(0,1),$$

such that for any ε, $0 < \varepsilon < \varepsilon_o$ the problem (2"), (3), (4) has a unique bounded classical solution in G^{sm}, satisfying the inequality $\|u\|_{F^{sm}} \le M$.

Proof. It is based on Yamaguchi's idea. The Picard iteration method is used. The sequence $\{u_n(t,\cdot)\}$ is constructed by an iteration scheme in the following way :

(17) $u_{n+1} + Au_{n+1} = \varepsilon g(t,\cdot,u_n(t,\cdot),Du_n(t,\cdot),\dots,D^m u_n(t,\cdot))$,

$n = 1,2,\dots$

(18) $u_o + Au_o = \varepsilon g(t,\cdot,0,0,\dots,0)$,

(19) $u_n(0) = \phi$, $D^\beta u_n|_\Gamma = 0$, $t\in R^+$, $|\beta| \le m-1$, $n = 0,1,2,\dots$.

By the assumptions A1, A2, A4, $\overline{A2}$, from the Theorem 1 it follows that each linear equation (17) with conditions (18) has a unique solution $u_n\in G^{sm}(\Omega)$, $n = 0,1,2,\dots$. We must show that $\{u_n\}$ converges to u in the space G^{sm} and $\|u\|_{F^{sm}}\le M$. The boundedness of $\{u_n\}$ will be proven by the induction. For n=0 it follows from (18), (19) and Theorem 1 that

$$\|u_o\|_{F^{sm}} \le C_o\left(\|\phi\|_{H^{sm}} + \frac{1}{|\lambda_1|} \sup_{t\in R^+} \|g(t,\cdot,0,\dots,0)\|_{H^{(s-1)m}}\right) \le$$

$$\le C_o\left(L + \frac{\varepsilon}{|\lambda_1|} C_1\tilde{H} \right) \le M.$$

Assume that $\|u_n\|_{F^{sm}} \le M$ for $n = 1,2,\dots k$. Since $\|u_k\|_{F^{sm}} \le M$ by the Sobolev inequality (see [1]) it follows that

$|u_k(t,x)| \le C(sm)M$, $|D^\alpha u_k(t,x)| \le C((s-1)m)M$ for $|\alpha|\le m$, where $C((s-1)m)$, $C(sm)$ are the Sobolev constants. Applying Theorem 1 to (17), (19) for n=k we have

$$(20)\quad \|u_{k+1}\|_{F^{sm}} \le C_o\Big(L + \frac{\varepsilon}{|\lambda_1|}\sup_{t\in R^+}\|g(t,\cdot,u_k(t,\cdot),Du_k(t,\cdot),\ldots$$

$$\ldots,D^m u_k(t,\cdot))\|_{H^{(s-1)m}}\Big).$$

Let $Q = C(sm)M$. From the Lemma, the assumption $\overline{A3}$ and induction assumption we obtain

$$(21)\quad \sup_{t\in R^+}\|g(t,\cdot,u_k(t,\cdot),Du_k(t,\cdot),\ldots\ldots,D^m u_k(t,\cdot))\|_{H^{(s-1)m}} \le$$

$$\le C_1\tilde{H}\Big(\|u_k\|_{F^{sm}} + 1\Big) \le C_1\tilde{H}(M+1).$$

The inequalities (20), (21), the assumption A4, the definition of ε_0, imply the desired estimate

$$\|u_{k+1}\|_{F^{sm}} \le C_o\Big(L + \frac{\varepsilon}{|\lambda_1|}\tilde{H}C_1(M+1)\Big) \le M .$$

In order to prove convergence $\{u_n\}$ to u in a Banach space $F^{sm}(\Omega)$ we must show that $\{u_n\}$ is a Cauchy sequence. Setting $v_n = u_{n+1} - u_n$ from (17), (19) we have

$$\dot{v}_n + Av_n = \varepsilon(g(t,\cdot,u_n(t,\cdot),Du_n(t,\cdot),\ldots\ldots,D^m u_n(t,\cdot)) +$$

$$- g(t,\cdot,u_{n-1}(t,\cdot),Du_{n-1}(t,\cdot),\ldots\ldots,D^m u_{n-1}(t,\cdot))) ,$$

$v_n(0) = 0$, $D^\beta v_n|_\Gamma = 0$, $|\beta| \le m-1$ for $t\in R^+$ and n=1,2,... .

Applying Theorem 1, Lemma to the above linear system and using the assumption $\overline{A3}$, we obtain estimate

$$\|v_n\|_{F^{sm}} \le \frac{\varepsilon}{|\lambda_1|}C_oC_2\sup_{t\in R^+}h_Q(t)\|u_n(t,\cdot) - u_{n-1}(t,\cdot)\|_{F^{sm}} \le$$

$$\cdot \frac{\varepsilon}{|\lambda_1|} C_o C_2 \tilde{H} \|v_{n-1}\|_{F^{sm}} .$$

This estimate and the definition ε_o guarantee that $\{u_n\}$ is a Cauchy sequence in $F^{sm}(\Omega)$ with $\theta \in (0,1)$ i.e.

$$\|u_{n+1} - u_n\|_{F^{sm}} \le \theta \|u_n - u_{n-1}\|_{F^{sm}} \le \dots \le \theta^n \|u_1 - u_o\|_{F^{sm}} .$$

Now we will prove that $\{u_n\}$ converges to u in $G^{sm}(\Omega)$. By (17), the triangle inequality of the norm, the properties of the operator A and Sobolev spaces $H^{sm}(\Omega)$, $s \ge 1$ we have

$$\|\dot{u}_r(t,\cdot) - \dot{u}_n(t,\cdot)\|_{H^{(s-2)m}} \le \tilde{C} \|u_r(t,\cdot) - u_n(t,\cdot)\|_{H^{sm}} +$$

$$+ \varepsilon \|g(t,\cdot,u_{r-1}(t,\cdot),Du_{r-1}(t,\cdot),\dots\dots,D^m u_{r-1}(t,\cdot)) +$$

$$- g(t,\cdot,u_{n-1}(t,\cdot),Du_{n-1}(t,\cdot),\dots\dots,D^m u_{n-1}(t,\cdot))\|_{H^{(s-2)m}} \le$$

$$\le \tilde{C}\|u_r(t,\cdot) - u_n(t,\cdot)\|_{H^{sm}} + C_2 \varepsilon \tilde{H} \|u_{r-1}(t,\cdot) - u_{n-1}(t,\cdot)\|_{H^{(s-1)m}},$$

where constant $\tilde{C}$ dependent only on m and a_{pq} .

The right-hand side tends to zero uniformly in $t \in R^+$ as $r,n \longrightarrow \infty$. This means that $\{\dot{u}_n\}$ converges in $C^o(R^+,K^{(s-2)m}(\Omega)$. Hence $\{u_n\} \subset G^{sm}(\Omega)$ converges to $u \in G^{sm}$ satisfying the inequality $\|u\|_{F^{sm}} \le M$. u is a solution of the problem (2"), (3), (4). It implies from the following inequality :

$$\|\dot{u}(t,\cdot) + Au(t,\cdot) - \varepsilon g(t,\cdot,u(t,\cdot),Du(t,\cdot),\dots,D^m u(t,\cdot))\|_{H^{(s-2)m}} =$$

$$= \|\dot{u}_{n+1}(t,\cdot) - u(t,\cdot) + A(u_{n+1}(t,\cdot) - u(t,\cdot)) +$$

$$- \varepsilon(g(t,\cdot,u_n(t,\cdot),Du_n(t,\cdot),\dots,D^m u_n(t,\cdot)) +$$

$$- g(t,\cdot,u(t,\cdot),Du(t,\cdot),\dots,D^m u(t,\cdot)))\|_{H^{(s-2)m}} \le$$

$$< C\|u_{n+1} - u\|_{G^{sm}} + \varepsilon C_2 \tilde{H} \|u_n(t,\cdot) - u(t,\cdot)\|_{H^{(s-1)m}}.$$

where C is a positive constant.

The right-hand side tends to zero for any $t \in R^+$, thus u is a solution of the problem (2"), (3), (4). The uniqueness of the solution u of the problem (2"), (3), (4) in $G^{sm}(\Omega)$ can be proved in a classical way using Theorem 1, Lemma and assumption $\overline{A2}$ (see [7], [12]).

Now it has been presented the example of the function $f(t,x,a)$ which satisfies hypotheses of Theorem 2.

Because

$D^p f = \sum_{\beta+\gamma \le p} C_{\alpha\beta\gamma} (D^\gamma D_a^\beta f)(Du)^{\alpha_1} \dots (D^p u)^{\alpha_p}$, where $\beta = \alpha_1 + \alpha_2 + \dots + \alpha_p$, $l + \alpha_1 + 2\alpha_2 + \dots + p\alpha_p = p$, $C_{\alpha\beta\gamma}$- constants, then if $f(t,x,a)$ satisfies $\overline{A1}$ and $\overline{A3}$ and $D^k D_a^l f$, $0 \le k+l \le ms+1$ belongs to $C_o^\infty(\Omega)$ for any bounded domain $B \in R^{N+1}$; then f satisfies $\overline{A1}$ - $\overline{A3}$.

Theorem 3. Let the assumptions A1, A2, A4, $\overline{A1}$ - $\overline{A3}$ be fulfilled then for any $M > C_o L$, there exists a positive constant ε_o (from the Theorem 2) such that any solution of the problem (2"), (3), (4) for any ε, $0 < \varepsilon < \varepsilon_o$ is asymptotically stable, i.e. for any two solutions $u_{1\varepsilon}$, $u_{2\varepsilon}$ of the problem (2"), (3), (4) with the initial conditions ϕ_1, ϕ_2 respectively, the following estimate holds for $t \in R^+$

$$\|u_{1\varepsilon}(t,\cdot) - u_{2\varepsilon}(t,\cdot)\|_{H^{sm}} \le C_o \|\phi_1 - \phi_2\|_{H^{sm}} \exp(-\tfrac{1}{2}(\lambda_1^2 - \varepsilon^2 C_o^2 C_2^2 \hat{H}^2)t).$$

Proof. The assumptions A1, A2, A4, $\overline{A1}$ - $\overline{A3}$ guarantee the existence and uniquence of a solution of the problem (2"), (3), (4). In virtue of the estimate (16) and the assumption 2 of Lemma, we have for the difference of two solutions $u_{1\varepsilon}$, $u_{2\varepsilon}$ of the problem (2"), (3), (4)

$$\|u_{1\varepsilon}(t,\cdot)-u_{2\varepsilon}(t,\cdot)\|^2_{H^{sm}} \le C_o^2\exp(-2\lambda_1^2 t)\|\phi_1-\phi_2\|^2_{H^{sm}} +$$

$$+\ \varepsilon^2C_o^2C_2^2\,\tilde{H}^2\exp(\lambda_1^2 t)\int_0^t \|u_{1\varepsilon}(t,\cdot)-u_{2\varepsilon}(t,\cdot)\|^2_{H^{sm}}\exp(\lambda_1^2\tau)\,d\tau.$$

By the Gronwall inequality it follows

$$\|u_{1\varepsilon}(t,\cdot)-u_{2\varepsilon}(t,\cdot)\|^2_{H^{sm}}\exp(\lambda_1^2 t) \le$$

$$\le C_o^2\|\phi_1-\phi_2\|^2_{H^{sm}}\exp(\,\varepsilon^2C_o^2C_2^2\,\tilde{H}^2 t\,).$$

From the above inequality, we obtain the estimate

$$\|u_{1\varepsilon}(t,\cdot)-u_{2\varepsilon}(t,\cdot)\|_{H^{sm}} \le C_o\|\phi_1-\phi_2\|_{H^{sm}}\exp(-\tfrac{1}{2}\,(\lambda_1^2-\varepsilon^2C_o^2C_2^2\,\tilde{H}^2)t),$$

which implies stability and asymptotical stability of any solution of the problem (2"), (3), (4).

Remark 1. If the assumptions A1, A2, A4, and besides

A3' $f=f(t,x)$, $f\in C^0(R^+,K^{(s-1)m}(\Omega))$, $\int_0^\infty |f(t,\cdot)|_{K^{(s-1)m}}dt = H$

hold, then the problem (2'), (3), (4) has a unique global solution $u\in G^{sm}(\Omega)$.

Remark 2. If the assumptions A1, A2, A4, $\overline{A1}$, $\overline{A2}$ and besides

$\overline{A3}'$ $\int_0^\infty h_Q(t)\,dt=\tilde{H}$ for every $Q>0$

hold, then the problem (2"), (3), (4) has a unique global solution $u\in G^{sm}(\Omega)$.

Remark 3. It can be proven the same properties of solutions of the problem (2)-(4) with others types boundary conditions.

REFERENCES

[1] R. Adams : Sobolev spaces. Academic Press, New York 1975.

[2] Ju. Berezansky : Expansion in eigenfunctions of selfadjoint operators, Kiev 1965 (in Russian), English transl., Amer. Math. Soc. Providence, R. I. 1968.

[3] G. Fichera : Existence theorems in elasticity. Berlin 1972.

[4] J. Kacur : Method of Rothe in Evolution Equations. Leipzig 1985.

[5] A. Haraux : Nonlinear Evolution Equations - Global Behaviour of Solutions, Lect. N. Math. 841, Springer-Verlag, Berlin 1981.

[6] B. Kawohl : Qualitative properties of solutions to semilinear equations, Expositiones Mathematicae, 1986.

[7] T. Kowalski, W. Sadkowski : Existence of global classical solutions a nonlinear hyperbolic equation with the mixed boundary conditions, Demonstratio Math. 19, (2) (1986) 487-502.

[8] J. L. Lions : Quelques methodes de resolution des problèmes aux limites non linèaires. Dunod Gauthier - Villars, Paris 1969.

[9] A. Pazy : Semigroups of Linear Operators and Applications to Partial Differential Equations, Springer - Verlag, New

York, Berlin, Heidelberg, Tokyo 1983.

[10] K. Rektorys : Variational Methods in Mathematics, Science and Engineering. Reidel Publishing Company, Dordrecht, Boston, London 1980.

[11] M. Slemrod : Global Existence, Uniqueness, and Asymptotic Stability of classical Smooth Solutions in One-Dimensional Non-linear Thermoelasticity, A.R.M.A. 76 (1981) 97-133.

[12] M. Yamaguchi : Existence and stability of global classical solutions of initial boundary value problem for semilinear wave equations, Funkcialaj Ekwacioj, 23 (1980) 289-308.

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AN AXIOM SYSTEM FOR THE LINE GEOMETRY

Dedicated to the memory
of Professor Edward Otto

Introduction

The set of lines of three-dimensional projective geometry PG_3 usually called the line geometry (cf.[2]) is always considered in the projective space which is treated as the theory of the universe of points and planes, with e.g. an incidence relation. The set of lines has a well-known interpretation in five-dimensional projective geometry (so-called the geometry on Plucker' quadric).

It would be interesting to define the line geometry as the axiomatic theory of the lines with certain relation. An attempt of the axiomatic approach of such a theory was shown below and this paper stands for the first part of research of axiomatic theory (designed by LG_3 in this paper) for the set lines mentioned above. Next it was shown that in LG_3 can be built the projective geometry PG_3(representation theorems) and, that set of lines built in such a way geometry with the properly defined relation is isomorphic with LG_3. This isomorphism allows to treat LG_3 as axiomatics of geometry on Plucker'quadric.

We shall use the one-sorted language with one binary predicate to describe this theory. The *variables* of this language will be called *lines* and denoted by capital Latin letters. The *predicate* will be denoted by the symbol "▬". The atomic formula $A\,\text{▬}\,B$ will be expressed as "A intersects B". Considering the volume of this paper, generally we leave out all proofs, but exceptionally, we give the proofs of theorems T16, T17, T18, T19, T20, T21 because they concern the representation of PG_3 in theory LG_3 discussed in this paper. On account of the representation mentioned above the theory may be treated as a one-sorted axiomatic approach to three-dimensional projective geometry.

We shall use the following abbreviations:

we shall write $\neq(A_1A_2\ldots A_n)$ instead of $\bigwedge_{i=1}^{n}\bigwedge_{j=1}^{n}(i\neq j \Rightarrow A_i\neq A_j)$,

$\text{▬}(A_1A_2\ldots A_n)$ instead of $\bigwedge_{i=1}^{n}\bigwedge_{j=1}^{n}A_i\,\text{▬}\,A_j$,

$(B_1B_2\ldots B_n)\,\text{▬}\,(A_1A_2\ldots A_n)$ instead of $\bigwedge_{i=1}^{n}\bigwedge_{j=1}^{n}B_i\,\text{▬}\,A_j$,

$A\not\text{▬}B$ instead of $\neg(A\,\text{▬}\,B)$,

where $\bigwedge_{i=1}^{n}\sigma_i$ denotes the conjunction $\sigma_1\wedge\sigma_2\wedge\ldots\wedge\sigma_n$.

The used symbols Ai, Ti, Ci, Di, Ri denote an axiom, theorem, corollary, definition, and remark (number i), respectively.

1. Axioms and main relations

The structure we shall discuss (a realisation of the above language) is of the form

$$\langle\mathcal{L},\text{▬}\rangle.$$

As axioms we assume the following propositions :

A1. $\exists AB(A\neq B \wedge A=B)$.

A2. $\forall AB(A=B \Rightarrow B=A)$.

A3. $\forall A(A=A)$.

A4. $\forall AB(A=B \Rightarrow \exists CD(=(ABC) \wedge =(ABD) \wedge C\neq D))$.

A5. $\forall ABCDE((=(ABCD) \wedge =(ABCE) \wedge \exists F(=(ABF) \wedge F\neq C)) \Rightarrow D=E \vee A=B)$.

A6. $\forall ABCDE(=(ABC) \wedge D=E \Rightarrow \exists F(F=(ABCDE)))$.

A7. $\forall ABCD(=(ABCD) \Rightarrow \exists E(=(ABCDE) \wedge \forall FG((=(ABF) \wedge =(CDG)) \Rightarrow$
$\Rightarrow E=(FG) \vee A=B \vee C=D)))$.

A8. $\forall ABCDE((=(ABC) \wedge =(ABD) \wedge D\neq C \wedge E=(AB)) \Rightarrow E=D \vee E=C \vee A=B)$.

A9. $\forall ABC\exists D(A\neq D \wedge B\neq D \wedge C\neq D)$.

The theory

$$LG_3 := \mathrm{Cn}\ (\{A1, A2, \ldots, A9\})$$

will be called the *line geometry*.

As an immediate consequence of A4 and A9 we obtain

Theorem 1. $\exists ABCD(A\neq B \wedge A\neq C \wedge A\neq D \wedge B\neq C \wedge B\neq D \wedge C\neq D)$.

Now we introduce the very important in our considerations relation **T**.

Definition 1. $\mathbf{T}(ABC) \overset{\text{df}}{\Leftrightarrow} (\neq(ABC) \wedge =(ABC) \wedge \exists D(=(ABD) \wedge C\neq D))$.

Intuitively, the lines which form a triangle or are concurrent but do not lie on the same plane satisfy the relation **T**. Using A1,A4 from D1 we may prove

Theorem 2. $\forall AB(A=B \wedge A\neq B \Rightarrow \exists CD(\mathbf{T}(ABC) \wedge \mathbf{T}(ABD)))$.

T is symmetric, i.e.

Theorem 3. $\forall ABC(\mathbf{T}(ABC) \Rightarrow \mathbf{T}(ACB) \wedge \mathbf{T}(BAC) \wedge \mathbf{T}(BCA) \wedge \mathbf{T}(CAB) \wedge \mathbf{T}(CBA))$.

This theorem is a consequence of D1, A2, A4, A5, T2.

For every three lines A,B,C satisfying **T** we define

Definition 2. $[ABC] \overset{df}{=} \{X\in\mathfrak{L} : \blacksquare(ABCX) \wedge \mathbf{T}(ABC)\}$.

The set $[ABC]$ will be called the *variety*. It will play a vital role in what follows. As a consequence of D2, A4, . T2 we obtain

Theorem 4. (a) $\forall AB(A\blacksquare B \wedge A\neq B \Rightarrow \exists CD([ABC]\neq[ABD]))$,
(b) $\forall A\ \exists BCD\ ([ABC]\neq[ABD])$.

T4(a) and T4(b) show that every two intersecting lines belong at least to two distinct varieties and every line belongs at least to two distinct varieties. Note that T4(b) results from A4 and T4(a). Using A5 we may prove that every variety may be represented by its three arbitrary lines satisfying **T**, i.e.

Theorem 5. $\forall ABCXYZ(X,Y,Z \in [ABC] \wedge \mathbf{T}(XYZ) \Rightarrow$
$\Rightarrow [ABC] = [XYZ])$.

We introduce now the relation **P**.

Definition 3. $\mathbf{P}(ABC) \overset{df}{\Leftrightarrow} (A\neq B \wedge \blacksquare(ABC)$
$\wedge\ \forall D(\blacksquare(ABD) \Rightarrow D\blacksquare C))$.

Intuitively, the lines are concurrent and lie on the same plane satisfy the relation **P**. Using A2, A3 from D3 we obtain

Theorem 6. (a) $\forall ABC(\mathbf{P}(ABC) \Rightarrow \mathbf{P}(BAC))$,
(b) $\forall AB(A\blacksquare B \wedge A\neq B \Rightarrow \mathbf{P}(ABB) \wedge \mathbf{P}(ABA))$.

T3 and T6(a) imply the following

Theorem 7. $\forall ABC(\mathbf{P}(ABC) \wedge A\neq C \Rightarrow \mathbf{P}(ACB))$,

from whence follows (analogous to T3)

Theorem 8. $\forall ABC(\mathbf{P}(ABC) \wedge A\neq C\neq B \Rightarrow \mathbf{P}(ACB) \wedge \mathbf{P}(BAC)$
$\wedge\ \mathbf{P}(BCA) \wedge \mathbf{P}(CAB) \wedge \mathbf{P}(CBA))$.

$\mathbf{P}$ is transitive, i.e.

Theorem 9. $\forall ABCD(\mathbf{P}(ABC) \wedge \mathbf{P}(ABD) \wedge A \neq C \Rightarrow \mathbf{P}(ACD))$.

T9 is a consequence of D3 and T8.

In further investigation there will be useful the notion of *pencil*, which we define as follows

Definition 4. $\langle AB \rangle \overset{df}{=} \{X \in \mathcal{L} : \mathbf{P}(ABX) \wedge A \neq B\}$.

Using T6(b), T8, T9 and considering a lot of cases we obtain (analogous to T5)

Theorem 10. $\forall ABXY\ (X,Y \in \langle AB \rangle \wedge X \neq Y \Rightarrow \langle AB \rangle = \langle XY \rangle)$.

The essential fact in our theory is the reciprocal position of varieties. The following theorem says about it.

Theorem 11. $\exists ABCDEFG\ ([DEG] \cap [ABD] = \{D\}$ $\wedge [DEG] \cap [ABC] = \emptyset \wedge [ABC] \cap [ABD] = \langle AB \rangle)$.

Proof. (a sketch). A1 together with T2 imply an existence of lines

$$A,B,C,D\colon\ \mathbf{T}(ABC) \wedge \mathbf{T}(ABD) \wedge C \neq D,$$

next from T2 it follows an existence of lines

$$\begin{aligned} E,F,G\colon\ &\mathbf{T}(BDE) \wedge \mathbf{T}(BDA) \wedge E \neq A,\\ &\mathbf{T}(ADF) \wedge \mathbf{T}(ADB) \wedge F \neq B,\\ &\mathbf{T}(DEG) \wedge \mathbf{T}(DEB) \wedge G \neq B. \end{aligned}$$

The obtained lines fulfil the conditions of the above theorem.∎

Note in addition that from T5 it follows

Corollary 1. $\forall ABCDEF\ (\mathbf{T}(ABC) \wedge \mathbf{T}(DEF) \wedge$ $\exists XYZ\ (X,Y,Z \in [ABC] \cap [DEF] \wedge \mathbf{T}(XYZ)) \Rightarrow [ABC] = [DEF])$.

2. Varieties

Let $\mathcal{V}$ denote the set of all varieties, i.e.

$$\mathcal{V} = \{\mathfrak{x} : \exists ABC\ (A,B,C \in \mathcal{L} \wedge \mathbf{T}(ABC) \wedge \mathfrak{x} = [ABC])\},$$

T1 with C1 say that

Remark 1. (1) Any two elements a, b belonging to $\mathcal{V}$ fulfil exactly one of the following conditions

(i) $a \cap b = \emptyset$,

(ii) $a \cap b = \{C\}$, for some line C,

(iii) $a \cap b = \langle DE \rangle$, for some lines D, E with $D\text{-}E$

(iv) $a = b$.

(2) each condition of (i) - (iv) is realized in $\mathcal{V}$.

From A4, A6 it follows the useful

Theorem 12. $\forall ABCD(\text{▬}(ABC) \Rightarrow \exists EF(E \neq F \wedge (EF)\text{▬}(ABCD)))$.

Using T12 and A7, A8 we may prove

Theorem 13. $\forall abc(|a \cap b|=1 \wedge |b \cap c|=1 \Rightarrow a=c \vee |a \cap c|=1)$.

Now we may define in $\mathcal{V}$ the following relation

$$\equiv_{\mathfrak{v}} : \equiv_{\mathfrak{v}} \subset \mathcal{V}^2 \wedge \forall ab(a \equiv_{\mathfrak{v}} b \overset{\text{df}}{\Leftrightarrow} |a \cap b|=1 \vee a=b).$$

By T13 it is easy to see that $\equiv_{\mathfrak{v}}$ is an equivalence relation, i.e.

Theorem 14. For every varieties a, b, c we have

(a) $a \equiv_{\mathfrak{v}} b$,

(b) $a \equiv_{\mathfrak{v}} b \Rightarrow b \equiv_{\mathfrak{v}} a$,

(c) $a \equiv_{\mathfrak{v}} b \wedge b \equiv_{\mathfrak{v}} c \Rightarrow a \equiv_{\mathfrak{v}} c$.

Thus we may consider the quotient set $\mathcal{V}/\equiv_{\mathfrak{v}}$. For the exact characterization of sets $\mathcal{V}$, $\mathcal{V}/\equiv_{\mathfrak{v}}$ it is very to use

Lemma 1. For every varieties a, b, c the following conditions:

(1) $|a \cap b|>1 \wedge |b \cap c|>1 \wedge |a \cap c|>1 \wedge a \neq b \wedge b \neq c \wedge a \neq c$,

(2a) $|a\cap b|>1 \wedge |b\cap c|>1 \wedge |a\cap c|=0 \wedge a\neq b \wedge b\neq c$,

(2b) $|a\cap b|>1 \wedge |b\cap c|=0 \wedge |a\cap c|>1 \wedge a\neq b \wedge a\neq c$,

(2c) $|a\cap b|=0 \wedge |b\cap c|>1 \wedge |a\cap c|>1 \wedge b\neq c \wedge a\neq c$,

(3a) $|a\cap b|>1 \wedge |b\cap c|=0 \wedge |a\cap c|=0 \wedge a\neq b$,

(3b) $|a\cap b|=0 \wedge |b\cap c|>1 \wedge |a\cap c|=0 \wedge b\neq c$,

(3c) $|a\cap b|=0 \wedge |b\cap c|=0 \wedge |a\cap c|>1 \wedge a\neq c$,

(4) $|a\cap b|=0 \wedge |b\cap c|=0 \wedge |a\cap c|=0$

are impossible.

To prove this lemma A6, A7, A8 and T13 have been used. The proof in cases (2a), (2b), (2c) and analogously in (3a), (3b), (3c) passes in a similar manner. Taking L1 we may show that

$$\forall abc \in \mathcal{V}\ (a \equiv_v b \vee b \equiv_v c \vee a \equiv_v c).$$

Thus we obtain

Theorem 15. $|\mathcal{V}/\equiv_v|=2$.

3. Representation

The space $\mathcal{V}$ is the sum of two sets

$$\mathcal{V}_1,\ \mathcal{V}_3:\ \{\mathcal{V}_1, \mathcal{V}_3\} = \mathcal{V}/\equiv_v.$$

Thus in LG_3 we construct the following structure

$$\langle \mathcal{V}_1, \mathcal{V}_3, I_\ell \rangle,$$

where I_ℓ is defined as follows

Definition 5. $\forall a \in \mathcal{V}_1\ b \in \mathcal{V}_3\ (a I_\ell b \overset{df}{\Leftrightarrow} a\cap b\neq\emptyset)$.

From D5 it results that $a I_\ell b$ and $b I_\ell a$ may be identified. We assume that

the elements of $\mathcal{V}_1$ will be called *points* and denoted by small Latin letters,

- the elements of $\mathcal{V}_3$ will be called *planes* and denoted by small Greek letters,

- the formulas $a|_\ell\alpha$ and $\alpha|_\ell a$ will be read "a and α are incidented".

We shall use one more abbreviation yet; namely, we shall write $a_1a_2\ldots a_n|_\ell\alpha_1\alpha_2\ldots\alpha_m$ instead of $\bigwedge\limits_{i=1}^{n}\bigwedge\limits_{j=1}^{m} a_i|_\ell\alpha_j$.

Now we shall prove that the structure $\langle\mathcal{V}_1,\mathcal{V}_3,|_\ell\rangle$ is a model of three-dimensional projective geometry PG_3 with the axiom system assumed by M Kordos in monograph [1]. The following axiom system is given in [1] (referred here in a little transcription)

B1. $\forall abc\alpha\beta\gamma\ (abc|_\ell\ \alpha\beta \wedge ab|_\ell\ \alpha\beta\gamma \Rightarrow a=b \vee \alpha=\beta \vee c|_\ell\ \gamma)$.

B2. $\forall abc\exists\delta\ (abc|_\ell\delta)$.

B3. $\forall\alpha\beta\gamma\exists d\ (\alpha\beta\gamma|_\ell d)$.

B4. $\exists a_1a_2a_3a_4a_5\alpha_1\alpha_2\alpha_3\alpha_4\alpha_5\ (a_i|_\ell\alpha_j \Rightarrow i-j = 1,2,3 \pmod 5))$.

First we want to realize two very useful constructions (exercise 1 and theorem T16).

Exercise 1. Let two non-intersecting lines A, B be given. Construct a tetrahedron $(ABCDEF)$, i.e. a figure such that the varieties : $[ACE]$, $[BDE]$, $[ADF]$, $[BCF]$ are planes and $[BDF]$, $[ACF]$, $[BCE]$, $[ADE]$ are points.

Solution. Take any two lines A,B such that $A-B$. A6 implies the existence of a line C satisfying $C-(AB)$. Using A4 we find

$$X,Y\colon X-(AC) \wedge Y-(AC) \wedge X\neq Y.$$

Thus we have $\mathbf{T}(ACX)$ and $\mathbf{T}(ACY)$. By T12 we choose

$$E\colon E-(ACX) \wedge E-B \wedge E\neq C.$$

It is easy to see that $E=A$. $\mathbf{P}(ACE)$ is impossible. Indeed, let us assume that $\mathbb{P}(ACE)$. Then in accordance with T8 and D3 we obtain

$$\mathbf{P}(ACE) \Rightarrow \mathbb{P}(CEA) \text{ and } B\,\text{▬}\,(CE) \wedge \mathbf{P}(CEA) \Rightarrow B\,\text{▬}\,A.$$

The last fact is in conflict with the assumption of the exercise. Therefore we have $\mathbf{T}(ACE)$, hence by T5 we have $[ACX] = [ACE]$. Analogously there exists F such that $[ACY] = [ACF]$. Let us assume that $[ACE]$ is a plane then $[ACF]$ must be a point (we may assume inversely, too). Notice that $\text{▬}(CFB)$, where as $\mathbb{P}(CFB)$ is impossible. Then we have $\mathbf{T}(CFB)$. Similarly $\mathbf{T}(CEB)$ holds. Since $[ACF]$ is a point then $[BCF]$ must be a plane and $[BCE]$ a point again. For A, E using A4, A8 we find Z: $Z\,\text{▬}\,(AE) \wedge Z \not{\text{▬}}\, C$.

In accordance with A6 it exists

$$D\colon D\,\text{▬}\,(AZE) \wedge D\,\text{▬}\,(FB).$$

Similarly as above we obtain $\mathbf{T}(ADE)$ and $\mathbf{T}(BDF)$ and $\mathbf{T}(ADF)$ and $\mathbf{T}(BDE)$. Thus we obtain four varieties: $[ADE]$, $[BDF]$, $[ADF]$, $[BDE]$ from whence $[ADE]$, $[BDF]$ are points and $[ADF]$, $[BDE]$ are planes. Hence we construct the tetrahedron $(ABCDEF)$ such that $[ACE]$, $[BDE]$, $[ADF]$, $[BCF]$ are planes and $[BDF]$, $[ACF]$, $[BCE]$, $[ADE]$ are points. ■

Theorem 16. $\forall A \in \mathcal{L}\ \exists abc \in \mathcal{V}_1 (A \in a \wedge A \in b \wedge A \in c \wedge (abc))$.

Proof. Let A be an arbitrary line. Using A9 we find B, such that A-B. Solving the exercise 1 we construct the tetrahedron $(ABCDEF)$ (Fig.1.).

By A9 there exists

$$Q\colon Q \not{\text{▬}}\, E \wedge Q \not{\text{▬}}\, A \wedge Q \not{\text{▬}}\, C,$$

using T12 we find

$$R,S\colon\ R\blacksquare(ACE) \wedge R\blacksquare Q \wedge S\blacksquare(ACE) \wedge S\blacksquare Q.$$

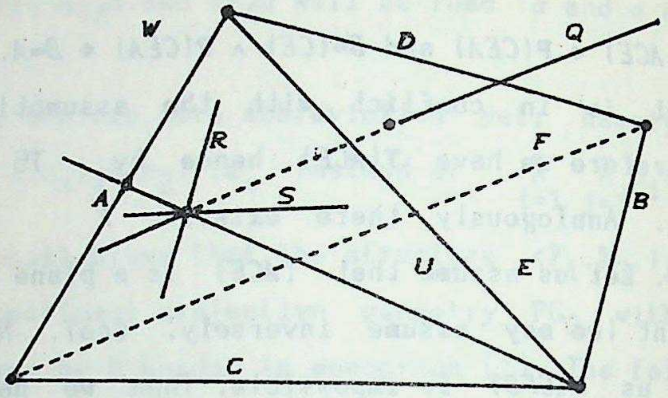

Figure 1.

As $[ACE]$ is a plane then $[RSQ]$ is a point and $[BCE]$ is a point too. A7 implies the existence

$$U\colon\ \{U\} = [RSQ] \cap [BCE] \cap [ACE].$$

Obviously $U \neq A$ and $U \neq C$ and $\blacksquare(UAC)$. $\mathbf{P}(UAC)$ is impossible. Indeed, if $\mathbf{P}(UAC)$ were fulfilled, then there would exist C, U such that $\{C,U\} \subset \langle CE \rangle \cap \langle AC \rangle$. Hence $\langle CE \rangle = \langle AC \rangle$. It is in conflict with the tetrahedron considered above. For A,U using A4 we find O, W such that $\blacksquare(AUO)$ and $\blacksquare(AUW)$ and $O \neq W$. Instead of O we may take C. As $[ACU]$ is a plane then $[AUW]$ must be a point. Notice that $[ACF] \neq [AUW] \neq [ADE]$. Indeed, if $[ACF] = [AUW]$ or $[AUW] = [ADE]$ then $U = C$ or $U = E$. This is contradictory with the assumption concerning the position of the line Q. Putting $a = [ACF]$, $b = [ADE]$, $c = [AUW]$ we prove the theorem.■

Now we are ready to prove the theorems concerning the representation of PG_3 in LG_3. Exactly we must prove that the propositions B1, B2, B3, B4 are true in LG_3.

Theorem 17. (B1) $\forall abc \in \mathcal{V}_1 \forall \alpha\beta\gamma \in \mathcal{V}_3 (abc|_\ell \alpha\beta \wedge ab|_\ell \alpha\beta\gamma \Rightarrow$ $\Rightarrow a=b \vee \alpha=\beta \vee c|_\ell \gamma)$.

Proof. Let $abc|_\ell \alpha\beta$ and $ab|_\ell \alpha\beta\gamma$. If $a|_\ell \alpha\beta$, then there exist

$$A_\alpha, A'_\alpha, A_\alpha, A'_\alpha : a\cap\alpha = \langle A_\alpha A'_\alpha \rangle \wedge a\cap\beta = \langle A_\alpha A'_\alpha \rangle.$$

Using A7 we find

$$G: G \in \langle A_\alpha A'_\alpha \rangle \cap \langle A_\alpha A'_\alpha \rangle.$$

In a similar manner for $b|_\ell \alpha\beta$ and $c|_\ell \alpha\beta$ we find two lines M, Z. Hence by D5 we may choose the following lines

$$G,E,F: E,G \in \alpha \wedge F,G \in \beta \wedge E,F,G \in a \wedge E\neq F\neq G,$$

$$M,K,L: K,M \in \alpha \wedge L,M \in \beta \wedge K,L,M \in b \wedge K\neq M\neq L,$$

$$Z,X,Y: X,Z \in \alpha \wedge Y,Z \in \beta \wedge X,Y,Z \in c \wedge X\neq Z\neq Y.$$

The following cases are possible.

P1: $G\neq M \vee M\neq Z \vee G\neq Z$. Then $|\alpha\cap\beta|>1$. It denotes that $\alpha=\beta$.□

P2: $G=M \wedge M=Z \wedge G=Z$. Then we have $G\in a$ and $G\in b$ and $G\in c$. Here there are cases:

P21: $a=b$.△

P22: $a\neq b$. Then G is a unique line which belongs to a and b simultaneously. But a and b incidenting with γ have two common pencils with it. These pencils have the common line G'. Obviously $G'=G$. Hence $G\in\gamma$. But $G\in c$, too. Thus $c\cap\gamma\neq\emptyset$, i.e. $c|_\ell \gamma$.△ □ ■

Theorem 18. (B2) $\forall abc \; \exists\delta \; (abc|_\ell \delta)$.

Proof. Three cases are possible.

P1: $a=b=c$. There exist

$$A, A', A'': a = [AA'A''].$$

For A, A' in accordance with A4 we find

$$E,F:\ \equiv(AA'E) \wedge \equiv(AA'F) \wedge E\neq F.$$

From A8 we obtain $A''\equiv E \vee A''\equiv F$. Assume that $A''\equiv E$ is true (in second case the proof passes analogously). Then we have $[AA'A''] = [AA'E]$. Since $a = [AA'E]$ is a point, then the variety $[AA'F]$ is a plane. It is easy to see that taking $\delta = [AA'F]$ we obtain $abc\,|_{\ell}\,\delta$.□

P2: $a=b\neq c \vee a\neq b=c \vee a=c\neq b$. It suffices to consider one case only (the others are symmetric). Let it be e.g. $a=b=c$. It exists

$$E:\ \{E\} = a\cap c.$$

We take

$$F:\ F \in b \wedge F\neq E.$$

Using A6 we choose

$$G:\ G\equiv E \wedge G\in c.$$

Note that $G\notin b$. Thus $G\neq E$ and $G\neq F$. $\mathbf{P}(EFG)$ is impossible. So $\mathbf{T}(EFG)$. As the plane δ we may take the plane $[EFG]$, because then it is satisfied $a\cap[EFG]\neq\emptyset$ and $b\cap[EFG]\neq\emptyset$ and $c\cap[EFG]\neq\emptyset$.□

P3: $a=b \wedge b=c \wedge a=c$. There exist

$$A,B,C:\ a\cap b=\{C\} \wedge b\cap c=\{A\} \wedge a\cap c=\{B\}.$$

It is $\mathbf{T}(ABC)$, because $\mathbf{P}(ABC)$ is impossible. Taking $\delta=[ABC]$ we prove the theorem.□ ■

Theorem 19. (B3) $\forall\alpha\beta\gamma\exists d(\alpha\beta\gamma\,|_{\ell}\,d)$.

The proof of this theorem as similar to the proof of theorem T18 we omitted.

Theorem 20. (B4) $\exists a_1a_2a_3a_4a_5\alpha_1\alpha_2\alpha_3\alpha_4\alpha_5(a_i\,|_{\ell}\,\alpha_j \Leftrightarrow$ $\Leftrightarrow$ i-j = 1,2,3(mod 5)).

Proof. Consider the tetrahedron $(ABCDEF)$ constructed in exercise 1 and the point $[QRS]$ constructed in the proof of T16. It exists

$$G\colon G\in[BDF]\cap[QRS].$$

For a line G using T16 we find a point

$$c\colon G\in c\wedge c\neq[BDF]\wedge c=[QRS].$$

Obviously the point c isn't incidented with any planes belonging to tetrahedron $(ABCDEF)$. Putting (Fig. 2)

$$a_1=[ACF],\ a_2=[ADE],\ a_3=[BDF],\ a_4=[BCF],\ a_5=c,$$

$$\alpha_1=[BDE],\ \alpha_2=[OSB],\ \alpha_3=[CWO],\ \alpha_4=[AUW],\ \alpha_5=[ADF]$$

we obtain B4.■

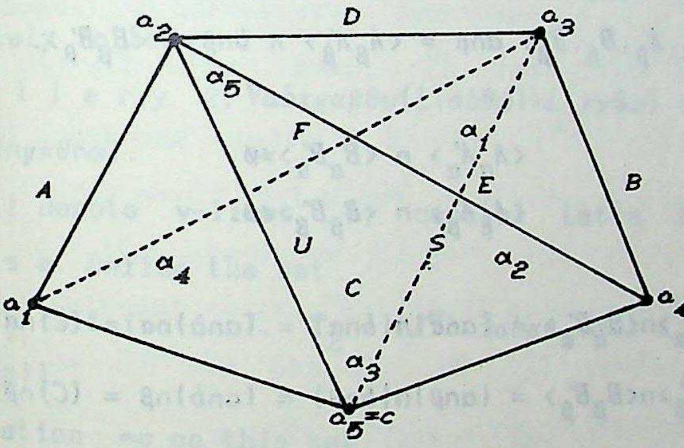

Figure 2.

This completes the proof of T20 and finishes the block of theorems concerning the representation of PG_3 in LG_3.

4. Isomorphism

In the last part we shall show that basing on the structure $\langle\mathcal{V}_1,\mathcal{V}_3,|_\ell\rangle$ we may construct the line structure $\langle\mathcal{V}_2,=_v\rangle$ and show that the structures $\langle\mathcal{V}_2,=_v\rangle$, $\langle\mathcal{L},=\rangle$ are isomorphic. First we shall prove some preliminary lemmas.

Lemma 2. $\forall A\exists bc\beta\gamma\,(b\cap c = \{A\} = \beta\cap\gamma \wedge b\neq c \wedge \beta\neq\gamma)$.

Proof. Take any line A. By A9 it exists B such that $A\neq B$. As the solution of exercise 1 we obtain tetrahedron $(ABCDEF)$. Putting $b=[ACF]$ and $c=[ADE]$ and $\beta=[ACE]$ and $\gamma=[ADF]$ we finish the proof. ■

Lemma 3. $\forall ab\alpha\beta\,(a\neq b \wedge \alpha\neq\beta \wedge ab|_\ell\alpha\beta \Leftrightarrow a\cap b = \alpha\cap\beta)$.

Proof. $\Rightarrow$: For the arbitrary points a,b and planes α,β there exist C,C' such that $a\cap b=\{C\}$ and $\alpha\cap\beta=\{C'\}$. From this assumption it follows the existence of lines

$$A_\alpha, A'_\alpha, B_\alpha, B'_\alpha\colon\ a\cap\alpha = \langle A_\alpha A'_\alpha\rangle \wedge b\cap\alpha = \langle B_\alpha B'_\alpha\rangle,$$

$$A_\beta, A'_\beta, B_\beta, B'_\beta\colon\ a\cap\beta = \langle A_\beta A'_\beta\rangle \wedge b\cap\beta = \langle B_\beta B'_\beta\rangle.$$

By A7 we have

$$\langle A_\alpha A'_\alpha\rangle \cap \langle B_\alpha B'_\alpha\rangle \neq \emptyset,$$
$$\langle A_\beta A'_\beta\rangle \cap \langle B_\beta B'_\beta\rangle \neq \emptyset.$$

more exactly

$$\langle A_\alpha A'_\alpha\rangle\cap\langle B_\alpha B'_\alpha\rangle = (a\cap\alpha)\cap(b\cap\alpha) = (a\cap b)\cap\alpha = \{C\}\cap\alpha,$$

$$\langle A_\beta A'_\beta\rangle\cap\langle B_\beta B'_\beta\rangle = (a\cap\beta)\cap(b\cap\beta) = (a\cap b)\cap\beta = \{C\}\cap\beta.$$

Thus $C\in\alpha$ and $C\in\beta$. Then $C\in\alpha\cap\beta$. Hence we have $C=C'$.

$\Leftarrow$: From the assumption we have

$$a\cap\alpha\neq\emptyset \wedge a\cap\beta\neq\emptyset \wedge b\cap\alpha\neq\emptyset \wedge b\cap\beta\neq\emptyset.$$

From D5 we have $a|_\ell\alpha$ and $a|_\ell\beta$ and $b|_\ell\alpha$ and $b|_\ell\beta$. At the end, we notice that the following alternative $a=b$ or $\alpha=\beta$ is impossible. ■

Lemma 4. (a) $\forall abcd\alpha\beta\pi(ab|_\ell\alpha\beta \wedge cd|_\ell\alpha\beta \wedge \pi|_\ell ab \Rightarrow \pi|_\ell cd)$.

(b) $\forall ab\rho\alpha\beta\gamma\delta\,(\alpha\beta|_\ell ab \wedge \gamma\delta|_\ell ab \wedge \rho|_\ell\alpha\beta \Rightarrow \rho|_\ell\gamma\delta)$.

Proof. (a) Let a,b,c,d be the arbitrary points and α,β,π planes. From L3 we have $a\cap b=\alpha\cap\beta=c\cap d$ and $a\cap b\subset\pi$. Thus $c\cap d\subset\pi$. Otherwise $c\cap\pi\neq\emptyset$ and $d\cap\pi\neq\emptyset$. Whence by D5 we obtain $\pi|_{\ell}c$ and $\pi|_{\ell}d$.

(b) In this case the proof pass analogously.■

Now, let a,b,α,β be such that $a\neq b$ and $\alpha\neq\beta$ and $ab|_{\ell}\alpha\beta$ and $a,b\in\mathcal{V}_1$ and $\alpha,\beta\in\mathcal{V}_3$. We define

Definition 6. $L(ab\alpha\beta)=\{\tau:\ \tau|_{\ell}ab\}\cup\{t:\ t|_{\ell}\alpha\beta\}$.

The set $L(ab\alpha\beta)$ will be called the *v-line*. From L4 it follows that every v-line may be represented by any its four elements: two distinct planes and two distinct points. From L3 immediately follows

Corollary 2. $\forall abxy\alpha\beta\theta\omega(L(ab\theta\omega)=L(xy\theta\omega)\Rightarrow$ $\Rightarrow a\cap b=\alpha\cap\beta=x\cap y=\theta\cap\omega)$.

We shall denote v-lines by capital Latin letters with superscripts v. Define the set

Definition 7. $\mathcal{V}_2=\{X^{v}:\ \exists ab\alpha\beta\ (a\neq b\wedge\alpha\neq\beta\wedge ab|_{\ell}\alpha\beta$ $\wedge X^{v}=L(ab\alpha\beta))\}$.

and the relation $=_{v}$ on this set

Definition 8. $\forall A^{v},B^{v}\in\mathcal{V}_2(A^{v}=_{v}B^{v}\overset{\mathrm{df}}{\Leftrightarrow}A^{v}\cap B^{v}\neq\emptyset)$.

We prove

Lemma 5. $\forall A^{v}B^{v}(\exists a\ (a\in A^{v}\cap B^{v})\Leftrightarrow\exists\alpha\ (\alpha\in A^{v}\cap B^{v}))$.

Proof. ⇒: Let A^{v}, B^{v} be any two v-lines and let a point a hold $a\in A^{v}\cap B^{v}$. Take a', b', a'', b'', α', β', α'', β'' such that $A^{v}=L(a'b'\alpha'\beta')$ and $B^{v}=L(a''b''\alpha''\beta'')$. From the assumption we have $a|_{\ell}\alpha'\beta'$ and $a|_{\ell}\alpha''\beta''$. Then from L3 there exist two lines C',

C'' such that $\{C'\} = \alpha' \cap \beta' \subset a$ and $\{C''\} = \alpha'' \cap \beta'' \subset a$ and $\{C'\} = a' \cap b' \subset a$ and $\{C''\} = a'' \cap b'' \subset a$. Obviously $C' \equiv C''$. By A4 and T4 (if $C' = C''$) and by T4 (if $C' \neq C''$) it exists α satisfying $C', C'' \in \alpha$. Hence we obtain $\alpha |_{\ell} aba'b'$. It means that $\alpha \in A^{v}$ and $\alpha \in B^{v}$.

$\Leftarrow$: Here the proof passes analogously.■

Now, let us consider the structures

$$\langle \mathcal{L}, \equiv \rangle \text{ and } \langle \mathcal{V}_2, \equiv_{v} \rangle .$$

Define the correspondence ℓ in the following way

Definition 9. $\ell : \mathcal{V}_2 1\ \mathcal{L}$ such that for any $X^{v} = L(ab\alpha\beta)$: $\ell(X^{v}) = Y$, where $\{Y\} = a \cap b (= \alpha \cap \beta)$.

From L2 it follows that ℓ is a surjection. We are going to prove that f is a one-to-one correspondence. Take a, b, a', b', α, β, α', β' such that $A^{v} = L(ab\alpha\beta)$ and $B^{v} = L(a'b'\alpha'\beta')$. Assume that $\ell(A^{v}) = \ell(B^{v})$. From D9 we obtain $a \cap b = \alpha \cap \beta = a' \cap b' = \alpha' \cap \beta'$. Taking an equality $\alpha \cap \beta = a' \cap b'$ and using L3 we obtain $\alpha\beta |_{\ell} a'b'$, taking $a \cap b = \alpha' \cap \beta'$ and using L3 we obtain $ab |_{\ell} \alpha'\beta'$. By L4 we obtain $A^{v} = L(ab\alpha\beta) = L(a'b'\alpha'\beta') = B^{v}$. As an immediate consequence of the definitions D2, D6, D8 and A7, L5 we obtain the implication $\forall X^{v} Y^{v} (X^{v} \equiv_{v} Y^{v} \Rightarrow \ell(X^{v}) \equiv \ell(Y^{v}))$.■

Now, we may formulate the main proposition of this part

Theorem 21. The structures $\langle \mathcal{V}_2, \equiv_{v} \rangle$, $\langle \mathcal{L}, \equiv \rangle$ are isomorphic.

REFERENCES

[1] M. Kordos : Podstawy geometrii rzutowej i rzutowo-metrycznej, Warszawa 1984 (Principles of Projective and Projective-Metrix Geometry, Warsaw 1984).

[2] J.G. Semple, L. Roth : Introduction to Algebraic Geometry, Clarendon Press Oxford 1986.

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REFERENCES

[1] K. Kordos: Podstawy geometrii rzutowej (rzutowo-metrycznej), Warszawa 1981 (Principles of Projective and Projective-Metric Geometry, Warsaw 1981).

[2] J.G. Semple, L. Roth: Introduction to Algebraic Geometry, Clarendon Press Oxford 1949.

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Tomasz Traczyk

A NOTE ON MAXIMAL HAMILTONIAN CYCLES IN THE SQUARE OF A GRAPH

Dedicated to the memory of Professor Edward Otto

We consider only finite graphs without loops or multiple edges. We use the terminology and notation of Behzad and Chartrand [1]. In particular, by V(G) and E(G) we denote the set of vertices and the set of edges, respectively, of a graph G. We form G^2, the *square* of a graph G, by adding to G a *new edge* uv whenever $\text{dist}_G(u,v)=2$. The original edges of G are referred to as *old edges* of G^2. If G^2 is hamiltonian then by *m(G)* we denote the maximum number of old edges in a hamiltonian cycle of G^2. We define also $m(K_2)=2$. A hamiltonian cycle C in G^2 is called *maximal* if C contains m(G) old edges.

Let v be a cut vertex of a graph G. For every component H of G-v, the subgraph induced in G by V(H)∪{v} is called a *v-fragment* of G. A *caterpillar* is a tree in which vertices of degree greater than 1 induce a path.

Here is the main result of this note:

Theorem 1. Let G be a graph with hamiltonian square and let v be a cutvertex of G If H is one of v-fragments of G and $F=G-(V(H)\setminus\{v\})$ then $m(G)\le m(F)+m(H)-2$.

Proof. By Lemma 1 of [5], there exists a maximal hamiltonian cycle $C=(u_0,u_1,\dots u_{n-1},u_0)$ in G^2 such that $V(H)$ or $V(F)$ induces a path in C. Without loss of generality we may assume that $V(H)$ induces a path $(u_0,\dots,u_k)$ in C and $v=u_t$ for some t, $0\le t\le k$. Consider the following two cases:

(i) $V(F)$ does not induce a path in C.

In this case we have $0<t<k$. This implies that u_0,u_k,u_{k+1} and u_{n-1} are all adjacent in G to v. Hence $(u_0,\dots,u_k,u_0)$ is a hamiltonian cycle in H^2 containing no less edges of H than C does and $(v,u_{k+1},u_{k+2},\dots,u_{n-1},v)$ is a hamiltonian cycle in F^2 containing 2 more edges of F than C does.

(ii) $V(F)$ induces a path in C.

This implies that $t=0$ or $t=k$. Without loss of generality we may assume that $t=0$. This means that both u_k and u_{k+1} are adjacent in G to v. Hence $(u_0,\dots,u_k,u_0)$ is a hamiltonian cycle in H^2 containing one more edge of H than C does and $(u_0,u_{k+1},u_{k+2},\dots,u_{n-1},u_0)$ is a hamiltonian cycle in F^2 containing one more edge of F than C does.

Notice that, due to our convention of $m(K_2)=2$, the theorem remains true in the cases $F=K_2$ and $F=H=K_2$.■

We conjectured in [6] that for every graph G with the hamiltonian square, $m(G)>0$. We proved also that this conjecture is equivalent to another one - that for every graph G with the hamiltonian square, $m(G)>1$. This fact can be now easily deduced from Theorem 1:

Corollary 2. The following statements are equivalent:

(i) For every graph G with hamiltonian square, $m(G)>0$

(ii) For every graph G with hamiltonian square, $m(G)>1$.

Proof. We only need to prove (i)⇒(ii). Suppose $m(G)=1$ for some graph G and v is an end vertex of the only old edge of a hamiltonian cycle of G^2. Let H be an isomorphic copy of G having only v in common with G and such that also in H^2, v is an endvertex of the only old edge of some hamiltonian cycle. Clearly $(G\cup H)^2$ is hamiltonian and by Theorem 1, $m(G\cup H)=0$.■

Combining the Hobbs' characterization of unicyclic graphs with hamiltonian squares ([5], Corollary 1A) with Theorem 1 and the fact that for every nontrivial caterpillar T, $m(T)=2$ (see [4]) one can easily obtain the following result.

Corollary 3. For every unicyclic graph G with the cycle C, if G^2 is hamiltonian then $m(G)=|V(C)|$.■

Fleischner proved in [2] that for every 2-connected graph G, $m(G)\geq 3$. We will show now that for every $k>1$ there exists a graph G with $m(G)=k$.

Proposition 4. (i) For every n and k, $n\geq k>3$, there exists a 2-connected graph G on n vertices with $m(G)=k$.
(ii) For every n and k such that $n\geq k>1$, there exists a graph G on n vertices with $m(G)=k$.

Proof. (i) Let G consists of a path P, $P=(v_1,v_2,\dots,v_{k-1})$ and vertices $v_k,v_{k+1},\dots,v_n$, each joined only to v_1 and v_{k-1}. G is 2-connected and it is easy to see that $m(G)=k$.

(ii) For every caterpillar T we have $m(T)=2$ and it follows from Corollary 3 that any unicyclic graph G with the hamiltonian square, whose only cycle has length 3, will do for k=3.■

K_3 apart, we have been unable to find a 2-connected graph G with m(G)=3 and we conclude the paper with the following

Conjecture 5. For every 2-connected graph G on more than 3 vertices, m(G)>3.

REFERENCES

[1] M. Behzad, G. Chartrand: Introduction to the Theory of Graphs, Allyn & Bacon, Boston, 1971.

[2] H. Fleischner: In the square of graphs, hamiltonicity and pancyclicity, hamiltonian connectedness and panconnectedness are equivalent concepts, Monatsh. Math. 82(1976),125-149.

[3] F. Harary, A. Schwenk: Trees with hamiltonian squares, Mathematika 18(1971),138-140.

[4] A. Hobbs: Hamiltonian squares of cacti, J. Combinatorial Theory, Ser. B, 26(1979), 50-65.

[5] A. Hobbs: Maximal hamiltonian cycles in squares of graphs, J. Combinatorial Theory, Ser. B, 26(1979), 35-49.

[6] T. Traczyk: On a problem in hamiltonian squares, in "Graphs, Hypergraphs and Matroids II" - Proc. 6th Regional Session of Mathematicians held in Zagań in 1986, ed by M. Borowiecki and Z. Skupień, Zielona Góra, 1987, 79-82.

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THE PONCELET CONFIGURATION OF CLOSING FOR NORMAL CURVES

*Dedicated to the memory
of Professor Edward Otto*

1. Introduction

The notion of Poncelet configuration for conics is known in specialized literature ([4], page 289). It is said that the conic k^2 is in the Poncelet configuration of closing with the conic s^2 if and only if tangent to the conic s^2 drawn from any point P of the conic k^2 intersect the conic k^2 additionally in the points Q, R of this quality that the straight line [QR] is tangent to s^2.

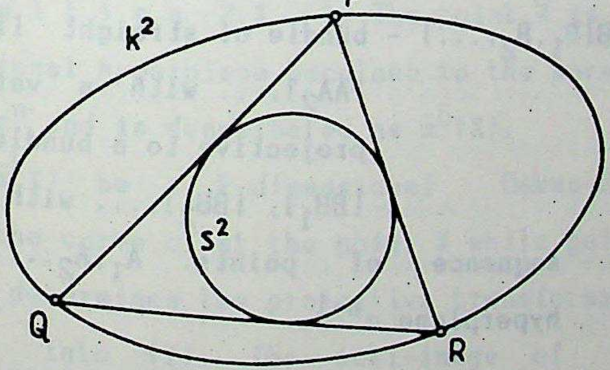

Figure 1.

It is easily proved that the sufficient and indispensable precondition for which the conic k^2 would be in the Poncelet configuration of closing with the conic s^2 is the existence of two triangles circumscribed on the conic s^2 and simultaneously inscribed into the conic k^2.

An analogous notion to the Poncelet configuration of closing for conics may be introduced for normal curves of any order.

Notations:

$\mathbf{P}^n$ - the projective space over the field of complex numbers C,

$[X_1X_2\ldots X_m]$ - hyperplane determined by the points $X_1, X_2, \ldots, X_m$,

$[\alpha^k\alpha^l]$ - hyperplane determined by k-dimensional hyperplane α^k and l-dimensional hyperplane α^l,

$[\alpha^k\alpha^l X_1X_2\ldots X_m]$ - hyperplane determined by hyperplanes α^k, α^l and by the points $X_1, X_2, \ldots, X_m$,

(X) - median bundle with vertex X,

(α^{n-2}) - pencil of hyperplanes with a hyperedge α^{n-2},

$\langle A_oA_1\ldots A_n\rangle$ - n-dimensional simplex with vertexes $A_o, A_1, \ldots, A_n$,

$A(A_1, A_2, \ldots) \barwedge B(B_1, B_2, \ldots)$ - bundle of straight lines $[AA_1]$, $[AA_2], \ldots$ with a vertex A and projective to a bundle of lines $[BB_1]$, $[BB_2], \ldots$ with a vertex B.

$\alpha^{n-1}(A_1, A_2, \ldots)$ - sequence of points $A_1, A_2, \ldots$ of the hyperplane α^{n-1}.

2. Determination of a normal curve with the help of osculant hyperplanes

A normal curve of a projective space $\mathbf{P}^n$ is an algebraic curve of the order n of that space which is not included in the space of a dimension n-1 ([3], page 5). We designate it by the symbol c^n. A normal curve c^n is a set of points of intersection of lines corresponding to each other in two projective bundles A and B of the projective space $\mathbb{P}^n$ ([1], page 323). The projection of a normal curve c^n onto a n-1-dimensional α^{n-2} hyperplane from the point $X \in C^n$, where $X \epsilon \alpha^{n-1}$, is a normal curve c^{n-1} of a space α^{n-1} ([2], page 94). Let α_1^{n-2} and α_2^{n-2} be two n-2-dimensional hyperplanes n-1-secants of the curve c^n. For each point $X \in c^n$ we will subordinate the n-1-dimensional $[\alpha_2^{n-2}X]$ hyperplane to the n-1-dimensional $[\alpha_1^{n-2}X]$ hyperplane. It is known that such a determined projection of the pencil α_1^{n-2} onto the pencil α_2^{n-2} is projective, $n \geq 2$ ([1], page 323).

Let us introduce the following inductive definition of an osculant hyperplane to the normal curve $c^n \subset \mathbb{P}^n$.

Definition 2.1. 1. The point X is determined as a zerodimensional hyperplane osculant to the normal curve c^n at the point $X \epsilon c^n$ and is denominated as $\bar{\alpha}^o(X)$.

2. Let $[\bar{\alpha}^k X]$ be a k-dimensional $0 \leq k \leq n-2$ hyperplane osculant to the curve c^n at the point X while point $Y \neq X$ i $Y \epsilon c^n$. The curve c^n determines the projective transformation f of the bundle (X) into (Y). The anti-image of a hyperplane $\alpha^{k+1} = [\bar{\alpha}^k(X)Y]$ in the projective transformation f is determined as k+1-dimensional hyperplane osculant to the c^n at the point X and is determined as $\bar{\alpha}^{k+1}(X)$.

Observe that this definition does not depend on the choice of $Y \in c^n$. It results from the following property of a normal curve: if Y_1, Y_2 are two distinct points of a normal curve c^n and $A_1, A_2, \ldots$ are arbitrary points that curve, then the bundles $Y_1(A_1, A_2, \ldots)$ and $Y_2(A_1, A_2, \ldots)$ are projective ([1], page 323). Let us prove the following:

Theorem 2.1. In the projective space $\mathbb{P}^n$, three different points A,B,C and all the hyperplanes which are osculant at the points A,B determine exactly one the normal curve c^n.

Proof. Let be given the points A,B,C and hyperplanes $\alpha_A^1 \subset \alpha_A^2 \subset \ldots \subset \alpha_A^{n-1}$, to which belongs point A and hyperplanes $\alpha_B^1 \subset \alpha_B^2 \subset \ldots \subset \alpha_B^{n-1}$ to which belongs point B. Point C does not belong to α_A^{n-1} nor to α_B^{n-1} $(n \geq 2)$. Let hyperplanes α_A^i be in the general position according to corresponding hyperplanes α_B^i for $1 \leq i \leq n-1$ it means: hyperplanes $\alpha_B^{\frac{n-1}{2}}$ and $\alpha_B^{\frac{n-1}{2}}$ are skew if n is odd and $\alpha_A^{\frac{n-2}{2}}$ and $\alpha_B^{\frac{n-2}{2}}$ are skew if n is even.

Let us take as vertices of bundles points A and B. The projective transformation of a bundle (A) into a bundle (B) will be determined if we subordinate accordingly n-1 projective pencils with hyperedges passing through B to the n-1 pencils of the n-1 dimensional hyperplanes with n-2-dimensional hyperedges passing through A (subordination is here understood as the determination of the projection between pencils) ([1], page 323).

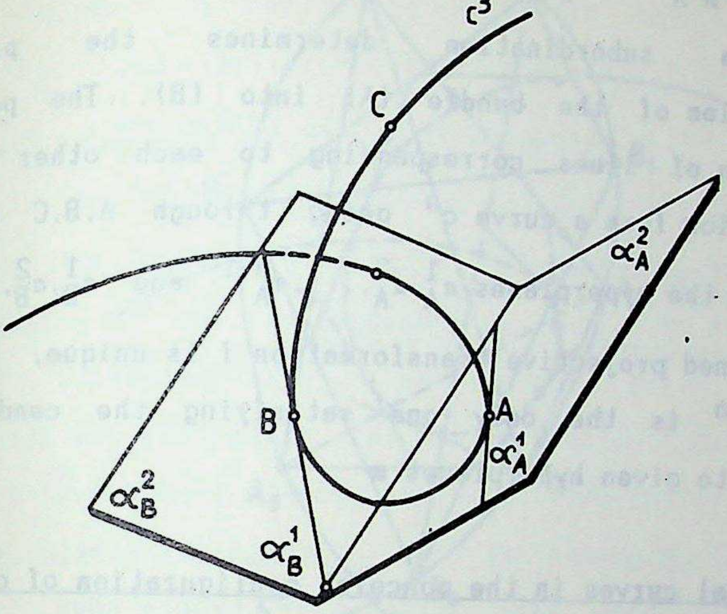

Figure 2.

Let us consider pencils of hyperplanes in the bundle (A):

$[A\ \alpha_B^{n-3}]\ ([A\ \alpha_B^{n-2}], [A\ \alpha_B^{n-3}C], [\alpha_A^1\alpha_B^{n-3}], \ldots),$

$[\alpha_A^1\alpha_B^{n-4}]\ ([\alpha_A^1\alpha_B^{n-3}], [\alpha_A^1\alpha_B^{n-4}C], [\alpha_A^2\alpha_B^{n-4}], \ldots),$

....

$[\alpha_A^{n-3}B]\ ([\alpha_A^{n-3}\alpha_B^1], [\alpha_A^{n-3}BC], [\alpha_A^{n-2}B], \ldots),$

$\alpha_A^{n-2}([\alpha_A^{n-2}B], [\alpha_A^{n-2}C],\ \alpha_A^{n-1}, \ldots).$

Let us subordinate correspondingly projective pencils of the dimensional hyperplanes n-1 in the bundle (B) to them:

$\alpha_B^{n-2}(\alpha_B^{n-1}, [\alpha_B^{n-2}C]\ [\alpha_B^{n-2}A], \ldots),$

$\alpha_B^{n-3}A]([\alpha_B^{n-2}A], [\alpha_B^{n-3}AC], [\alpha_B^{n-3}\alpha_A^1], \ldots),$

....

$\alpha_B^1\alpha_A^{n-4}]\ [\alpha_B^2\alpha_A^{n-4}], [\alpha_B^1\alpha_A^{n-4}C], [\alpha_B^1\alpha_A^{n-3}], \ldots.$

$[B\ \alpha_A^{n-3}]$ $([\alpha_B^1\alpha_A^{n-3}], [B\ \alpha_A^{n-3}C], [B\ \alpha_A^{n-2}], \ldots)$.

Such a subordination determines the projective transformation of the bundle (A) into (B). The points of intersection of lines corresponding to each other in this transformation form a curve c^n going through A, B, C which is osculant to the hyperplanes $\alpha_A^1, \alpha_A^2, \ldots, \alpha_A^{n-1}$ and $\alpha_B^1, \alpha_B^2, \ldots, \alpha_B^{n-1}$. The determined projective transformation f is unique, therefore the curve c^n is the only one satisfying the condition of osculation to given hyperplanes.■

3. Normal curves in the poncelet configuration of closing

Let us formulate the following definition:

Definition 3.1: 1. The simplex $\langle A_0A_1\ldots A_n\rangle$ is inscribed into the normal curve c^n if and only if all the vertexes of this simplex belong to the curve c^n.

2. The simplex $\langle A_0A_1\ldots A_n\rangle$ is circumscribed on the normal curve c^n if and only if all n-1-dimensional faces of this simplex are osculant to the curve c^n.

Definition 3.2: The normal curve c_1^n is in the Poncelet configuration of closing with the curve c_2^n if and only if there are two simplexes n-dimensional inscribed into the curve c_1^n and simultaneously circumscribed onto the curve c_2^n.

Theorem 3.1: If two simplexes n-dimensional $\langle A_0A_1\ldots A_n\rangle$ and $\langle B_0B_1\ldots B_n\rangle$ inscribed into the normal curve $c_1^n \subset \mathbf{P}^n$, they are circumscribed on another curve c_2^n.

Proof.

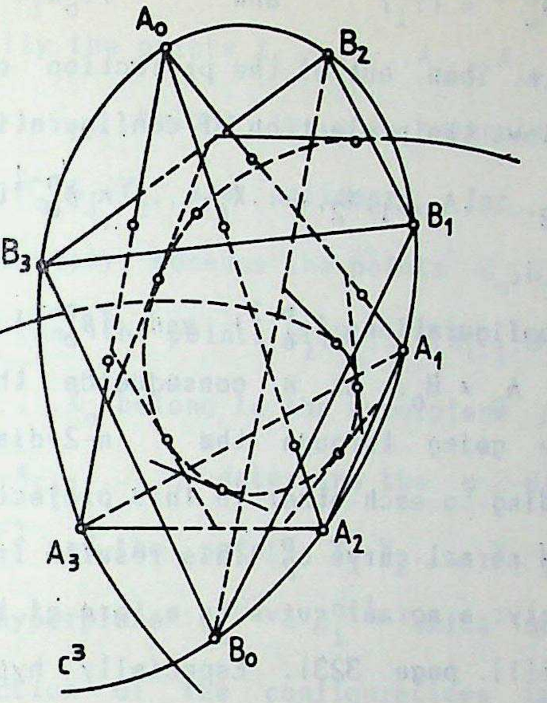

Figure 3.

Let α_i^{n-1} denote the n-1-dimensional hyperplane assigned by the points $A_o, A_1, \ldots, A_{i-1}, A_{i+1}, \ldots, A_n$. Similarly β_i^{n-1} denotes the hyperplane n-1-dimensional assigned by the points $B_o, B_1, \ldots, B_{i-1}, B_{i+1}, \ldots, B_n$. From the property of a normal curve mentioned in 2., it follows that median bundles (A_o) and (B_o) projecting appropriately the vertexes $A_1, A_2, \ldots, A_n$, $B_1, B_2, \ldots, B_n$ of the given simplexes inscribed into the curve c_1^n are projective:

$A_o(A_1, A_2, \ldots, A_n, B_1, B_2, \ldots, B_n, \ldots) \barwedge B_o(A_1, A_2, \ldots, A_n, B_1, B_2, \ldots, B_n, \ldots)$.

Let us cut the median bundle (A_o) with the hyperplane α_o^{n-1} and the bundle (B_o) with the hyperplane β_o^{n-1}. Let

$[A_oA_i] \cap \beta_o^{n-1} = \{Y_i\}$ and $[B_oB_i] \cap \alpha_o^{n-1} = \{x_i\}$, for $i=1,2,\dots,n$. Then, out of the projection of bundles (A_o) and (B_o), follows the projection of configurations:

$$\alpha_o^{n-1}(A_1,A_2,\dots,A_n,X_1,X_2,\dots,X_n,\dots) \barwedge \beta_o^{n-1}(Y_1,Y_2,\dots,Y_n,B_1,B_2,\dots \dots,B_n,\dots).$$

The configurations (α_o^{n-1}) and (β_o^{n-1}) are not perspective because $A_o \neq B_o$. As a consequence the n-1-dimensional hyperplane going through the n-2-dimensional hyperplane corresponding to each other in this projection are osculant to a determined normal curve c_2^n. This results from a dual property to the property: a normal curve is a form of two simple projective bundles ([1], page 323). Especially hyperplanes α_o^{n-1} and β_o^{n-1} are osculant to the curve c_2^n. Actually the hyperplane $\gamma_o^{n-2} = \alpha_o^{n-1} \cap \beta_o^{n-1}$ belonging to the hyperplane α_o^{n-1} conforms to $\varepsilon_o^{n-2} \subset \beta_o^{n-1}$. Analogously the hyperplane $\delta_o^{n-2} \subset \alpha_o^{n-1}$ corresponds to the hyperplane γ_o^{n-2} belonging to the hyperplane β_o^{n-1}. So, in accordance with the definition 2.1., the hyperplane β_o^{n-1} which contains the hyperplane ε_o^{n-2} corresponds to the hyperplane $\gamma_o^{n-1} \in (A_o)$ which includes γ_o^{n-2}. As a consequence β_o^{n-1} is osculant to c_2^n. The other way round the hyperplane α_o^{n-1} which includes δ_o^{n-2} conforms with the hyperplane $\bar{\gamma}_o^{n-1}(B_o)$ which includes γ_o^{n-2} and that is why α_o^{n-1} is also osculant to c_2^n.

Hyperplanes $\alpha_1^{n-1}, \alpha_2^{n-1}, \dots, \alpha_n^{n-1}, \beta_1^{n-1}, \beta_2^{n-1}, \dots, \beta_n^{n-1}$ are also osculant to c_2^n. Actually the points $A_1, A_2, \ldots, A_{i-1}, A_{i+1}, \ldots, A_n$, $Y_1, Y_2, \ldots, Y_{i-1}, Y_{i+1}, \ldots, Y_n$ belong to one hyperplane α_i^{n-1} because the points A_o, A_j, Y_j are colinear, for $i \neq j$ and $i, j = 1, 2, \ldots, n$. Analogously, because the points B_o, B_j, X_j, are collinear, it follows that the points $B_1, B_2, \ldots, B_{i-1}, B_{i+1}, \ldots, B_n$ $, X_1, X_2, \ldots, X_{i-1}, X_{i+1}, \ldots, X_n$ belong to the hyperplane β_i^{n-1}. The points $A_1, A_2, \ldots, A_{i-1}, A_{i+1}, \ldots, A_n$ determine the n-2-dimensional hyperplane $\delta_i^{n-2} \subset \alpha_i^{n-1}$ and the points $Y_1, Y_2, \ldots, Y_{i-1}, Y_{i+1}, \ldots$ $\ldots, Y_n$ determine the hyperplane $\varphi_i^{n-2} \subset \beta_i^{n-1}$ which corresponds to it in the projection of the configurations (α_o^{n-1}) and (β_o^{n-1}). The hyperplanes δ_i^{n-2} and φ_i^{n-2} determine the hyperplane α_i^{n-1}. Similarly the points $B_1, B_2, \ldots, B_{i-1}, B_{i+1}, \ldots, B_n$ determine the n-2-dimensional hyperplane $\varepsilon_i^{n-2} \subset \beta_i^{n-1}$ and the points $X_1, X_2, \ldots, X_{i-1}, X_{i+1}, \ldots, X_n$ determine the hyperplane $\psi_i^{n-2} \subset \alpha_i^{n-1}$ which corresponds to it in the projection of the configurations α_o^{n-1} and β_o^{n-1}. The hyperplane ε_i^{n-2} and ψ_i^{n-2} determine the hyperplane β_i^{n-1}. The hyperplane α_i^{n-1} and β_i^{n-1} are determined appropriately by a pair of n-2-dimensional hyperplanes corresponding to each other in the projection of the configurations (α_o^{n-1}) and (β_o^{n-1}). Thus both are osculant to a normal curve c_2^n, for $i=1,2,\ldots,n$. As a consequence all the n-1-dimensional faces of the simplexes $\langle A_o A_1 \ldots A_n \rangle$ and $\langle B_o B_1 \ldots B_n \rangle$ are osculant to the curve c_2^n.■

4. Spherical surface of a normal curve with a curvature

In this part of the paper we will take into consideration the Euclidean space W^n complemented by improper elements. In order to generalize the consideration we also admit complex forms of the space W^n. It is known that two groups of points $A_o, A_1, \dots, A_n$ and $B_o, B_1, \dots, B_n$ determine the involution of the order n+1 on the normal curve c^n ([4], page 182).

From the theorem 3.1. follows:

Theorem 4.1. The group of points n+1 belongs to the involution of the order n+1 on the normal curve c_1^n if and only if is a set of vertexes of a dimensional simplex n, whose all faces n-1-dimensional are osculant to the normal curve c_2^n which is in the Poncelet configuration of closing with the curve c_1^n.

The involution of the order n+1 on the normal curve $c^n \subset W^n$ may be used to determine the notion of the so-called spherical surface with a curvature in any point of the curve c^n.

Let hyperplanes $\alpha_D^{n-2}, \alpha_D^{n-1}$ will be appropriately n-2 and n-1-dimensional hyperplanes osculant to the curve c^n in the point D. Let the straight line k will be perpendicular to α_D^{n-2} at the point D included in α_D^{n-1}. The set of spherical surfaces n-1-dimensional, which are tangent to α_D^{n-2} in the point D with midpoints belonging to the line k, forms a pencil. Each of the spherical surfaces has also n+1 common points with the curve c^n apart from the point D. In this way we obtain a monoparameter set of n+1 pointed groups on the normal curve c^n forming the involution of the order n+1 on that curve.

n-1-Dimensional hyperplanes determined by any of the points from the group of n+1 points which belongs to the involution of the order n+1 on the curve c^n, are osculant to another curve c^{-n} which is in the Poncelet configuration of closing with the curve c^n.

The spherical surface n-1-dimensional determined by that group of n+1 points, to which point D belongs, is called a spherical surface osculant to the normal curve c^n in that point. That spherical surface cuts the hyperplane osculant to α_D^{n-1} in n-2-dimensional spherical surface, which is called a curvature spherical surface n-2-dimensional of the curve c^n in the point D.

The points D_1 and D_2 denote points of intersection of a normal curve c^3 with a sphere. Let us farther consider the pencil of spherical surfaces n-1-dimensional determined by a curvature spherical surface n-2-dimensional. Each of the spherical surfaces of that pencil is osculant to the curve c^n in the point D and cuts that curve in n points. Those groups of n intersection points of the curve c^n with each of the spherical surfaces of a pencil from a mono-parameter set of groups of n points, that is to say an involution of the order n on the curve c^n.

That group of n points, to which point D belongs, determines the spherical surface belonging to the pencil. We designate that spherical surface as a curvature spherical surface n-1-dimensional of a curve c^n in point D.

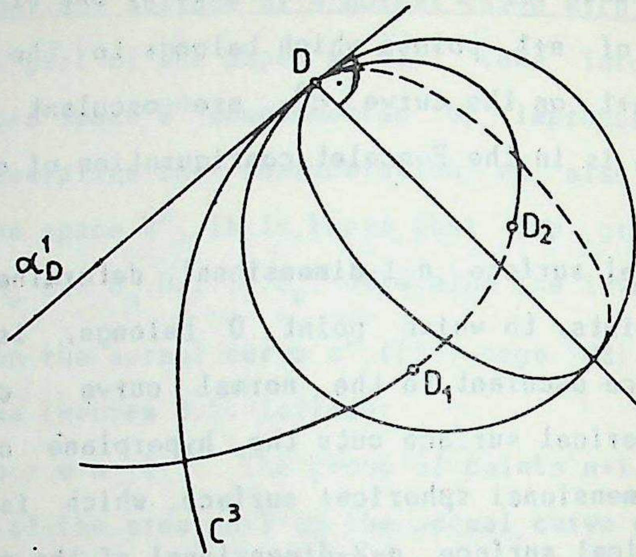

Figure 4.

REFERENCES

[1] E. Bertini: Einführung in die projektive Geometrie mehrdimensionaler Räume, Seidel und Sohn, Wien 1924.

[2] J. Kaczmarek: Generalization of Hermite collineation to the n-dimensional space, Ann. Polon. Math., 23, 93-104, 1970.

[3] E. Otto: Własności charakterystyczne krzywych normalnych, Zeszyty Nauk. Geometria Wykreślna V, 5-7, Warszawa-Poznań 1967.

[4] R. Sturm: Die Lehre von den geometrischen Verwandtschaften, t.I, Leipzig und Berlin Teubner, 1908.

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Krzysztof Witczyński

A GEOMETRICAL PROOF OF THE GENERALIZED PAPPUS THEOREM

Dedicated to the memory
of Professor Edward Otto

The well-known theorem of Pappus has been generalized in [1]. The proof of that theorem was algebraic i.e. the considered projective space was constructed over any field. However it may be interesting, whether there exists a proof which does not depend on coordinates. The aim of this note is to present such a proof, quite geometrical, of the generalized Pappus theorem.

The projective space P^n will be treated here as a set of points with distinguished classes of its subsets $\{H^m\}$ $m = -1,0,\dots,n$. We shall assume that elements of these classes, called by us m-dimensional subspaces, satisfy the incidence axioms from [2]. Furthemore we shall assume that there is a unique projectivity in P^1 which transforms three given distinct points into three other given distinct points.

$N-1$ - dimensional subspaces will be called hyperplanes.

N o t a t i o n.

$Z(H_1,\dots,H_m)$ - the join of subspaces $H_1,\dots,H_m$

$A_{(i)} = A\setminus\{a_i\}$ $i = j,j+1,\dots,m$, where $A = \{a_j,a_{j+1},\dots,a_m\}$ is an arbitrary set of points.

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GP^n - the group of nonsingular collineations in P^n.

Lemma. For each $f \in GP^n$ and each hyperplane H_1 there exists a hyperplane H_p (called by us Pappus hyperplane) such that for every set of linearly independent points $A = \{a_1, \ldots, a_n\}$ contained in H_1 the inclusion

$$\bigcap_{i=1}^{n} Z(A_{(i)}, f(a_i)) \subset H_p$$ holds.

Proof. Call $H_2 = f(H_1)$, $F = H_1 \cap H_2$, $F_1 = f(F)$, $F_o = f^{-1}(F)$, $b_i = f(a_i)$ $i=1,\ldots,n$. Since in the case $H_1 = H_2$ Lemma is trivial, we may assume that $H_1 \neq H_2$. Notice that if $n=1$, then $H_1 = a_1$, $H_2 = b_1$, $A_{(i)} = \emptyset$ and $Z(A_{(i)}, b_1) = b_1 = H_2$. Hence the hyperplane $H_p = H_2$ satisfies the thesis of Lemma.

First we shall consider the case

1° $F \neq F_1$.

Assume that, with $n \geq 2$, in the space of dimension not greater than n-1 Lemma holds.

Let $\bar{T}$ be a nonsingular projective transformation of H_2 into itself such that if $x \in F$, then $\bar{T}(x) = f(x)$. We shall prove that $H_p^{n-1} = Z(H_p^{n-2}, F_o)$, where H_p^{n-2} is the Pappus hyperplane for the couple $\bar{T}$ and F.

First of all, we shall show that $\dim Z(H_p^{n-2}, F_o) = n-1$ which is equivalent to the equality

(1) $F \cap F_o = F \cap H_p^{n-2}$.

For this purpose we choose linearly independent points

$p_1,\dots,p_{n-1}$ in such a way that $p_1 \in F \cap F_o$, $p_i \in F \setminus F_o$ for $i=2,\dots,n-1$ and $f(p_1), p_2, \dots, p_{n-1}$ are linearly independent. Call $P = \{p_1,\dots,p_{n-1}\}$ and $Z_i = Z(P_{(i)}, f(p_i))$ $i=1,\dots,n-1$. From the assumptions we obtain $\dim Z_i = n-2$ and $Z_i \neq Z_j$ for $i \neq j$ (since $Z(Z_i, Z_j) = H_2$). The above implies that $\dim Z(Z_1 \cap Z_2) = n-3$, $\dim(Z_1 \cap Z_2 \cap Z_3) = n-4$ (because the point p_3 belongs to $Z_1 \cap Z_2$ and at the same time it does not belong to Z_3) ...., $\dim(Z_1 \cap \dots \cap Z_{n-1}) = 0$. Since the point p_1 belongs to each of the subspaces Z_i, we have $\bigcap_{i=1}^{n-1} Z_i = p_1$. Hence $p_1 \in H_p^{n-2}$ (inductive assumption). Consequently $F \cap F_o \subset F \cap H_p^{n-2}$ and, since $\dim(F \cap H_p^{n-2}) = n-3 = \dim(F \cap F_o)$, the equality (1) holds.

Let S_i, S_o denote the subspaces $Z(A_{(i)}, b_i)$ and $\bigcap_{i=1}^{n} S_i$, respectively. We shall study the nature of the subspace S_o in each of the following cases:

1) $a_i \notin F \cup F_o$ for $i=1,\dots,n$.

As in the proof of the equality (1) we show that $\dim S_o = 0$ i.e. S_o is a point

2) $a_i \in F$ for $i=1,\dots,k$ $1 \le k \le n-1$ and $a_i \notin F_o$ for $i=1,\dots,n$. In view of the relations $b_i \notin F$, all i, we have $\dim S_i = n-1$ for $i=1,\dots,n$ and $S_i \neq S_j$ for $i \neq j$. Then one can check easily that S_o is a point. Notice however that if $k = n-1$, then $S_o \in H_p^{n-2}$. We infer this from the fact that the subspace $\bigcap_{i=1}^{n-1} Z(A_{(i)}, b_i)$ is a

point of H_p^{n-2} (inductive assumption) and from the equality $Z(A_{(n)}, b_n) = H_2$.

3) $a_i \in F_o$ for $i=1,\dots,k$ $1 \le k \le n-1$ and $b_j \in Z(A_{(j)})$ for $j=1,\dots,l$ $0 \le l \le k$ (the equality $l=0$ means that $b_j \notin Z(A_{(j)})$, all j). In this last case we have $\dim S_i = n-1$ for $i=l+1,\dots,n$ and at the same time $S_i = H_1$ for $i=l+1,\dots,k$. If $i=1,\dots,l$ then obviously $S_i = Z(A_{(i)})$. Hence $S_o \subset H_1$. One check easily that $\dim \bigcap_{i=k+1}^{n} S_i = k$. It implies that $\dim (H_1 \cap \bigcap_{i=l+1}^{n} S_i) = k-1$. Since $C_o = Z(a_1,\dots,a_k) \subset S_i$ for $i=l+1,\dots,n$ and $\dim C_o = k-1$, then $C_o = H_1 \cap \bigcap_{i=l+1}^{n} S_i$. Consider the subspaces $C_1 = C_o \cap S_1, \dots, C_l = C_{l-1} \cap S_l$. It is clear that $\dim C_i = k-i-1$ for $i=1,\dots,l$, hence $\dim S_o = k-l-1$. From the inclusions $Z(a_{l+1},\dots,a_k) \subset S_i$, all i, and from the equality $\dim Z(a_{l+1},\dots,a_k) = k-l-1$ it follows that $S_o = Z(a_{l+1},\dots,a_k) \subset F_o$. Thus we have shown that Lemma is true (with the additional assumption $F \neq F_1$) for such point systems $a_1,\dots,a_n$ that at least one of these points belongs to F_o or $Z(A_{(i)}) \subset F$ for some i $1 \le i \le n$. Take into account such a point system $a_1,\dots,a_n$ in H_1 that

(2) $a_i \notin F_o$ and $Z(A_{(i)}) \not\subset F$ for $i=1,\dots,n$.

Then, as it has already been shown, S_o is a point. Replace in $A = \{a_1,\dots,a_n\}$ the point a_n by a variable point x_n describing the straight line $Z(a_1,a_n)$. Consequently the point $x'_n = f(x_n)$

will describe the line $Z(b_1,b_n)$. Call $c_{ij}= Z(a_i,a_j) \cap F_o$ for $i<j$. According to the previous notations we shall use the following symbols:

$A(x_n) = \{a_1,\dots,a_{n-1},x_n\}$, $A_{(i)}(x_n) = A(x_n)\setminus\{a_i\}$, $S_i(x_n) = Z(A_{(i)}(x_n),b_i)$ for $i=1,\dots,n-1$, $S_n(x_n) = Z(A_{(n)},x_n)$,

$S_o(x_n) = \bigcap_{i=1}^{n} S_i(x_n)$.

Assume first that the point c_{1n} is such that

(3) $$f(c_{1n}) \in Z(A_{(n)}).$$

Notice that $S_o(x_n)$ is always a point when x_n describes the line $Z(a_1,a_n)$. Obviously, we exclude here two positions of x_n, namely $x_n= a_1$ (because of the inequality $\dim Z(A(x_n= a_1)) < n-1$) and $x_n= c_{1n}$ (because of the equality $S_o(x_n= c_{1n}) = \emptyset$).

One can see easily that $S_i(x_n) = Z(A_{(i)},b_i)$ for $i=2,\dots,n-1$. Hence the subspace $\alpha = \bigcap_{i=2}^{n-1} S_i(x_n)$ is a plane. Since $a_n\in Z(a_1,c_{1n})$, b_n belongs to $Z(b_1,f(c_{1n}))$. From this, in view of (3), we obtain the equality $Z(A_{(n)},b_n) = Z(A_{(n)},b_1)$. The relations $x_n'\in Z(b_1,b_n)$ and $Z(A_{(n)}(x_n)) = Z(A_n)$ imply $S_n(x_n) = S_n$.

Thence the subspace $T = \bigcap_{i=2}^{n} S_i(x_n)$ is a straight line independent of the position of x_n. Evidently $S_o(x_n) = T \cap S_1(x_n)$. Since $S_1(x_n) \cap S_n = Z(a_2,\dots,a_{n-1},b_1)$ and $T \subset S_n$, therefore $T \cap Z(a_2,\dots,a_{n-1},b_1)$ is a point independent of the position of x_n. Hence $S_o(x_n) = S_o$. Thus we have shown that if the condition (3) holds, then $S_o(x_n)$ is a fixed point independent of the position of x_n on the line $Z(a_1,a_n)$.

We have to consider the remaining case

(4) $$f(c_{1n}) \notin Z(A_{(n)}).$$

Then, as previously, we obtain the equalities $S_i(x_n) = S_i$ for $i=2,\dots,n-1$ and $\bigcap_{i=2}^{n-1} S_i(x_n) = \alpha$. The projectivity between the ranges of points $(x_n) \barwedge (x'_n)$ implies a projectivity, say π, between the pencils of hyperplanes $(S_1(x_n)) \barwedge (S_n(x_n))$. The vertex subspace $Z(a_2,\dots,a_{n-1},b_1)$ of the first of them cuts the vertex subspace $Z(A_{(n)})$ of the second in $Z(a_2,\dots,a_{n-1}) = Z^*$. Since the hyperplane $Z(A_{(n)},b_1)$ is self-corresponding in π, π is a perspectivity. The intersection subspaces of corresponding elements of π form a pencil with Z^* as a vertex subspace. From the inclusion $Z^* \subset (S_1(x_n) \cap S_n(x_n))$ and from the fact that $S_o(x_n)$ is a point it follows that $\dim Z^* \cap \alpha \le 0$. However if we had $\dim Z^* \cap \alpha = 0$; then the point $S_o(x_n)$ would be independent of the position of x_n. On the other hand, it can be easily seen that $S_o(x_n = a_n) = S_o$ and $S_o(x_n = c_{1n}) = c_{1n} \neq S_o$. We have therefore $Z^* \cap \alpha = \emptyset$. If follows from the above that the hyperplane contaning the pencil (Z^*) cuts α in a straight line, say L_α. Similarly, each subspace belonging to this pencil cuts α in one of the points $S_o(x_1)$.

Finally, if the point x_n describes the line $Z(a_1,a_n)$, then the respective point $S_o(x_n)$ describes the line L_α (in case (4)) or coincides with the fixed point S_o (in case (3)). Obviously if we replace, in the above considerations, the line $Z(a_1,a_n)$ by an other line $Z(a_i,a_j)$ $i\neq j$, then we obtain the same results.

Let now $a_1,\dots,a_n$ be given points in H_1 such that $a_i \in F$ for $i=1,\dots,a_k$, $0 \le k \le n-2$. Call $d_{ij} = F \cap Z(a_i,a_j)$ for $j=k+1,\dots,n$, $i\neq j$. Suppose first $d_{in}\notin F_o$, all i. Replace the point a_1 by a variable point x_1 which describes the line $Z(a_1,a_n)$. Then the respective point $S_o(x_1)$ describes the line $Z(S_o,c_{1n})$, and, if $x_1= d_{1n}$, then the point $S_o(x_1= d_{1n})$ takes a position p_1 on $Z(S_o,c_{1n})$. If we repeat the above proceeding n-1 times, we arrive at the point system $d_{1n},\dots,d_{n-1n},a_n$ which is associated with a point P_{n-1} contained in the hyperplane $Z(S_o,F_o)$. However, as it has been earlier shown, the point p_{n-1} must belong to H_p^{n-2}. By the above, $Z(S_o,F_o) = Z(p_{n-1},F_o) = = Z(H_p^{n-2},F_o)$. Hence Lemma is true in this case.

Assume now that among (n-1) points d_{ij} associated with each of the points $a_{k+1},\dots,a_n$ at least one belongs to $F\cap F_o$. Take into account a point $\bar{a}_n$ such that $\bar{a}_n\notin \{ F\cup F_o\cup Z(F\cap F_o,a_{k+1})\cup\dots\cup Z(F\cap F_o,a_n)\}$. It is clear that the points $\bar{d}_{in} = F\cap Z(\bar{a}_n,a_i)$ $i=1,\dots,n-1$ do not belong to $F\cap F_o$. Applying a finite number of displacements of the points a_i along the lines

$Z(a_i, a_j)$ we can arrive from the point system $a_1, \ldots, a_n$ at a point system as that above, $a_1, \ldots, a_{n-1}, \overline{a}_n$ which satisfies the assumptions of the previous case. This completes the proof of Lemma with the additional assumption $F \neq F_1$.

We have still to consider the second case:

2° $F = F_1$

If $a_i \in F$ for $i=1,\ldots,k$ and $b_i \in Z(A_{(i)})$ for $i=1,\ldots,l$, where $0 \le l \le k$, then, as previously, we show that $S_o = Z(a_{l+1}, \ldots, a_k) \subset F$. If conversely, $a_i \notin F$ for $i=1,\ldots,n$, then S_o is a point which does not belong to H_1.

Let $a_1, \ldots, a_n$ be such a point system in H_1 that $a_i \notin F$ for $i=1,\ldots,n$. We shall show that $H_p^{n-1} = Z(S_o, F)$. In order to prove it, notice that if we replace the point a_n by variable point x_n of the line $Z(a_1, a_n)$, then, as in 1°, the point $S_o(x_n)$ will describe the line $Z(S_o, c_{1n})$, where $c_{1n} = F \cap Z(a_1, a_n)$. By a finite number of such displacements we can replace the points $a_1, \ldots, a_n$ by arbitrary other linearly independent points $\overline{a}_1, \ldots, \overline{a}_n$, where $\overline{a}_i \notin F$ for $i=1,\ldots,n$. Clearly, the respective points $S_o(x_i)$ belong always to $Z(S_o, F)$. This ends the proof of Lemma.

Take now into account the collineation f^{-1}. Clearly there exists a unique Pappus hyperplane $\overline{H}_p$ associated with the couple f^{-1}, H_2. One can see easily that $f(\overline{H}_p^{n-2}) = H_p^{n-2}$, where $\overline{H}_p^{n-2} = H_1 \cap \overline{H}_p$ and $H_p^{n-2} = H_2 \cap H_p$.

Let now $A = \{a_o, \dots, a_n\}$ and $B = \{b_o, \dots, b_n\}$ be two sets of points lying on two hyperplanes H_1 and H_2, respectively. We shall introduce some auxiliary notation.

Let

$S_{ij} = Z(A \setminus \{a_i, a_j\}, b_i)$, where $i \neq j$, $i, j = 0, \dots, n$,

$$S_j = \bigcap_{\substack{i=0 \\ i \neq j}}^{n} S_{ij} \qquad j = 0, \dots, n .$$

Notice that if $Z(A_{(i)}) = H_1$ and $Z(B_{(i)}) = H_2$ for all i, then there exists a unique projective transformation $f: H_1 \longrightarrow H_2$ such that $f(a_i) = b_i$ for $i = 0, \dots, n$. In view of Lemma, we obtain hence the following generalization of the theorem of Pappus:

Theorem. Let H_1 and H_2 be two hyperplanes in the space P^n. Let next $A = \{a_o, \dots, a_n\}$ and $B = \{b_o, \dots, b_n\}$ be two sets of points. If $A \subset H_1$, $B \subset H_2$ and $\dim Z(A_{(i)}) = \dim Z(B_{(i)}) = n-1$ for $i = 0, \dots, n$, then $\dim Z(S_o, \dots, S_n) \leq n-1$.

Remarks.

If we denote by H_p and $\overline{H}_p$ the Pappus hyperplanes corresponding to the couples f_1, H_1 and f_1^{-1}, H_2, respectively, where $f_1: P^n \longrightarrow P^n$ is an extension of f, then $Z(S_o, \dots, S_n) \subset H_p$ (in general $Z(S_o, \dots, S_n) = H_p$).

In the case $n=2$ we obtain the well-known theorem of Pappus. Then the lines H_p and $\overline{H}_p$ coincide.

REFERENCES

[1] K. Witczyński: Generalized Pappus Theorem in the Projective Space P^n, Bull. Acad. Polon. Sci. Serie sci.math.,astr. et phys. 6 (1979).

[2] W. V. D. Hodge, D. Pedoe: Methods of algebraic geometry, Cambridge 1947.

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COMPLEX-TYPE DESCRIPTION OF 3-DIMENSIONAL HYPERBOLIC GEOMETRY

Dedicated to the memory
of Professor Edward Otto

Introduction

The aim of this paper is to construct a field on the horizon of 3-dimensional hyperbolic space and coordinatize the space over the field we obtain. Our construction is a natural analogue of Hilbert's "End-Calculus". Let's recall the classical construction (compare [1], [5]). For given ends α, β we denote $\alpha \sqcup \beta$ the line joining α and β. Given lines L, M we denote $L \oplus M$ the bisector line; it is unique whenever K, M are parallel or hyperparallel. Denote by σ_k the symmetry with axis K. Let $0, 1, \infty$ be three distinct ends. Then we define for ends $\alpha, \beta \neq \infty$

$$\alpha + \beta = \gamma \quad \text{iff} \quad \gamma \sqcup \infty = \sigma_{(\infty \sqcup \alpha) \oplus (\infty \sqcup \beta)}(\infty \sqcup 0).$$

We obtain $\quad -\alpha = \gamma \quad \text{iff} \quad \gamma \sqcup \infty = \sigma_{0 \sqcup \infty}(\alpha \sqcup \infty)$, i.e.

$$-\alpha = \gamma \quad \text{iff} \quad \gamma = \alpha = 0 \quad \text{or} \quad \alpha \sqcup \gamma \perp 0 \sqcup \infty.$$

$\alpha > 0$ iff $0, \infty$ does not separate $1, \alpha'$ (the horizon can be projectively ordered).

For $\alpha, \beta > 0$ we put

$$\alpha \cdot \beta = \gamma \quad \text{iff} \quad \gamma > 0 \quad \text{and}$$

$$\gamma\cup-\gamma = \sigma_{(\alpha\cup-\alpha)\oplus(\beta\cup-\beta)}(1\cup-1)$$

and extend this by $\alpha\cdot(-\beta) = (-\alpha)\cdot\beta = -\alpha\cdot\beta$, $(-\alpha)\cdot(-\beta) = \alpha\cdot\beta$. Drawing the figures we see that the construction involves two pencils of lines - lines parallel in the direction ∞, and lines orthogonal to $O\cup\infty$. Even subgroup of the first one is invariant under the even subgroup of the second - this gives distributivity. Operations +, · form groups because of simple properties of pencils (see [3] or [4]).

The above construction can be generalized to 3-dimensional hyperbolic geometry; the field we obtain is in fact a complex-type field (see [2]).

Basic notions and notations

The geometry we shall be concerned with is the 3-dimensional hyperbolic geometry. It is considered as the theory of all 3-dimensional Kleins Models, i.e. structures with universe K consisting of points inside non ruled quadric V in an ordered projective space coordinatized by some ordered Euclidean field. Lines and planes in the model are interpreted as intersections of K with projective lines and planes. Let us fix one such model; all the constructions will be presented in this arbitrary but fixed structure. One can visualise V as a sphere (in some coordinate system), this will be helpful to understand the construction. The horizon of our model is simply the set V; its elements, ends, will be usually denoted α, β, γ, δ,... . To every two ends α, β there corresponds uniquely a line $\alpha\cup\beta$ having α, β as ends. For given (hyperbolic) plane Q we denote by ∂Q the set of all ends of lines contained in Q, clearly Q determines Q uniquely. Moreover ∂Q is a circle on V. For any three ends α, β, γ we denote by $c(\alpha,\beta,\gamma)$ the circle on V determined by α, β, γ. Then $P(\alpha,\beta,\gamma)$ denotes the plane Q such that $\partial Q = c(\alpha,\beta,\gamma)$. Just from the construction we obtain

Fact 1. $\langle V; \{\partial Q: Q\text{-hyp. plane}\}, \in \rangle$ is a Möbius plane.

Using an order we can define a half-line $\overrightarrow{a\alpha}$ with origin a and direction (end) α, and a (hyperbolic) half-plane. If Q' is a half-plane, then $\partial Q'$ denotes its boundary, the set of ends of halflines contained in Q'. Then $\partial Q'$ forms a "half-circle". For three ends α, β, γ we denote by $hc(\alpha,\beta;\gamma)$ this half-circle of $c(\alpha,\beta,\gamma)$ with ends α, β, which contains γ; analogously $HP(\alpha,\beta;\gamma)$ denotes this halfplane of $P(\alpha,\beta,\gamma)$ with boundary line $\alpha u \beta$ which contains on the horizon.

Let us choose $O, E, \infty \in V$-three distinct ends. They will be used as parameters in constructing field. In the surrounding projective space we consider planes Z_o, Z'_o tangent to V in O, ∞. Next we consider the stereographical projection λ with pole ∞, from $V \setminus \{\infty\}$ onto $Z = Z_o \setminus Z'_o$. Clearly $Z_o \setminus Z'_o$ can be considered as an affine plane. But we know even more.

Fact 2. $\mathbf{Z} = \langle Z, \lambda(\{ c : \infty \in c,\ c\text{-circle on } V \}), \lambda(\{ c : \infty \notin c,\ c\text{-circle on } V \})\rangle$ is an Euclidean plane with (euclidean) circles.

This fact will be crucial in justifying further investigations.

The construction of (complex) field on ends

For hyperbolic line L we denote by σ_L the line symmetry with axis L, analogously if Q is a plane then σ_Q denote the reflection on Q. Its known for $L = \alpha \sqcup \beta$ that $\lambda(\sigma_L)$ is a inversion with centers $\lambda(\alpha)$, $\lambda(\beta)$. Analogously $\lambda(\sigma_Q)$ is a inversion on $\lambda(\partial Q)$.

Let $M = \alpha \sqcup \infty$, $L = \beta \sqcup \infty$. Then there exists unique line N with end ∞, such that $\sigma_N(L) = M$; we denote $N = L \oplus M$.

Now we define addition on $V \setminus \{\infty\}$.

Definition 3. $\alpha, \beta, \gamma \neq \infty \longrightarrow \alpha + \beta = \gamma :\longleftrightarrow \gamma \sqcup \infty = \sigma_{(\alpha \sqcup \infty) \oplus (\beta \sqcup \infty)}(0 \sqcup \infty)$.

One can notice that for every $\delta \neq \infty$ the function $\lambda(\sigma_{\delta \sqcup \infty})$ is a central symmetry on Z, with center $\lambda(\delta)$. Therefore $\lambda(+)$ is a usual (affine) addition in $\mathbb{Z}$. This proves

Lemma 4. $\langle V \setminus \{\infty\}, 0, + \rangle$ is an abelian group.

If Q, R are planes, $Q, R \perp M$, M is a line then there is exactly one plane S such that $\sigma_S(Q) = R$; we denote $S = Q \oplus R$. If Q, R are planes and $Q, R \supseteq M$ then there are two, such bisecting planes. But if we consider two half-planes Q', R' with common boundary line M then there is exactly one plane S with $\sigma_S(Q') = R'$; we denote also $S = Q' \oplus R'$.

Given any end δ we denote

Q_δ : Q_δ a plane, $Q_\delta \perp 0 \sqcup \infty$, $\delta \in \partial(Q)$,

Q'_δ : $HP(0, \infty\, ; \delta)$.

Now we define multiplication

Definition 5. $\alpha, \beta \neq 0, \infty \longrightarrow$

$$\alpha \cdot \beta := \partial(\sigma_{Q_\alpha \oplus Q_\beta}(Q_E)) \cap \partial(\sigma_{Q'_\alpha \oplus Q'_\beta}(Q'_E))$$

$$\alpha\cdot 0 = 0\cdot\alpha = 0.$$

One can notice that $\lambda(Q_\delta)$ is a circle in $\mathbb{Z}$, with center $0 = \lambda(0)$, and $\lambda(Q'_\delta)$ is a half line with origin o. Then we see that $\cdot$ is a multiplication in a complex field, as a $\sqrt{-E}$ on V one can take I such that $I \in \partial(Q_E)$, $E\cup -E \perp I\cup -I$. The point $\lambda(I)$ will serve as a "imaginary unit" with respect to $\lambda(\cdot)$ on $\mathbb{Z}$. Finally we put

Definition 6. $\alpha \neq 0 \longrightarrow$

$\bar{\alpha} = \gamma : \longleftrightarrow \alpha \in \partial P(0,E,\infty)$ & $\alpha = \gamma$ or $\alpha\cup\gamma \perp P(0,E,\infty)$.

More easily we can write $\bar{\alpha} = \sigma_{P(0,E,\infty)}(\alpha)$.

All the above considerations give us

Theorem 7. $\mathbb{Z} = \langle V \setminus \{\infty\}, 0, E, +, \cdot, ^{-} \rangle$ is a complex field satisfying $(\exists x)[\, x^2 = -1 \wedge \bar{x} = -x \,]$.

Remark 8. It is seen that Definition 3 is exactly the same as this used in Hilbert's construction. But we also can define multiplication in a way which will be more similar to those of Hilbert. Let $\delta \neq 0, \infty$. Then $\delta\cup -\delta \perp 0\cup\infty$. Denote by L'_δ the halfline $L'_\delta = \overrightarrow{(\delta\cup -\delta) \cap (0\cup\delta), \delta}$. For any two ends $\alpha, \beta \neq 0, \infty$ there is exactly one line M such that $\sigma_M(L'_\alpha) = L'_\beta$, $M \perp 0\cup\infty$. We denote $M = L'_\alpha \oplus L'_\beta$. Then we can put

$\alpha, \beta \neq 0, \infty \longrightarrow$

$\alpha\cdot\beta = \gamma : \longleftrightarrow L'_\gamma = \sigma_{L'_\alpha \oplus L'_\beta}(L'_E)$.

Analytical interpretation of hyperbolic geometrical notions

To describe hyperbolic geometry in terms of the field we have constructed above it suffices to give analytical interpretation of points, lines, incidence and orthogonality. To do so we shall carefully analyze λ on $K \cup V$ and fix the field on $\mathbb{Z}$. Denote $V_0 = V \setminus \{\infty\}$. Lines correspond to pairs of ends, therefore lines are interpreted as elements of $\left\{\{\alpha,\beta\} : \alpha \neq \beta, \quad \alpha,\beta \in V_0\right\} \cup V_0$. Then we have natural interpretation $\{\alpha,\beta\} \longmapsto \alpha \cup \beta$, $\alpha \longmapsto \infty \cup \alpha$. Let $a \in K$, $a \notin 0 \cup \infty$. There is unique line L_a such that $a \in L_a \perp 0 \cup \infty$. Let $L_a = \alpha \cup \beta$. There is unique line M_a such that $a \in M_a$, M_a has ∞ as an end. Let $M_a = \infty \cup \gamma$. Then a determines a triple α, β, γ. It's seen that $\alpha = -\beta$ and $\gamma \neq 0$, γ lies between α, β. Therefore every point a corresponds to $\langle \{\alpha,\beta\},\gamma \rangle$ such that $\alpha = -\beta$, $|\alpha - \beta| = |\alpha - \gamma| + |\gamma - \beta|$. The point a determined by $\{\alpha,\beta\},\gamma$ will be denoted by $p_{\alpha,\beta;\gamma}$. We have $p_{\alpha,\beta;\gamma} \in \alpha' \cup \beta'$ iff α, β, α', β' are cocincular in $\mathbb{Z}$ and γ lies between α', β'. This means $p_{\alpha,\beta;\gamma} \in \alpha' \cup \beta' \longleftrightarrow |\alpha' - \beta'| = |\alpha' - \gamma| + |\gamma - \beta'|$ &

$$\begin{vmatrix} |\alpha|^2 & \alpha & \bar{\alpha} & 1 \\ |\beta|^2 & \beta & \bar{\beta} & 1 \\ |\alpha'|^2 & \alpha' & \bar{\alpha}' & 1 \\ |\beta'| & \beta' & \bar{\beta}' & 1 \end{vmatrix} = 0.$$

Analogously $p_{\alpha,\beta;\gamma} \in \alpha' \cup \infty \longleftrightarrow \gamma = \alpha'$.

Obviously circles and lines in $\mathbb{Z}$ are interpreted as hyperbolic planes - orthogonality of such planes correspond to usual (euclidean) orthogonality. Therefore we have interpreted all primitive notions of hyperbolic geometry in terms of complex field.

Practically this means that one can investigate 3-dimensional hyperbolic geometry analytically with the help of complex fields.

REFERENCES

[1] D. Hilbert : Grundlagen der Geometrie, Leipling, 1938.

[2] E. Kusak, K. Prażmowski : The Analytical Geometry without Coordinates, Zeszyty Nauk. Geometria, N. 13, Poznań 1985.

[3] G. Lewandowski : On Vectors in Affine Space of Arbitrary Dimension, Zeszyty Nauk. Geometria, N. 13, Poznań 1985.

[4] K. Prażmowski : Few Remarks on the Algebraic Construction of a Pencil and a Congruence, Demonstratio Math., 17, (1984).

[5] P. Szasz : Dirpermissible introduction of Weierstrass homogenous coordinates...[in] The` Axiomatic Method... North-Holland Publishing Company, Amsterdam 1959 pp. 97-113.

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In practically this means that one can investigate n-dimensional hyperbolic geometry analytically with its help over Euclidean fields.

REFERENCES

[1] H i l b e r t : Grundlagen der Geometrie, Leipzig, 1930.

[2] [illegible] : [illegible] Geometry without Conditions [illegible] Geometrie, [illegible] Poznań 1965.

[3] [illegible] : On Factors in Affine Space of Arbitrary Dimension, Zeszyty Naukowe Geometria 13, Poznań 1979.

[4] K. P r a ż m o w s k i : Some Remarks on the Algebraic Construction of a Parcel and a Convergence, Demonstratio Math. 17 (1984).

[5] W. S z m i e l e w : Directed Introduction of Generalized homogeneous coordinates, (in:) The Axiomatic Method, North-Holland Publishing Company, Amsterdam 1959 pp. 97-134.

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INVESTIGATION OF INFINITE SYSTEMS OF DIFFERENTIAL EQUATIONS

Dedicated to the memory of Professor Edward Otto

This paper deals with the existence and uniqueness of the solutions of certain uncountable systems of ordinary differential equations in locally convex topological spaces. It is based on the ideas of paper [4] in which a countable system of a differential equations in Fréchet spaces was investigated.

1. Basic definitions and notations

Let $\left\{\mathbf{X}_s : s \in \mathbf{S}\right\}$ where $\mathbf{S}$ is a measurable set of real parameters, be a family of real Banach spaces. In the sequel $\mathbf{S}$ will be the interval $[0,1]$. By Ω we denote the cartesian product of the spaces, $\Omega = \mathop{\mathbf{P}}_{s\in\mathbf{S}} \mathbf{X}_s$. An element of Ω will be denoted by $\xi = \left\{\mathbf{x}_s\right\}$, $\mathbf{x}_s \in \mathbf{X}_s$ for $s \in \mathbf{S}$. In the set Ω in a natural way we introduce addition of elements, multiplication by real scalars, thus Ω becomes a linear space.

In the space Ω we introduce the projection type topology (Tichonov topology) defined by an uncountable family of seminorms $\mathbf{p}_s$, $s \in \mathbf{S}$. Thus if $\|.\|_s$ is the norm in $\mathbf{X}_s$, then $\mathbf{p}_s(\xi) = \|\mathbf{x}_s\|_s$, for $\xi \in \Omega$ and $\xi = \left\{\mathbf{x}_s\right\}$, $x_s \in \mathbf{X}_s$. The convergence in Ω is pointwise.

Let $\xi^n = \left\{\mathbf{x}_s^n\right\}$, $\xi = \left\{\mathbf{x}_s\right\}$. $\xi^n \longrightarrow \xi$ in Ω if, for every $\mathbf{s} \in \mathbf{S}$, $\mathbf{x}_s^n \longrightarrow \mathbf{x}_s$ in $\mathbf{X}_s$. Ω is locally convex, sequentially complete, in general it is not metrisable (see [1], [2], [3]).

We denote by $\mathcal{F}$ the set of all functions $\eta : [0,\tau] \longrightarrow \Omega$ continuous on $[0,\tau]$, $0<\tau<\infty$, with values in Ω. If $\eta \in \mathcal{F}$, then $\eta(\mathbf{t}) = \left\{\mathbf{y}_s(\mathbf{t})\right\}$, $s \in [0,1]$, where $\mathbf{y}_s : [0,\tau] \longrightarrow \mathbf{X}_s$ is a continuous function of $\mathbf{t}$. Just like in Ω we introduce a linear structure in $\mathcal{F}$. Next we introduce in $\mathcal{F}$ a Tichonov topology by an uncountable family of seminorms $\mathbf{q}_s$, $s \in [0,1]$, defined in the following way : for $\eta \in \mathcal{F}$, $\eta = \left\{\mathbf{y}_s\right\}$, $s \in [0,1]$,

$$\mathbf{q}_s(\eta) = \max_{t\in[0,\tau]} \mathbf{p}_s(\eta(\mathbf{t})) = \max_{t\in[0,\tau]} \|\mathbf{y}_s(\mathbf{t})\|_s \ , \ s \in [0,1].$$

The space $\mathcal{F}$ with this topology is, like Ω, linear, locally convex and sequentially complete.

Now we introduce several other notations.

Let $\mathbf{M}(s,r)$, $\mathbf{M} : [0,1]\times[0,1] \longrightarrow \mathbf{R}$, be a nonnegative function measurable in both of variables $\mathbf{r}$ and $\mathbf{s}$. We define an operator $\mathbf{M}$ acting on functions $\mathbf{p}_s(\xi)$ in the following way

$$\mathbf{M}\,\mathbf{p}_s(\xi) = \int_0^1 \mathbf{M}\,(s,r)\,\mathbf{p}_r(\xi)\,dr.$$

By $\mathbf{M}^0$ we shall denote the identity operator $\mathbf{M}^0\mathbf{p}_s(\xi) = \mathbf{p}_s(\xi)$ and by $\mathbf{M}^n$ the n-th iteration of $\mathbf{M}$, thus

$$\mathbf{M}^n\mathbf{p}_s(\xi) = \int_0^1 \mathbf{M}(s,r_1)\int_0^1 \mathbf{M}(r_1,r_2)\dots\int_0^1 \mathbf{M}(r_{n-1},r_n)\mathbf{p}_{r_n}(\xi)dr_n\dots dr_2dr_1 .$$

If for the functions $\mathbf{p}_s(\xi)$ the values $\mathbf{M}^k\mathbf{p}_s$, $k \in N$, exist and the series $\sum_{k=0}^{\infty}\frac{\mathbf{M}^k\mathbf{p}_s(\xi)}{k!}$ is convergent for every $s \in [0,1]$, then by $\mathbf{e}^{\mathbf{M}}$ we shall denote the operator defined by

$$\mathbf{e}^{\mathbf{M}}\,\mathbf{p}_s(\xi) = \sum_{k=0}^{\infty}\frac{\mathbf{M}^k\mathbf{p}_s(\xi)}{k!} .$$

In the same way we define the above operators for functions $q_s(\eta)$. With these integral operators we define some subspaces of Ω and $\mathcal{F}$ which play a crucial part in the sequel. Denote by $S(\mathbf{M}) \subset \Omega$ the set

$$\mathbf{S}(\mathbf{M}) := \left\{ \xi \in \Omega : \mathbf{M}\,\mathbf{p}_s(\xi) < \infty ,\ \forall\, s \in [0,1] \right\}.$$

This means that for every $s \in [0,1]$ there exists a finite integral $\int_0^1 \mathbf{M}(s,r)\,\mathbf{p}(\xi)\,dr$ which is a bounded function of s.

For a function $\eta : [0,\tau] \longrightarrow \Omega$ we define the set $\mathcal{S}(\mathbf{M})$ in the following way

$$\mathcal{S}(\mathbf{M}) := \left\{ \eta \in \mathcal{F} : \mathbf{M}\,\mathbf{q}_s(\eta) < \infty ,\ \forall\, s \in [0,1] \right\}.$$

For a function η for which the operators $\mathbf{M}^n$ and $\mathbf{e}^{\mathbf{M}}$ are defined we introduce the sets

$$\mathbf{U}(\mathbf{M}) := \bigcap_{n=1}^{\infty} \mathcal{S}(\mathbf{M}^n) ,$$

$$\mathbf{Z}(\mathbf{M}) := \left\{ \eta \in \mathbf{U}(\mathbf{M}) : \lim_{n\longrightarrow\infty} \frac{1}{n!}\mathbf{M}^n\ \mathbf{q}_s(\eta) = 0,\ \forall\, s \in [0,1] \right\},$$

$$\mathcal{S}(e^{\mathbf{M}}) := \left\{ \eta \in \mathbf{U}(\mathbf{M}) : \sum_{n=0}^{\infty} \frac{1}{n!} \mathbf{M}^n \; \mathbf{q}_s(\eta) < \infty \; , \; \forall \, s \in [0,1] \right\}.$$

The sets of type $\mathbf{S}(\mathbf{M})$ are linear subspaces of Ω. The sets $\mathcal{S}(\mathbf{M})$, $\mathbf{U}(\mathbf{M})$, $\mathbf{Z}(\mathbf{M})$, $\mathcal{S}(e^{\mathbf{M}})$ are linear subspaces of $\mathcal{F}$, moreover $\mathcal{S}(e^{\mathbf{M}}) \subset \mathbf{Z}(\mathbf{M}) \subset \mathbf{U}(\mathbf{M}) \subset \mathcal{S}(\mathbf{M})$.

2. The Cauchy problem for uncountable system of ordinary differential equations

Let $\mathbf{V}$ be a nonempty subset of Ω and let there be given a function $\mathbf{F} : [0,\tau] \times \mathbf{V} \longrightarrow \Omega$, where

$$\mathbf{F} = \left\{ \mathbf{F}_s \right\}_{s \in [0,1]} \quad \text{and} \quad \mathbf{F}_s : [0,\tau] \times \mathbf{V} \longrightarrow \mathbf{X}_s \; .$$

Let $\xi_0 \in \mathbf{V} \subset \Omega$, where $\xi_0 = \left\{ \mathbf{x}_s^0 \right\}$, $s \in [0,1]$, and $\mathbf{x}_s^0 \in \mathbf{X}_s$. We shall deal with a Cauchy problem for an uncountable system of ordinary differential equations, denoted by a single equation

$$\text{(2.1)} \qquad \begin{cases} \dfrac{\mathbf{d}\eta(t)}{\mathrm{d}t} = \mathbf{F}(t, \eta(t)), \\ \eta(0) = \xi_0, \end{cases}$$

where $\eta(t) = \left\{ \mathbf{y}_s(t) \right\}$, $s \in [0,1]$, and

$$\frac{\mathbf{d}\eta(t)}{\mathrm{d}t} = \left\{ \frac{\mathrm{d}\mathbf{y}_s(t)}{\mathrm{d}t} \right\}, \quad s \in [0,1].$$

We understand $\dfrac{\mathrm{d}\mathbf{y}_s(t)}{\mathrm{d}t}$ as the value in $\mathbf{t}$ of the derivative of the function $\mathbf{y}_s : [0,\tau] \longrightarrow \mathbf{X}_s$ in the ordinary sense, the derivative $\frac{\mathrm{d}\eta}{\mathrm{d}t}$ is then defined in the sense of the convergence given by the Tichonov topology in Ω.

Definition 1. Let $\mathbf{M}(s,r) \geq 0$ be a function integrable in each of the parameters in $[0,1]$ and let $\mathbf{V} \subset \Omega$ and $\mathbf{S}(\mathbf{M}) \subset \mathbf{V}$. If the function $\mathbf{F} : [0,1] \times \mathbf{V} \longrightarrow \Omega$ satisfies the following conditions:

(i) $\mathbf{F}(.,\xi) : [0,\tau] \longrightarrow \Omega$ is a continuous function for every $\xi \in \mathbf{V}$,

(ii) $\forall\, t \in [0,1]$, $\xi_1, \xi_2 \in \mathbf{S}(\mathbf{M})$

$$\mathbf{p}_s[\, \mathbf{F}(t,\xi_1) - \mathbf{F}(t,\xi_2)\,] \leq \int_0^1 \mathbf{M}(s,r)\mathbf{p}_r(\xi_1-\xi_2)dr,$$

then the function $\mathbf{F}$ is said $\mathbf{M}$ - regular on $[0,\tau] \times \mathbf{V}$ and the Cauchy problem (2.1) is said $\mathbf{M}$ - regular on $[0,\tau] \times \mathbf{V}$.

The condition (ii) plays in our problem the role of the Lipschitz condition.

Lemma 1. Let $\mathbf{F}$ be $\mathbf{M}$ - regular on $[0,\tau] \times \mathbf{V}$, let $\eta \in \mathcal{S}(\mathbf{M})$ and $\mathbf{f}(t) = \mathbf{F}(t,\eta(t))$, $t \in [0,\tau]$. Then the function $\mathbf{f} : [0,\tau] \longrightarrow \Omega$ is continuous.

Proof. Let $\mathbf{t} \in [0,\tau]$ be fixed and $t+\Delta t \in [0,\tau]$. We have

$$\mathbf{p}_s[\, \mathbf{f}(t+\Delta t) - \mathbf{f}(t)\,] = \mathbf{p}_s[\mathbf{F}\big(t+\Delta t, \eta(t+\Delta t)\big) - \mathbf{F}\big(t,\eta(t)\big)] \leq$$

$$\leq \mathbf{p}_s[\mathbf{F}\big(t+\Delta t, \eta(t+\Delta t)\big) - \mathbf{F}\big(t+\Delta t,\eta(t)\big)] +$$

$$+ \mathbf{p}_s[\mathbf{F}\big(t+\Delta t, \eta(t)\big) - \mathbf{F}\big(t,\eta(t)\big)\,].$$

From the continuity of $\mathbf{F}$ in the first argument it follows that $\forall\, \varepsilon_s > 0\ \exists\, \mathbf{h}(s,\varepsilon_s) > 0$ such that, if $|\Delta t| < \mathbf{h}(s,\varepsilon_s)$, then $\mathbf{p}_s[\, \mathbf{F}(t+\Delta t,\eta(t)) - \mathbf{F}(t,\eta(t))\,] < \dfrac{\varepsilon_s}{2}$.

From condition (ii) of Definition 1 we have

$$\mathbf{p}_s[\mathbf{F}\big(t+\Delta t, \eta(t+\Delta t)\big) - \mathbf{F}\big(t+\Delta t,\eta(t)\big)] \leq$$

$$\leq \int_0^1 \mathbf{M}(s,r)\mathbf{p}_r\Big(\eta(t+\Delta t) - \eta(t)\Big)\,dr.$$

Consider for fixed $\eta \in \mathcal{G}(\mathbf{M})$, $\mathbf{t}$ and $\Delta \mathbf{t}$ small, the integral

$$I(\Delta t) = \int_0^1 \mathbf{M}(s,r)\mathbf{p}_r\Big(\eta(t+\Delta t) - \eta(t)\Big)dr.$$

We have $\mathbf{p}_r\Big(\eta(t+\Delta t) - \eta(t)\Big) \leq 2\mathbf{q}_r(\eta)$ and as $\eta \in \mathcal{G}(\mathbf{M})$, thus there exists $\int_0^1 \mathbf{M}(s,r)2\mathbf{q}_r(\eta)\,dr$ and $I(\Delta t) \leq \int_0^1 \mathbf{M}(s,r)2\mathbf{q}_r(\eta)dr$.

Given a fixed $\varepsilon_s > 0$ we have :

$\mathbf{1}^0$. From the theorem of absolute continuity of the Lebesgue integral for $\varepsilon_s > 0$ there exists $\delta(s,\varepsilon_s) > 0$, such that, for every measurable $\mathcal{U} \subset [0,1]$ such that $\mu\mathcal{U} < \delta$, we have $\int_{\mathcal{U}} \mathbf{M}(s,r)2\mathbf{q}_r(\eta)dr < \frac{\varepsilon_s}{4}$.

$\mathbf{2}^0$. If we take $\mathbf{c} \in \mathbf{R}$ such that $\frac{1}{\mathbf{c}}\int_0^1 \mathbf{M}(s,r)2\mathbf{q}_r(\eta)dr < \delta$, then from the Tshebyshev inequality we get

$$\mu\left\{ r \in [0,1] : \mathbf{M}(s,r)2\mathbf{q}_r(\eta) \geq \mathbf{c} \right\} \leq \delta.$$

Taking $\mathcal{U} = \left\{ r \in [0,1] : \mathbf{M}(s,r)2\mathbf{q}_r(\eta) \geq \mathbf{c} \right\}$ we have $\mu\mathcal{U} < \delta$ and thus $\int_{\mathcal{U}} \mathbf{M}(s,r)2\mathbf{q}_r(\eta)\,dr < \frac{\varepsilon_s}{4}$. In the set $[0,1]\setminus\mathcal{U}$ we have $\mathbf{M}(s,r)2\mathbf{q}_r(\eta) < \mathbf{c}$. As the integrant in this set converges to 0 with $\Delta t \longrightarrow \mathbf{0}$, so there exists $\mathbf{h}^*(s,\varepsilon_s) > 0$ such that for $|\Delta \mathbf{t}| < \mathbf{h}^*$ there is

$$\int_{[0,1]\setminus\mathcal{U}} \mathbf{M}(s,r)\mathbf{p}_r\Big(\eta(t+\Delta t) - \eta(t)\Big)dr < \frac{\varepsilon_s}{4}.$$

Thus

$$I(\Delta t) \le \int_{\mathcal{U}} \mathbf{M}(s,r)2\mathbf{q}_r(\eta)dr + \int_{[0,1]\setminus\mathcal{U}} \mathbf{M}(s,r)\mathbf{p}_r\Big(\eta(t+\Delta t)-\eta(t)\Big)\,dr <$$

$$< \frac{\varepsilon_s}{4} + \frac{\varepsilon_s}{4} = \frac{\varepsilon_s}{2}\ .$$

Taking $\bar{h}(s,\varepsilon_s) \le \min\Big\{ h(s,\varepsilon_s),\ h^*(s,\varepsilon_s) \Big\}$ we get $\mathbf{p}_s\Big(\mathbf{f}(t+\Delta t) - \mathbf{f}(t)\Big) < \varepsilon_s$ if $|\Delta t| < \bar{h}(s,\varepsilon_s)$, for every $s \in [0,1]$, which proves the continuity of the function $\mathbf{f}$ in the topology of $\mathcal{F}$.

Lemma 1 allows us to define the integral of function $\mathbf{f}(t)$ on $[0,\tau]$, $\tau < \infty$. If $\mathbf{f}(t) = \Big\{\mathbf{f}_s(t)\Big\}$, $s \in [0,1]$, $t \in [0,\tau]$, and $\mathbf{f} : [0,\tau] \longrightarrow \Omega$, then $\int_0^t \mathbf{f}(u)du := \Big\{ \int_0^t \mathbf{f}_s(u)du \Big\}$, $s \in [0,1]$.

Each of the integrals $\int_0^t \mathbf{f}_s(u)\,du$ is well defined for every $s \in [0,1]$ as the integral of a continuous function on $[0,\tau]$ with values in the Banach space $\mathbf{X}_s$.

Definition 2. The sequence of functions $\eta_k \in \mathcal{F}$, $k = 0,1,2$ ..., is said a Picard sequence for a Cauchy problem (2.1), if $\eta_{k+1}(t) = \xi_0 + \int_0^t \mathbf{F}\Big(u,\eta_k(u)\Big)du$, $t \in [0,1]$, $k = 0,1,2,$ ... .

Definition 3. **P** - solution (solution in the sense of Picard) of a Cauchy problem (2.1) is an arbitrary limit in $\mathcal{F}$ of a Picard sequence for this problem.

Definition 4. A classical solution of the Cauchy problem (2.1) is a function $\eta(\mathbf{t})$ such that

$$\eta(t) = \xi_0 + \int_0^t \mathbf{F}\big(u,\eta(u)\big)du \ , \quad t \in [0,\tau] \ .$$

3. The existence and uniqueness

In the sequel we shall assume that the function $\mathbf{F}(t,\eta(t))$ is $\mathbf{M}$ - regular and the Cauchy problem (2.1) is $\mathbf{M}$ - regular on $[0,\tau] \times \mathbf{V}$.

Lemma 2. If $\eta_0, \eta_1 \in \mathbf{U}(\mathbf{M})$, where $\eta_1(t) = \int_0^t \mathbf{F}\big(u,\eta_0(u)\big)du$, then the Picard sequence $\left\{\eta_k\right\}$ for problem (2,1) is well defined and $\eta_k \in \mathbf{U}(\mathbf{M})$ for $k=0,1,2,\ldots$

Proof. Let $\eta_0, \eta_1, \ldots, \eta_n$, $n \ge 1$, be elements of $\mathbf{U}(\mathbf{M})$. As $\mathbf{U}(\mathbf{M}) \subset \mathcal{G}(\mathbf{M})$ then from Lemma 1 the function $\mathbf{F}(t,\eta_n(t))$ is continuous in $[0,\tau]$ and the function $\int_0^t \mathbf{F}\big(u,\eta_n(u)\big)du$ is defined and continuous on $[0,\tau]$. Thus $\eta_{n+1}(t) = \xi_0 + \int_0^t \mathbf{F}\big(u,\eta_n(u)\big)du$ exists and is continuous. For every $s \in [0,1]$ we have

$$\mathbf{p}_s\big(\eta_{n+1}(t) - \eta_n(t)\big) = \mathbf{p}_s\left\{ \int_0^t \mathbf{F}\big(u,\eta_n(u)\big)du - \int_0^t \mathbf{F}\big(u,\eta_{n-1}(u)\big)du \right\} \le$$

$$\le \int_0^t \mathbf{p}_s\Big[\mathbf{F}\big(u,\eta_n(u)\big) - \mathbf{F}\big(u,\eta_{n-1}(u)\big)\Big] \, du \le$$

$$\le \int_0^t \Big[\int_0^1 \mathbf{M}(s,r)\mathbf{p}_r\big(\eta_n(u) - \eta_{n-1}(u)\big) \, dr \Big] \, du \le$$

$$\le \int_0^t \Big[\int_0^1 \mathbf{M}(s,r)\mathbf{q}_r(\eta_n - \eta_{n-1})dr \Big]du \le \tau \, \mathbf{M} \, \mathbf{q}_s(\eta_n - \eta_{n-1}).$$

Thus $\mathbf{q}_s(\eta_{n+1} - \eta_n) \le \tau\, \mathbf{M}\, \mathbf{q}_s(\eta_n - \eta_{n-1})$. Applying to both sides the operator $\mathbf{M}^n$ for n = 1,2, ... we get

$\mathbf{M}^n\mathbf{q}_s(\eta_{n+1} - \eta_n) \le \tau\, \mathbf{M}^{n+1}\mathbf{q}_s(\eta_n - \eta_{n-1})$ and, as $\eta_n - \eta_{n-1} \in \mathbf{U}(\mathbf{M})$, we have $\eta_{n+1} \in \mathbf{U}(\mathbf{M})$.

Lemma 3. Let $\left\{\eta_n\right\}$, $\left\{\bar{\eta}_n\right\}$, n = 0,1,2, ..., be Picard sequences for the problem (2.1) such that $\left\{\eta_n\right\} \subset \mathbf{U}(\mathbf{M})$ $\left\{\bar{\eta}_n\right\} \subset \mathbf{U}(\mathbf{M})$. Then for each $\mathbf{n}$ we have the following inequality

$$\mathbf{q}_s(\eta_n - \bar{\eta}_n) \le \frac{\tau^n}{n!}\, \mathbf{M}^n\mathbf{q}_s(\eta_0 - \bar{\eta}_0). \tag{3.1}$$

Proof. Denote by $\Delta_j(t)$ the following set (j - dimensional simplex)

$$\Delta_j(t) = \begin{cases} 0 \le t_1 \le t \\ 0 \le t_2 \le t_1 \\ \dots\dots\dots\dots \\ 0 \le t_j \le t_{j-1} \end{cases} \quad t \in [0,\tau].$$

Using the following obvious equality

$$\int_0^t\int_0^{t_1} \dots \int_0^{t_{j-1}} dt_j dt_{j-1} \dots dt_1 = \int_{\Delta_j(t)} dt_j dt_{j-1} \dots dt_1 = = \frac{t^j}{j!}$$

we have for $s \in [0,1]$

$$\mathbf{p}_s\Big(\eta_n(t) - \bar{\eta}_n(t)\Big) = \mathbf{p}_s\left\{\int_0^t\Big[\mathbf{F}\big(u,\eta_{n-1}(u)\big) - \mathbf{F}\big(u,\bar{\eta}_{n-1}(u)\big)\Big]du\right\} \le$$

$$\le \int_0^t\Big[\int_0^1 \mathbf{M}(s,r)\mathbf{p}_r\big(\eta_{n-1}(u) - \bar{\eta}_{n-1}(u)\big)\, dr\Big]\, du \le$$

$$\le \int_0^t\int_0^{t_1}\cdots\int_0^{t_{n-1}}\Big[\int_0^1 \mathbf{M}(s,r_1)\int_0^1 \mathbf{M}(r_1,r_2)\cdots\int_0^1 \mathbf{M}(r_{n-1},r_{n-2})\times$$

$$\times\, \mathbf{p}_{r_n}\Big(\eta_0(t_n)-\bar{\eta}_0(t_n)\Big)\,dr_n dr_{n-1}\ldots dr_1\Big]dt_n dt_{n-1}\ldots dt_1\le$$

$$\le \int_{\Delta_n(t)} dt_n\ldots dt_2 dt_1\, \mathbf{M}^n\mathbf{q}_s(\eta_0-\bar{\eta}_0)\le \frac{\tau^n}{n!}\mathbf{M}^n\mathbf{q}_s(\eta_0-\bar{\eta}_0)$$

and thus $\mathbf{q}_s(\eta_n-\bar{\eta}_n)\le \frac{\tau^n}{n!}\mathbf{M}^n\mathbf{q}_s(\eta_0-\bar{\eta}_0)$.

Corollary . If $\left\{\eta_n\right\}\subset \mathbf{U}(\mathbf{M})$ is a Picard sequence for the problem (2.1), then for all $n\ge 0$ we have the following inequalities

$$\mathbf{q}_s(\eta_{n+m}-\eta_n)\le\sum_{j=n}^{n+m-1}\frac{\tau^j}{j!}\mathbf{M}^j\mathbf{q}_s(\eta_1-\eta_0), \tag{3.2}$$

$$\mathbf{q}_s(\eta_n-\eta_0)\le\sum_{j=0}^{n-1}\frac{\tau^j}{j!}\mathbf{M}^j\mathbf{q}_s(\eta_1-\eta_0), \tag{3.3}$$

$$\mathbf{q}_s(\eta_n)\le\sum_{j=0}^{n-1}\frac{\tau^j}{j!}\mathbf{M}^j\mathbf{q}_s(\eta_1). \tag{3.4}$$

Theorem 1. (the existence of the **P** - solution)
If $\eta_0,\ \eta_1\in\mathcal{S}(\mathbf{e}^{\mathrm{M}\tau})$, where $\eta_1(t)=\xi_0+\int_0^t\mathbf{F}\big(u,\eta_0(u)\big)du$, then there exists a **P** - solution of the problem (2.1).

Proof. From the definition we have

$$\mathcal{S}(\mathbf{e}^{\mathrm{M}\tau})=\left\{\eta\in\mathbf{U}(\mathbf{M})\ :\ \sum_{n=0}^{\infty}\frac{\tau^n}{n!}\mathbf{M}^n\mathbf{q}_s(\eta)<\infty\ ,\ \forall\ s\in[0,1]\right\}.$$

From the assumption $\eta_0,\ \eta_1\in\mathcal{S}(\mathbf{e}^{\mathrm{M}\tau})\subset\mathbf{U}(\mathbf{M})$ and from Lemma 2 the sequence $\left\{\eta_n\right\}$, $n = 0,1,2, \ldots$, is well defined. For $\mathbf{n} \geq 0$, $\mathbf{m} \geq 0$ we have the inequality

$$\mathbf{q}_s(\eta_{n+m} - \eta_n) \leq \sum_{j=n}^{n+m-1} \frac{\tau^j}{j!} \mathbf{M}^j \mathbf{q}_s(\eta_1 - \eta_0).$$

Because $\eta_1 - \eta_0 \in \mathcal{G}(e^{M\tau})$ we have

$$\forall s \in [0,1] \quad \forall \varepsilon_s > 0 \quad \exists \mathbf{n}_0(s,\varepsilon_s) \quad \forall \mathbf{n} > \mathbf{n}_0 \quad \forall \mathbf{m} \in \mathbf{N}$$

$$\sum_{j=n}^{n+m-1} \frac{\tau^j}{j!} \mathbf{M}^j \mathbf{q}_s(\eta_1 - \eta_0) < \varepsilon_s \quad \text{and} \quad \mathbf{q}_s(\eta_{n+m} - \eta_n) \leq \varepsilon_s$$

for $\mathbf{n} > \mathbf{n}_0(s,\varepsilon_s)$, $\mathbf{m} \in \mathbf{N}$. Thus the sequence $\left\{\eta_n\right\}$ is a Cauchy sequence in $\mathcal{F}$. As the space $\mathcal{F}$ is sequentially complete , there exists a function $\eta \in \mathcal{F}$ such that $\lim\limits_{n\to\infty} \eta_n = \eta$ which is the P - solution of the considered problem, according to the definition.

Theorem 2. (the uniqueness of the **P** - solution) Let $\left\{\eta_n\right\}$ and $\left\{\bar{\eta}_n\right\}$, $n = 0,1,2, \ldots$, be two Picard sequences for the Cauchy problem (2.1), $\left\{\eta_n\right\} \subset \mathbf{U}(\mathbf{M})$, $\left\{\bar{\eta}_n\right\} \subset \mathbf{U}(\mathbf{M})$. If $\lim\limits_{n\to\infty} \eta_n = \eta$ and $(\eta_0 - \bar{\eta}_0) \subset \mathbf{Z}(\mathbf{M}\tau)$, then $\lim\limits_{n\to\infty} \bar{\eta}_n = \eta$.

Proof. By definition of Z(M), there is

$$\mathbf{Z}(\mathbf{M}) = \left\{ \eta \in \mathbf{U} : \frac{\tau^n}{n!} \mathbf{M}^n \mathbf{q}_s(\eta) \longrightarrow 0 \ \text{ if } n \longrightarrow 0 \right\} .$$

We have $\mathbf{q}_s(\bar{\eta}_n - \eta) = \mathbf{q}_s(\bar{\eta}_n - \eta_n + \eta_n - \eta) \leq \mathbf{q}_s(\bar{\eta}_n - \eta_n) + \mathbf{q}_s(\eta_n - \eta)$.

Using the inequality (3.3) we get

$$\mathbf{q}_s(\bar{\eta}_n - \eta) \le \frac{\tau^n}{n!}\mathbf{M}^n\mathbf{q}_s(\bar{\eta}_0 - \eta_0) + \mathbf{q}_s(\eta_n - \eta).$$

From the assumption that $\lim\limits_{n\to\infty} \eta_n = \eta$ and $(\bar{\eta}_0 - \eta_0) \subset \mathbf{Z}(\mathbf{M}\tau)$ we conclude that the right - hand side of the above inequality converges to $\mathbf{0}$ with $n \longrightarrow 0$ and thus $\lim\limits_{n\to\infty} \bar{\eta}_n = \eta$ which proves the uniqueness of the $\mathbf{P}$ - solution.

The next two theorems deal with the existence and uniqueness of a classical solution.

Theorem 3. (the existence of the classical solution) If $\eta_0, \eta_1 \in \mathcal{G}(\mathbf{M}e^{\mathbf{M}\tau})$, where $\eta_1(t) = \xi_0 + \int_0^t \mathbf{F}\big(u, \eta_0(u)\big)du$, $t \in [0,\tau]$, then, there exists in the set $\mathcal{G}(\mathbf{M})$ a classical solution η of problem (2.1) which is a limit in $\mathcal{F}$ of a Picard sequence $\left\{\eta_n\right\}$, $n = 0,1,2, \ldots$, and for this solution

$$(3.5) \qquad \mathbf{q}_s(\eta - \eta_0) \le \mathbf{e}^{\mathbf{M}\tau}\, \mathbf{q}_s(\eta_1 - \eta_0).$$

Proof. By assumption, $\eta_0, \eta_1 \in \mathcal{G}(\mathbf{M}e^{\mathbf{M}\tau}) \subset \mathcal{G}(\mathbf{e}^{\mathbf{M}\tau}) \subset \mathbf{U}(\mathbf{M})$. From Theorem 1 there exists the $\mathbf{P}$ - solution η of the problem (2.1) such that $\lim\limits_{n\to\infty} \eta_n = \eta$. Applying the inequality (3.3) we have

$$\mathbf{q}_s(\eta_n - \eta_0) \le \sum_{j=0}^{n-1} \frac{\tau^j}{j!}\mathbf{M}^j\mathbf{q}_s(\eta_1 - \eta_0)$$

and passing in this inequality to the limit with $n \longrightarrow \infty$ we get

$$\mathbf{q}_s(\eta - \eta_0) \le \sum_{j=0}^{\infty} \frac{\tau^j}{j!} \mathbf{M}^j \mathbf{q}_s(\eta_1 - \eta_0) = \mathbf{e}^{\mathbf{M}\tau} \mathbf{q}_s(\eta_1 - \eta_0).$$

From this inequality we have

$$\mathbf{q}_s(\eta) \le \mathbf{q}_s(\eta_0) + \mathbf{e}^{\mathbf{M}\tau} \mathbf{q}_s(\eta_1 - \eta_0). \tag{3.6}$$

Applying the operator $\mathbf{M}$ to both sides of (3.6) we get $\mathbf{M}\mathbf{q}_s(\eta) \le \mathbf{M}\,\mathbf{q}_s(\eta_0) + \mathbf{M}\,\mathbf{e}^{\mathbf{M}\tau}\mathbf{q}_s(\eta_1 - \eta_0) < \infty$. Thus $\eta \in \mathcal{S}(\mathbf{M})$. Next we shall prove that η is a classical solution . As $\eta \in \mathcal{S}(\mathbf{M})$, from Lemma 1 it follows that $\int_0^t \mathbf{F}\big(u,\eta(u)\big)du$ is defined on the interval $[0,\tau]$ and thus

$$\mathbf{p}_s\Big[\int_0^t \mathbf{F}\big(u,\eta(u)\big)du - \int_0^t \mathbf{F}\big(u,\eta_n(u)\big)du \Big] \le \tau\, \mathbf{M}\, \mathbf{q}_s(\eta - \eta_n). \tag{3.7}$$

Applying to the right - hand side of (3.7), inequality (3.2) with $m = 0$ we get

$$\mathbf{q}_s\Big[\int_0^t \mathbf{F}\big(u,\eta(u)\big)du - \int_0^t \mathbf{F}\big(u,\eta_n(u)\big)du \Big] \le \sum_{j=n}^{\infty} \frac{\tau^{j+1}}{j!} \mathbf{M}^{j+1} \mathbf{q}_s(\eta_1 - \eta_0).$$

Passing to the limit with $n \longrightarrow \infty$ we get

$$\int_0^t \mathbf{F}\big(u,\eta(u)\big)du - \int_0^t \mathbf{F}\big(u,\eta_n(u)\big)du \longrightarrow 0 \text{ with } n \longrightarrow \infty \text{ in } \mathcal{F},$$

$$\int_0^t \mathbf{F}\big(u,\eta_n(u)\big)du \longrightarrow \int_0^t \mathbf{F}\big(u,\eta(u)\big)du \text{ with } n \longrightarrow \infty \text{ in } \mathcal{F}.$$

Finally , as $\eta_{n+1} = \xi_0 + \int_0^t \mathbf{F}\big(u,\eta_n(u)\big)du$, then taking the limit with $n \longrightarrow \infty$ we get $\eta(t) = \xi_0 + \int_0^t \mathbf{F}\big(u,\eta(u)\big)du$, thus η is a classical solution of the problem (2.1) .

Remark 1. The above method gives us the existence of a global solution on $[0,\tau]$ where $\tau > 0$ is an arbitrary real number.

Theorem 4. (The uniqueness of the classical solution). If there exists a classical solution η of the problem (2.1) and $\eta \in \mathbf{Z}(\mathbf{M}\tau)$, then it is unique.

Proof. Let $\eta, \bar{\eta}_n$ be two classical solutions of the problem (2.1) and $\eta, \bar{\eta} \in \mathbf{Z}(\mathbf{M}\tau)$. Let $\eta \neq \bar{\eta}$. This means that there exists $\gamma \neq 0$, $\gamma \in [0,\tau]$, such that $\eta(\gamma) \neq \bar{\eta}(\gamma)$. For a fixed γ we have :

$$\eta(\gamma) - \bar{\eta}(\gamma) = \int_0^{\gamma} \Big[\mathbf{F}\big(u,\eta(u)\big) - \mathbf{F}\big(u,\bar{\eta}(u)\big) \Big]\, du \ ,$$

$$\mathbf{p}_s\big(\eta(\gamma) - \bar{\eta}(\gamma)\big) \le \int_0^{\gamma} \Big[\int_0^1 \mathbf{M}(s,r)\,\mathbf{p}_r\big(\eta(u) - \bar{\eta}(u)\big)\,dr \Big] du =$$

$$= \int_0^{\gamma} \Big[\mathbf{M}\, \mathbf{p}_s\big(\eta(u) - \bar{\eta}(u)\big) \Big] du \le \int_0^{\gamma} \int_0^{\Omega} \mathbf{M}^2 \mathbf{p}_s\big(\eta(\gamma_2) - \bar{\eta}(\gamma_2)\big)\, d\gamma_2\, d\gamma_1 \le$$

$$\dots \le \int_{\Delta_n(\gamma)} \mathbf{M}^n \mathbf{p}_s\big(\eta(\gamma_n) - \bar{\eta}(\gamma_n)\big)\, d\gamma_n \dots d\gamma_2\, d\gamma_1 \le$$

$$\le \frac{\gamma^n}{n!} \mathbf{M}^n \mathbf{q}_s(\eta - \bar{\eta}) \le \frac{\tau^n}{n!} \mathbf{M}^n \mathbf{q}_s(\eta - \bar{\eta}_n).$$

Finally we have

$$(3.8) \qquad 0 \le \mathbf{p}_s\big(\eta(\gamma) - \bar{\eta}(\gamma)\big) \le \frac{\tau^n}{n!} \mathbf{M}^n \mathbf{q}_s(\eta - \bar{\eta}_0).$$

By the assumption, $\bar{\eta} - \eta \in \mathbf{Z}(\mathbf{M}\tau)$, thus the right - hand side of (3.8) converges to $\mathbf{0}$ with $n \longrightarrow \infty$. . We get $\bar{\eta}(\gamma) = \eta(\gamma)$, which contradicts the earlier assumption that γ is a point at which the solutions are different.

REFERENCES

[1] A. Alexiewicz : Functional Analysis, Warsaw, 1969. (in Polish).

[2] R. Engelking : General Topology Outline , Warsaw , 1965. (in Polish).

[3] A. I. Kolmogorov, S. W. Fomin : Function Theory Elements and Functional Analysis, Moscow, 1976. (in Russian).

[4] K. Moszyński, A. Pokrzywa : Sur les systèmes infinis d'équations différrentielles ordinaires dans certains espaces de Fréchet , Dissertationes Math., 115 (1974).

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REFERENCES

[1] A. Alexiewicz : Functional Analysis, Warsaw, 1969 (in Polish).

[2] R. Engelking : General Topology Outline , Warsaw 1969 (in Polish).

[3] A. N. Kolmogorov, S. V. Fomin : Function Theory Elements and Functional Analysis, Moscow, 1976 (in Russian).

[4] K. Moszyński, A. Pokrzywa : Sur les systèmes infinis d'équations différentielles ordinaires dans certains espaces de Fréchet, Dissertationes Math., 115 (1974).

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SOME RELATIONS BETWEEN ALMOST SYMPLECTIC, PSEUDO-RIEMANNIAN AND ALMOST PRODUCT STRUCTURES ON DIFFERENTIAL SPACES

Dedicated to the memory
of Professor Edward Otto

In this paper some natural relations between almost product and pseudo-Riemannian structures on a differential space in the sense of Sikorski [3] are studied. One has showed that only pseudo-Riemannian metric g, with signature (n,n), and almost symplectic form ω may be mutually t-compactible on 2n-dimensional differential space, where t is an almost product structure on (M,C) of dimension (n,n) solely possible. One has showed also, that such three structures mutually connected may be given equivalently by some tensor field T on (M,C) of type (2,0).

0. Let C be a non-empty set of real functions defined on a set M. We will denote by τ_C the weakest topology on M such that all function in C are continuous. For any subset A of M we denote by C_A the set of all real functions f defined on A and such that for every point $p \in A$ there exists a neighbourhood $U \in \tau$ of p and a function $g \in C$ such that $f|A \cap U = g|A \cap U$.

The set C is said to be a differential structure on M provided that

(a) the set C is closed with respect to localization, i.e., $C = C_M$,

(b) the set C is closed with respect to composition with smooth functions, i.e., if $\alpha_1, \dots, \alpha_n \in C$ and ω is smooth real function defined on the n-dimensional Euclidean space R^n, then $\omega \circ (\alpha_1, \dots, \alpha_n) \in C$.

By a differential space we shall mean any pair (M,C), where M is a set and C is a differential structure on M [3]. If (M,C) is a differential space and A is an arbitrary non-empty subset of M then (A, C_A) is also a differential space. By a vector tangent to (M,C) at a point $p \in M$ we shall mean any linear mapping $v\colon C \longrightarrow R$ such that

$$v(\alpha \cdot \beta) = v(\alpha) \cdot \beta(p) + \alpha(p) \cdot v(\beta) \quad \text{for all } \alpha, \beta \in C.$$

We will denote by T_pM the set of all tangent vectors at a given point $p \in M$ and by TM the disjoint sum of all linear spaces T_pM, $p \in M$.

Let (M,C) and (N,D) be differential spaces. A mapping $f\colon M \longrightarrow N$ is said to be smooth provided $\alpha \circ f \in C$ for every $\alpha \in D$.

By $\mathfrak{X}(M)$ we denote the C-module of all smooth vector fields tangent to (M,C). We say that a differential space (M,C) has differential dimension n if $\dim T_pM = n$ for any point $p \in M$ and for any point $p \in M$ there exists an open set $U \ni p$ in τ_C and C_U-basis $X_1, \dots, X_n \in \mathfrak{X}(U)$ of C_U-module $\mathfrak{X}(U)$.

1. Let (M,C) be a differential space of constant

differential dimension n and t be a smooth tensor field of type (1,1). For any $x \in M$ let

$$P_x = \frac{1}{2}(\delta_x + t_x) \quad \text{and} \quad Q_x = \frac{1}{2}(\delta_x - t_x), \tag{1}$$

where δ is a unit tensor field on (M,C), this means $\delta_x = \delta | T_xM = \mathrm{id}_{T_xM}$. Next let

$$W_x = \mathrm{Im}\, P_x \quad \text{and} \quad U_x = \mathrm{Im}\, Q_x \quad \text{for } x \in M, \tag{2}$$

as well as let

$$W = \bigcup_{x \in M} W_x \quad \text{and} \quad U = \bigcup_{x \in M} U_x. \tag{3}$$

From (1) we get

$$P_x + Q_x = \delta_x \quad \text{and} \quad P_x - Q_x = t_x \quad \text{for any } x \in M. \tag{4}$$

Identically as in the theory of differentiable manifolds, a smooth tensor field t on M of type (1,1) is called an almost product structure on M iff $t^2 = \delta$.

One can easily prove

Proposition 1. Let t be a smooth tensor field on M of type (1,1). Then the following conditions are equivalent:

(i) t is an almost product structure on M,

(ii) $P_x^2 = P_x$ (resp. $Q_x^2 = Q_x$) for any $x \in M$,

(iii) $P_x \circ Q_x = Q_x \circ P_x = 0_x$ for any $x \in M$,

(iv) $t_x \circ P_x = P_x$ (resp. $t_x \circ Q_x = -Q_x$) for any $x \in M$.

(v) $\mathrm{Im}\, P_x = \ker Q_x$ (resp. $\mathrm{Im}\, Q_x = \ker P_x$) for any $x \in M$.

Let us observe that if t is an almost product structure on M then for any $x \in M$ $T_xM = W_x \oplus U_x$ as well as $TM = W \oplus U$, this means that TM is the Whitney sum of subbundle W and U defined in

(3).

Let t be an almost product structure on M and let $\dim W_x = r$ and $\dim U_x = s$ for $x \in M$. Then evidently $r + s = \dim T_xM$. The pair (r,s) is called the dimension of t. Of course if $t=0$ or $t=-\delta$ then t is called a trivial almost product structure on M.

2. Let now $\dim M = 2n$ and Φ be a smooth tensor field on M of type $(2,0)$. Assume that Φ is a non-degenerate, symmetric or skew-symmetric tensor field. Hence by assumption Φ is a pseudo-Riemannian metric on M if Φ is symmetric, and Φ is an almost symplectic form on M if Φ is skew-symmetric.

Now assume that on M is given an almost product structure t. An almost product structure t on M is said to be a Φ-symmetric almost product on M if .

$$\Phi(t_x(v),u) = \Phi(v,t_x(u)) \tag{5}$$

for any $x \in M$ and $v,u \in T_xM$, and t is said to be a Φ-skew-symmetric almost product structure on M if

$$\Phi(t_x(v),u) = -\Phi(v,t_x(u)) \tag{6}$$

for any $x \in M$ and $v,u \in T_xM$.

Evidently (5) is equivalent to

$$\Phi(v,u) = \Phi(t_x(v),t_x(u))$$

for any $x \in M$ and $v,u \in T_xM$. Therefore, in this case, one says that Φ is compactible with t.

It is easy to prove

Lemma 2. An almost product structure t on M is Φ-symmetric iff

(7) $$\Phi(P_x(v),u) = \Phi(v,P_x(u))$$

and t is Φ-skew-symmetric iff

(8) $$\Phi(v,u) = \Phi(P_x(v),u) + \Phi(v,P_x(u))$$

for any $x \in M$ and $v,u \in T_xM$, where $P = \frac{1}{2}(\delta + t)$.

Observe that Lemma 2 holds if instead of P in (7) and (8) we put Q, where $Q = \frac{1}{2}(\delta - t)$.

3. Now let ω be an almost symplectic form on a differential space (M,C). Then the pair ((M,C),ω) we will call an almost symplectic differential space.

So let (M,ω) be an almost symplectic differential space and let $TM = W \oplus U$ be the Whitney sum of ω-Lagrangian subbundles W and U, this means that $\omega(v,u) = 0$ for any $v,u \in W_x$ or $v,u \in U_x$ and $x \in M$. Denote by

$$P\colon TM \longrightarrow W \quad \text{and} \quad Q\colon TM \longrightarrow U$$

the projections onto W and U respectively, parallel to the complementary subbundles. By straightforward calculations one can show the following identities

(9) $$\omega(v,u) = \omega(P_x(v),u) + \omega(v,P_x(u))$$

(9') $$\omega(v,u) = \omega(Q_x(v),u) + \omega(v,Q_x(u))$$

for any $x \in M$ and $v,u \in T_xM$. Hence $t = P - Q$ is an ω-skew symmetric almost product structure on M.

4. Let now g be a pseudo-Riemanian metric with the signature (n,n) on a differential space (M,C). Then the pair ((M,C),g) will be called a pseudo-Riemannian differential space with the

signature (n,n).

So let (M,g) be a pseudo-Riemannian differential space with the signature (n,n) and let $TM = W \oplus U$ be the Whitney sum of g-isotropic subbundles W and U, this means that $g(v,u) = 0$ for any $v,u \in W_x$ or $v,u \in U_x$, $x \in M$. Next denote by

$$P : TM \longrightarrow W \quad \text{and} \quad Q : TM \longrightarrow U$$

the projections onto W and U respectively, parallel to the complementary subbundles. By straightforward calculations one can show the following identities:

$$g(v,u) = g(P_x(v),u) + g(v,P_x(u)) \tag{10}$$

$$g(v,u) = g(Q_x(v),u) + g(v,Q_x(u)) \tag{10'}$$

for any $x \in M$ and $v,u \in T_xM$. Hence $t = P - Q$ is a g-skew-symmetric almost product structure on M.

5. Let as the above Φ be a nondegerate symmetric or skew-symmetric tensor field on M of type (2,0), and let t be a tensor field on M of type (1,1) of the maximal rank $2n = \dim M$. Put

$$\Psi(v,u) = \Phi(t_x(v),u) \tag{11}$$

for any $x \in M$ and $v,u \in T_xM$. From (11) it follows that Ψ is a smooth nondegenerate tensor field on M of type (2,0).

Definition 1. A tensor field Ψ on M, defined by (11), is said to be t-compactible with Φ if

$$\Phi(v,u) = \Psi(t_x(v),u) \tag{12}$$

for any $x \in M$ and $v,u \in T_xM$.

From (11) and (12) it follows immediately

Corollary 3. A tensor field Ψ on M, defined by

(11), is t-compactible with Φ iff $t^2 = \delta$, this means, iff t is an almost product structure on M.

Using (5) and (6) one can easily prove

Proposition 4. Let Ψ be a t-compactible with Φ and let t be Φ-symmetric. Then Ψ is a symmetric (skew-symmetric) tensor field on M iff Φ is a symmetric (resp. skew-symmetric) tensor field on M.

Proposition 5. Let Ψ be a t-compactible with Φ and let t be a Φ-skew-symmetric. Then Ψ is a symmetric (skew-symmetric) tensor field on M iff Φ is a skew-symmetric (resp. symmetric) tensor field on M.

Observe that putting in (11) $t = 2P - \delta$ we get the equivalent identity

$$\Psi(v,u) = 2\Phi(P_x(v),u) - \Phi(v,u). \tag{11}$$

Next putting in (11) $t = \delta - 2Q$ we get the equivalent identity

$$\Psi(v,u) = -2\Phi(Q_x(v),u) + \Phi(v,u) \tag{11'}$$

for any $x \in M$ and $v,u \in T_xM$.

Using now Lemma 2, (8) and (4) one can prove

Lemma 6. An almost product structure t on M is Φ-skew-symmetric iff the subbundles W and U of TM, defined by (3), are Φ-isotropic.

Proof. Assume that t is Φ-skew-symmetric almost product structure on M. Hence by definition

$$\Phi(t_x(v),u) = -\Phi(v,t_x(u))$$

or equivalently by (8)

$$\Phi(v,u) = \Phi(P_x(v),u) + \Phi(v,P_x(u)) \tag{12}$$

for any $x \in M$ and $v,u \in T_xM$. Putting now in (12) $v,u \in W_x$, $x \in M$ we get

$\Phi(v,u) = 0$. Next putting in (12) instead of P, $\delta - Q$ and $v,u \in U_x$ we get $\Phi(v,u) = 0$, because $P^2 = P$ and $Q^2 = Q$. Hence W and U are Φ-isotropic subbundles of TM.

Conversely, let t be an almost product structure on M and let $TM = W \oplus U$ be the Whitney sum of Φ-isotropic subbundles W and U. Now for an arbitrary $x \in M$ and $v,u \in T_xM$ we have the identity

$$\Phi(v,u) = \Phi(P_x(v),P_x(u)) + \Phi(P_x(v),Q_x(u)) +$$
$$+ \Phi(Q_x(v),P_x(u)) + \Phi(Q_x(v),Q_x(u)).$$

Hence, by assumption and because $P_x(v) \in W_x$ and $Q_x(v) \in U_x$ for any $v \in T_xM$, we get

$$\Phi(v,u) = \Phi(P_x(v),Q_x(u)) + \Phi(Q_x(v),P_x(u)). \tag{13}$$

Putting in (13) $Q = \delta - P$ we get

$$\Phi(v,u) = \Phi(P_x(v),u) + \Phi(v,P_x(u))$$

or equivalently

$$\Phi(t_x(v),u) = -\Phi(v,t_x(u)) \quad \text{for any } x \in M \text{ and } v,u \in T_xM.$$

From (11) we get immediately

Corollary 7. If tensor fields Ψ and Φ are t-compactible and t is Φ-skew-symmetric almost product structure on M, then the subbundles W and U of TM are Ψ and Φ-isotropic at the same time. Moreover t has dimension (n,n), where $\dim M = 2n$.

The above considerations lead to the following

Corollary 8. Let (M,ω) be an almost symplectic differential space of constant differential dimension $2n$. Then only a pseudo-Riemannian metric g with the signature (n,n) may be t-compactible with ω. Furthermore t is then ω-skew-symmetric almost product structure on M of dimension (n,n).

Corollary 9. Let (M,g) be a pseudo-Riemannian differential space of constant differential dimension 2n, of the signature (n,n). Then only an almost symplectic structure ω on M may be t-compactible with g. Furthermore t is then g-skew-symmetric almost product structure on M of dimension (n,n).

6. Let now (M,ω) be an almost symplectic differential space of constant differential dimension 2n and let t be ω-skew-symmetric almost product structure on M. Let us put

$$T(v,u) = 2\omega(P_x(v),u) \tag{14}$$

for any $x \in M$ and $v,u \in T_xM$, where $P = \frac{1}{2}(\delta + t)$. By definition T is a smooth tensor field on M of type (2,0). We will prove

Proposition 10. A skew-symmetric part of tensor field T, defined by (14) coincide with ω, this means that

$$\omega(v,u) = \frac{1}{2}(T(v,u) - T(u,v)) \tag{15}$$

for any $v,u \in T_xM$ and $x \in M$, whereas a symmetric part of T is a pseudo-Riemannian metric g on M with the signature (n,n), this means that

$$g(v,u) = \frac{1}{2}(T(v,u) + T(u,v)) \tag{16}$$

for any $v,u \in T_xM$ and $x \in M$.

Proof. From (14) and (9) for $\omega = \Phi$ we get

$$T(v,u) - T(u,v) = 2\omega(P_x(v),u) - 2\omega(P_x(u),v) =$$
$$= 2(\omega(P_x(v),u) + \omega(v,P_x(u))) = 2\omega(v,u)$$

for any $x \in M$ and $v,u \in T_xM$. Hence we get (15).

Similarly, from (14) and (11) for $\omega = \Phi$ we get

$$T(v,u) + T(u,v) = 2\omega(P_x(v),u) + 2\omega(P_x(u),v) =$$
$$= 2\omega(P_x(v),u) - \omega(v,u) + 2\omega(P_x(u),v) - \omega(u,v) =$$
$$= g(v,u) + g(u,v) = 2g(v,u)$$

for any $x \in M$ and $v,u \in T_xM$. Hence we get (16).

Observe that we have the identity

$$g(v,u) = \omega(t_x(v),u)$$

what shows that g and ω are t-compactible.

7. Let now (M,g) be a pseudo-Riemannian differential space of constant differential dimension $2n$ and of signature (n,n). Assume that t is g-skew-symmetric almost product structure on M. Putting

$$\tilde{T}(v,u) = 2g(P_x(v),u), \tag{17}$$

for any $x \in M$ and $v,u \in T_xM$, we define a tensor field $\tilde{T}$ on M of type $(2,0)$.

Similarly as Proposition 10 we prove

Proposition 11. The symmetric part of tensor field $\tilde{T}$, defined by (17), coincide with g, this means that

$$g(v,u) = \frac{1}{2}\,(\tilde{T}(v,u) + \tilde{T}(u,v)), \tag{18}$$

for any $x \in M$ and $v,u \in T_xM$, whereas a skew-symmetric part of $\tilde{T}$ is an almost symplectic form on M. This means that

$$\omega(v,u) = \frac{1}{2}\,(\tilde{T}(v,u) - \tilde{T}(u,v))$$

for any $x \in M$ and $v,u \in T_xM$. Moreover g and ω are t-compactible, this means that $\omega(v,u) = g(t_x(v),u)$ for any $x \in M$ and $v,u \in T_xM$.

Observe now that we have also the identities

$$\tilde{T}(v,u) = 2g(P_x(v),u) = 2g(P_x(v),u) - g(v,u) + g(v,u) =$$

$$= \omega(v,u) + g(v,u) = \omega(v,u) + 2\omega(P_x(v),u) - \omega(v,u) =$$

$$= 2\omega(P_x(v),u) = T(v,u)$$

for any $x \in M$ and $v,u \in T_xM$. Hence $\tilde{T} = T$.

8. Let (M,C) be a differential space of constant differential dimension $2n$, and let T be a smooth tensor field on M of type $(2,0)$.

Definition 2. A smooth tensor field T on M of type $(2,0)$ is said to be an SRP-structure on M, if the following conditions are satisfied:

(i) rank $T_x = n$ for any $x \in M$,

(ii) rank $A(T_x) = 2n$ for any $x \in M$,

where $A(T)$ denotes a skew-symmetric part of T.

If T is a SRP-structure on a differential space M, then the pair (M,T) is called SRP-differential space.

From Definition 2 it follows immediately

Corollary 12. If (M,T) is an SRP-differential space, then $A(T)$ is an almost symplectic form on M.

It is easy to observe that if (M,T) is a SRP-differential space then there exists a smooth tensor field P on M of type $(1,1)$ such that

(19) $$T(v,u) = 2\omega(P_x(v),u)$$

for any $x \in M$ and $v,u \in T_xM$, where $\omega = A(T)$.

From (19) it follows that rank $P_x = n$ for any $x \in M$. Now we will prove

Proposition 13. The tensor field P on M of type $(1,1)$, defined by (19) is a projection of TM onto subbundle W parallel to complementary subbundle U, where $W = \bigcup_{x\in M} W_x$ and $U = \bigcup_{x\in M} U_x$, $W_x = \operatorname{Im} P_x$ and $U_x = \operatorname{Ker} P_x$ for $x \in M$.

Proof. From (19) it follows immediately that rank $P_x = n$ for any $x \in M$ and

$$(20) \qquad \omega(v,u) = \omega(P_x(v),u) + \omega(v,P_x(u))$$

for any $x \in M$ and $v,u \in T_xM$. Moreover $\dim W_x = \dim U_x = n$. It is easy to observe that the subbundle U of TM is ω-Lagrangian.

Let $TM = \overline{W} \oplus U$ be the Whitney sum of ω-Lagrangian subbundles, and let $P\colon TM \longrightarrow W$ be a projection. Now it is easy to show the identity

$$(21) \qquad \omega(v,u) = \omega(\overline{P}_x(v),u) + \omega(v,P_x(u))$$

for any $x \in M$ and $v,u \in T_xM$.

From (20) and (21) we get the identity

$$\omega((P_x - \overline{P}_x)(v),u) + \omega(v,(P_x - \overline{P}_x)(u)) = 0$$

for any $x \in M$ and $v,u \in T_xM$. Hence, for any $x \in M$, $u \in U_x$ and $v \in T_xM$ we get the identity

$$(22) \qquad \omega((P_x - \overline{P}_x)(v),u) = 0.$$

From (22) it follows that for any $x \in M$ and $v \in T_xM$

$$(P_x - \overline{P}_x)(v) \in U_x.$$

Hence for any $x \in M$ $P_x^2 = P_x \circ \overline{P}_x$ and consequently $P_x^2 = P_x$, this means that $P\colon TM \longrightarrow W$ is a projection onto subbundle W parallel to complementary subbundle U. Moreover W and U are then $\omega = A(T)$-Lagrangian subbundles.

From Proposition 13 it follows immediately

Corollary 14. On every SRP-differential space (M,T) there exists an almost product structure $t = 2P - \delta$, where P is a projection of TM onto A(T)-Lagrangian subbundle W parallel to the complementary A(T)-Lagrangian subbundle U. Moreover from (20) it follows that t is ω-skew-symmetric.

Now let us put

(23) $$g(v,u) = \omega(t_x(v),u)$$

for any $x \in M$ and $v,u \in T_xM$. From (23) by the above corollary and Corollary 3 we get

Corollary 15. A tensor field g on M, defined by (23), where t is the natural almost product structure on M, determined by T, is a pseudo-Riemannian metric on M of signature (n,n), and g is t-compactible with ω.

Using (19) and (23) one can show that

$$g(v,u) = (S(T))(v,u) = \frac{1}{2}(T(v,u) + T(u,v))$$

for any $x \in M$ and $v,u \in T_xM$.

Recapitulating we get

Proposition 16. Let (M,T) be SRP-differential space of constant differential dimension 2n. Then:

(i) A(T) is an almost symplectic structure on M,

(ii) S(T) is a pseudo-Riemannian metric on M of signature (n,n),

(iii) there exist on M the natural almost product structure t of dimension (n,n) such that ω and g are t-compactible,

(iv) the almost product structure t on, induced by T, determines a decomposition of tangent bundle TM into the Whitney sum of ω-Lagrangian subbundles W and U, which are g-isotropic too.

REFERENCES

[1] R. Abraham, J. E. Marsden: Foundations of Mechanics, 1967 Benjamin, New York 1967.

[2] S. Kobayashi, K. Nomizu: Foundations of Differential Geometry, vol. I, New York - London 1963, vol. II, New York - London - Sydney 1969.

[3] R. Sikorski: Introduction to Differential Geometry, PWN Warszawa 1972 (in Polish).

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ON CURVATURE STRUCTURES AND PROJECTIONS

Dedicated to the memory
of Professor Edward Otto

The underlying paper continues and extends investigations of [1] and [2] where relations between certain tensors (curvature tensors) of type (4,0) and families of projections have been studied. It deals with curvature tensors and more general with curvature structures (fields of curvature tensors) of type (2p,0) and projections generated by them and gives some applications to differential geometry.

1. Throughout the paper let V be a linear space of finite dimension $m > 1$ and p be a natural number with $2 \le p \le m$. For skew-symmetric p-forms ω on V one can easily prove the following

Theorem 1. If $a_1,\dots,a_p,b \in V$ and $\bar{a}_i = \alpha_i^k a_k$ 1) $i \in \{1,\dots,p\}$, then

$$\omega(\bar{a}_1,\dots,\bar{a}_p) = \det(\alpha_i^k)\,\omega(a_1,\dots,a_p)$$

and

---

1) We use the Einstein sum convention.

$$\sum_{i=1}^{p} \omega(\bar{a}_1,\dots,\bar{a}_{i-1},b,\bar{a}_{i+1},\dots,\bar{a}_p)\bar{a}_i =$$

$$\det(\alpha_i^k) \sum_{i=1}^{p} \omega(a_1,\dots,a_{i-1},b,a_{i+1},\dots,a_p)a_i \, .$$

2. Let $\alpha^p(V)$ be the set of all tensors A on V of type (2p,0) such that for every $a_1,\dots,a_p,b_1,\dots,b_p \in V$ and every $i,j \in \{1,\dots,p\}$ with $i \neq j$

(1) $\quad A(a_1,\dots,a_p,b_1,\dots,b_p) = A(b_1,\dots,b_p,a_1,\dots,a_p)$

and

(2) $$A(\dots,a_i,\dots,a_j,\dots,b_1,\dots,b_p) =$$
$$= -A(\dots,a_j,\dots,a_i,\dots,b_1,\dots,b_p)$$

are fulfilled. Let $\Lambda^p(V)$ denote the linear space of all p-vectors on V. There exists a one-to-one correspondence between $\alpha^p(V)$ and the set of all symmetric bilinear forms $\mathring{A}$ on $\Lambda^p(V)$ which is defined by

$$A(a_1,\dots,a_p,b_1,\dots,b_p) = \mathring{A}(a_1\wedge\dots\wedge a_p,b_1\wedge\dots\wedge b_p).$$

For a given tensor $A \in \alpha^p(V)$, $\mathring{A}$ will always denote this symmetric bilinear form on $\Lambda^p(V)$.

3. A bilinear form ν on a linear space is said to be nondegenerate if the following holds: If $\nu(a,b) = 0$ for every element a of the linear space, then b = 0. A tensor $A \in \alpha^p(V)$ is said to be nondegenerate if $\mathring{A}$ is nondegenerate.

Theorem 2. For $A \in \alpha^p(V)$ the following are equivalent:

1. A is nondegenerate.

2. If for every $a_1\wedge\ldots\wedge a_p\in \Lambda^p(V)$, $\mathring{A}(a_1\wedge\ldots\wedge a_p,\mu) = 0$, then $\mu = 0$.

Let $\{e_1,\ldots,e_m\}$ be a basis of V and let I denote the set of all p-tuples $(i_1,\ldots,i_p)$ with $i_1,\ldots,i_p\in \{1,\ldots,m\}$ and $i_1<\ldots< i_p$. For every $\iota = (j_1,\ldots,j_p)$, $\kappa = (k_1,\ldots,k_p)$ of I let $e_\iota = e_{j_1}\wedge\ldots\wedge e_{j_p}$ and $A_{\iota\kappa} = A(e_{j_1},\ldots,e_{j_p},e_{k_1},\ldots,e_{k_p})$.
$\{e_\iota;\ \iota\in I\}$ is a basis of $\Lambda^p(V)$ and the numbers $A_{\iota\kappa}$ are the coordinates of A with respect to $\{e_1,\ldots,e_m\}$. Obviously

$$\mathring{A}(\lambda^\iota e_\iota,\mu^\kappa e_\kappa) = \lambda^\iota\mu^\kappa A_{\iota\kappa}.$$

Corollary. The assertions of Theorem 2 are equivalent to every one of the following assertions:

3. If for every $\iota\in I$, $A_{\iota\kappa}\mu^\kappa = 0$, then $\mu = \mu^\kappa e_\kappa = 0$.

4. $\det(A_{\iota\kappa})\neq 0$.

The Corollary is independent of the special choice of the basis $\{e_1,\ldots,e_m\}$ of V.

4. Let g be a symmetric bilinear form on V. Then

$$A(a_1,\ldots,a_p,b_1,\ldots,b_p) = \begin{vmatrix} g(a_1,b_1)\ldots g(a_1,b_p) \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ g(a_p,b_1)\ldots g(a_p,b_p)\end{vmatrix}.$$

defines a tensor $A\in\alpha^p(V)$. Each such tensor A is called simple and g is said to generate A.

Theorem 3. Let $A\in\alpha^p(V)$ be simple and g be a generating symmetric bilinear form. A is nondegenerate if and only if g is nondegenerate.

Proof. It is easy to see that there exists a basis $\{e_1,\dots,e_m\}$ of V and an integer n with $0 \le n \le m$ such that

$$g(e_j,e_k) = \begin{cases} \gamma_j \neq 0 & \text{if } j = k \le n \\ 0 & \text{otherwise} \end{cases} .$$

Hence we have

$$A_{\iota\kappa} = \begin{cases} \gamma_{j_1}\dots\gamma_{j_p} \neq 0 & \text{if } \iota = \kappa = (j_1,\dots,j_p) \text{ with } j_p \le n \\ 0 & \text{otherwise} \end{cases}$$

and thus $\det(A_{\iota\kappa}) \neq 0$ if and only if $n = m$. Since g is nondegenerate if and only if $n = m$, the theorem is proved.

Theorem 4. Assume $m \ge p + 1$. Let g, g′ be positive definite symmetric bilinear forms on V, generating tensors A and A′ of $\alpha^p(V)$, respectively. Then $g = \alpha g'$ if and only if $A = \alpha^p A'$.

Proof. It is obvious that $g = \alpha g'$ implies $A = \alpha^p A'$. To prove the converse, take a basis $\{e_1,\dots,e_m\}$ of V with respect to which both g and g′ have the canonical form (cf. [5], p. 273). For arbitrary $i \in \{1,\dots,m\}$ let $\gamma_i = g(e_i,e_i)$ and $\gamma'_i = g'(e_i,e_i)$. If $i, i_1,\dots,i_{p-1}$ are distinct elements of $\{1,\dots,m\}$, then

$$\gamma_i\gamma_{i_1}\dots\gamma_{i_{p-1}} = \alpha^p\gamma'_i\gamma'_{i_1}\dots\gamma'_{i_{p-1}}$$

hence

$$\gamma_i = \alpha^p \; \frac{\gamma'_{i_1}\dots\gamma'_{i_{p-1}}}{\gamma_{i_1}\dots\gamma_{i_{p-1}}}\;\gamma'_i .$$

Because of $m \ge p + 1$ from this it follows that there exists a constant β such that

$$\gamma_i = \beta\gamma'_i$$

for every $i \in \{1,\dots,m\}$ Obviously $\beta = \alpha$ which proves the theorem.

5. Let $A' \in \alpha^q(V)$ with $2 \leq q < p$ and let g be a symmetric bilinear form on V. Then a tensor $A = \overset{p}{A'} \in \alpha^p(V)$ is defined by

$$A(a_1,\dots,a_p,b_1,\dots,b_p) =$$

$$= \sum_{\delta,\theta\in\pi(p)} \operatorname{sign}(\delta)\operatorname{sign}(\theta)\, g(a_{\delta(1)},b_{\theta(1)}\cdots g(a_{\delta(p-q)},b_{\theta(p-q)}) \times$$

$$\times A'(a_{\delta(p-q+1)},\dots,a_{\delta(p)},b_{\theta(p-q+1)},\dots,b_{\theta(p)})$$

where $\pi(p)$ denotes the set of all permutations of $\{1,\dots,p\}$. If A' is simple and generated by g, then also A is simple and generated by $(p!(p-q)!)^{1/p}\, g$.

6. For arbitrary $a_1,\dots,a_p$ let $\operatorname{lin}\{a_1,\dots,a_p\}$ denote the linear space generated by $a_1,\dots,a_p$. If $a_1,\dots,a_p$ are linearly independent, then $\operatorname{lin}\{a_1,\dots,a_p\}$ is called a p-plane.

Let $A \in \alpha^p(V)$. A p-vector $a_1\wedge\dots\wedge a_p \in \Lambda^p(V)$ and a p-plane $\operatorname{lin}\{a_1,\dots,a_p\}$ are said to be A-isotropic if $A(a_1,\dots,a_p,a_1,\dots,a_p) = 0$. From Theorem 1 it follows that this definition is independent of the special representation of the p-vector respectively of the p-plane. Obviously a p-plane is A-isotropic if and only if the restriction of A to this p-plane is degenerate. If A is simple and g is a generating symmetric bilinear form on V, then because of Theorem 3 a p-plane is A-isotropic if and only if the restriction of g to this p-plane is degenerate.

If $a_1\wedge\ldots\wedge a_p$ is A-isotropic and $b_1\wedge\ldots\wedge b_p$ is non-A-isotropic, then $(a_1+\varepsilon b_1)\wedge\ldots\wedge(a_p+\varepsilon b_p)$ for every sufficiently small $\varepsilon\neq 0$ is non-A-isotropic, which follows by means of

$$A(a_1+\varepsilon b_1,\ldots,a_p+\varepsilon b_p,a_1+\varepsilon b_1,\ldots,a_p+\varepsilon b_p) =$$

$$\varepsilon^{2p}A(b_1,\ldots,b_p,b_1,\ldots,b_p)+\varepsilon^{2p-1}(\ldots)+\ldots .$$

7. Let $A\in\alpha^p(V)$. A vector $b\in V$ is said to be A-orthogonal to a p-vector $a_1\wedge\ldots\wedge a_p\in\Lambda^p(V)$ or to a p-plane $\mathrm{lin}\{a_1,\ldots,a_p\}$ if

$$A(b,b_2,\ldots,b_p,a_1,\ldots,a_p) = 0$$

for arbitrary vectors $b_2,\ldots,b_p$ which are linear combinations of the a_i, $i\in\{1,\ldots,m\}$. From Theorem 1 it follows that this definition is independent of the special representation of the p-vector respectively of the p-plane. Obviously b is A-orthogonal to the p-vector $a_1\wedge\ldots\wedge a_p\in\Lambda^p(V)$ or to the p-plane $\mathrm{lin}\{a_1,\ldots,a_p\}$ if and only if

$$A(a_1,\ldots,a_{i-1},b,a_{i+1},\ldots,a_p,a_1,\ldots,a_p) = 0$$

for every $i\in\{1,\ldots,p\}$.

Theorem 5. Let W be a non-A-isotropic p-plane and $W^{\perp}$ be the linear space of all vectors which are A-orthogonal to W. Then $V = W\oplus W^{\perp}$.

Proof. Let $W = \mathrm{lin}\{a_1,\ldots,a_p\}$. Every vector $a = \alpha^j a_j$ of W which belongs to $W^{\perp}$, for arbitrary $i\in\{1,\ldots,p\}$ fulfills

$$A(a_1,\dots,a_{i-1},\alpha^j a_j,a_{i+1},\dots,a_p,a_1,\dots,a_p) = 0$$

hence $\alpha^i = 0$ which proves $W \cap W^\perp = \{0\}$. As is easy to see, for every $a \in V$ there exists exactly one vector a' with $a - a' \in W^\perp$, namely $a' = \alpha^j a_j$ with

$$\alpha^j = \frac{A(a_1,\dots,a_{j-1},a,a_{j+1},\dots,a_p,a_1,\dots,a_p)}{A(a_1,\dots,a_p,a_1,\dots,a_p)}$$

for every $j \in \{1,\dots,p\}$. This proves our theorem.

Theorem 5 becomes incorrect if we replace W by an A-isotropic p-plane. Obviously in this case $W^\perp \supseteq W$. To illustrate what may happen else let us consider two examples, where $m = 3$, $p = 2$ and where A is simple and g denotes a generating symmetric bilinear form. Let $\{e_1,e_2,e_3\}$ be a basis of V and for arbitrary $i,j \in \{1,2,3\}$ let $g_{ij} = g(e_i,e_j)$.

1) Define g by $(g_{ij}) = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$. Then the 2-plane $W = \mathrm{lin}\{e_1,e_2\}$ is A-isotropic and because of

$$A(e_1,\alpha_1 e_1 + \alpha_2 e_2 + \alpha_3 e_3,e_1,e_2) = \alpha_3 A(e_1,e_3,e_1,e_2) =$$

$$= \alpha_3(g_{11}g_{23}g_{12}g_{13}) = \alpha_3,$$

we get $W^\perp = W$.

2) Define g by $(g_{ij}) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$. Then again the 2-plane $W = \mathrm{lin}\{e_1,e_2\}$ is A-isotropic. But because of

$$A(e_3,e_2,e_1,e_2) = g_{13}g_{22} - g_{12}g_{23} = 0$$

and

$$A(e_1,e_3,e_1,e_2) = g_{11}g_{23} - g_{12}g_{13} = 0$$

we have $W^{\perp} = V$.

Theorem 6. Let $A \in \alpha^p(V)$ be simple and g be a generating symmetric bilinear form. Let $W = \mathrm{lin}\{a_1,\dots,a_p\}$ be a non-A-isotropic p-plane. Then a vector $a \in V$ is A-orthogonal to W if and only if a is g-orthogonal to W.

Proof. We may assume that $\{a_1,\dots,a_p\}$ is such that

$$g_{jk} = g(a_j,a_k) = \begin{cases} \gamma_j \neq 0 & \text{if } j = k \leq n \\ 0 & \text{otherwise} \end{cases},$$

where from the non-A-isotropy of W we get $n = p$. For every $i \in \{1,\dots,p\}$ we have

$$A(a_1,\dots,a_{i-1},a,a_{i+1},\dots,a_p,a_1,\dots,a_p) =$$

$$= \begin{vmatrix} \gamma_1 & 0 & \dots & 0 \\ 0 & \gamma_2 & \dots & 0 \\ . & . & \dots & . \\ g(a,a_1) & g(a,a_2) & \dots & g(a,a_p) \\ . & . & \dots & . \\ 0 & 0 & \dots & \gamma_p \end{vmatrix} =$$

$$= \gamma_1 \dots \gamma_{i-1} g(a,a_i)\, \gamma_{i+1} \dots \gamma_p .$$

By means of this the assertion of our theorem becomes obvious.

8. For a given $A \in \alpha^p(V)$ let $W = \mathrm{lin}\{a_1,\dots,a_p\}$ be a non-A-isotropic p-plane in V. By means of

$$(3)\quad \mathrm{pr}_W^A(c) = \sum_{i=1}^{p} \frac{A(a_1,\dots,a_{i-1},c,a_{i+1},\dots,a_p,a_1,\dots,a_p)}{A(a_1,\dots,a_p,a_1,\dots,a_p)}\, a_i$$

we define a mapping $pr_W^A : V \longrightarrow W$. Theorem 1 ensures that pr_W^A is independent of the special representation of W.

Theorem 7. pr_W^A is a projection of V onto W such that for every $c \in V$, $c - pr_W^A(c)$ is A-orthogonal to W.

Proof. Obviously pr_W^A is linear and because of $pr_W^A(a_i) = a_i$, $i \in \{1,\dots,p\}$, pr_W^A maps V onto W. By means of

$$\sum_{i,j=1}^{p} A(a_1,\dots,a_{j-1},a_i,a_{j+1},\dots,a_p,a_1,\dots,a_p) \times$$

$$\times \frac{A(a_1,\dots,a_{i-1},c,a_{i+1},\dots,a_p,a_1,\dots,a_p)}{A(a_1,\dots,a_p,a_1,\dots,a_p)}\, a_j =$$

$$= \sum_{i=1}^{p} \frac{A(a_1,\dots,a_{i-1},c,a_{i+1},\dots,a_p,a_1,\dots,a_p)}{A(a_1,\dots,a_p,a_1,\dots,a_p)}\, a_i$$

we see that $pr_W^A pr_W^A(c) = pr_W^A(c)$ for every $c \in V$. Hence pr_W^A is a projection. For arbitrary $c \in V$ and $i \in \{1,\dots,p\}$

$$A(a_1,\dots,a_{j-1},c - \sum_{j=1}^{p} \frac{A(a_1,\dots,a_{j-1},c,a_{j+1},\dots,a_p,a_1,\dots,a_p)}{A(a_1,\dots,a_p,a_1,\dots,a_p)}\, a_j,$$

$$a_{i+1},\dots,a_p,a_1,\dots,a_p) = 0$$

which shows that $c - pr_W^A(c)$ is A-orthogonal to W.

9. By $\mathfrak{C}^p(V)$ we denote the linear subspace of $\mathfrak{a}^p(V)$ consisting of all tensors A, called curvature tensors, which fulfill the following Bianchi identity

$$(4)\qquad \sum_{j=1}^{p+1} (-1)^j A(a_1,\dots,\hat{a}_j,\dots,a_p,a_{p+1},a_j,b_2,\dots,b_p) \neq 0 \quad {}^{2)}$$

By means of a suitable basis of V it is easy to be seen that each simple tensor of $\alpha^p(V)$ and under the assumption $A' \in \mathfrak{C}^p(V)$ also the tensor $A = \overset{p}{A'}$ of section 5 belong to $\mathfrak{C}^p(V)$. Moreover, curvature structures considered in section 10 are fields of curvature tensors.

Theorem 8. For fixed $A \in \mathfrak{C}^p(V)$ all the assertions

$$E_r \begin{cases} \text{There exists a } (p+r)\text{-tuple } a_1,\dots,a_{p-r},b_{p-r+1},\dots \\ \dots,b_p,c_{p-r+1},\dots,c_p \text{ such that} \\ A(a_1,\dots,a_{p-r},b_{p-r+1},\dots,b_p,a_1,\dots,a_{p-r},c'_{p-r+1},\dots,c_p) = 0 \end{cases}$$

for arbitrary integer r with $0 \le r \le p$ are equivalent.

Remarks. 1. E_o means that there exists a non-A-isotropic p-vector.

2. E_p means that $A \neq 0$.

Proof of Theorem 8. Obviously $E_r \Rightarrow E_{r+1}$ for every r with $0 \le r < p$. Hence we have only to show that $E_r \Rightarrow E_{r-1}$ for every r with $0 < r \le p$. Thus assume that E_r is true. If $r=1$, then by means of

$$A(a_1,\dots,a_{p-1},b+c,a_1,\dots,a_{p-1},b+c) =$$
$$= A(a_1,\dots,a_{p-1},b,a_1,\dots,a_{p-1},b) + A(a_1,\dots,a_{p-1},c,$$
$$a_1,\dots,a_{p-1},c) + 2A(a_1,\dots,a_{p-1},b,a_1,\dots,a_{p-1},c)$$

the validity of E_{r-1} becomes obvious. Now let $r > 1$ and assume

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[2] The symbol ^ in the formula means as usual that the corresponding point a_j is omitted.

E_{r-1} is not fulfilled. Then because of

$$A(\dots,b+c,\dots,b+c,\dots) = A(\dots,b,\dots,b,\dots) + A(\dots,c,\dots,c,\dots) + A(\dots,b,\dots,c,\dots) + A(\dots,c,\dots,b,\dots),$$

for every fixed $a_1,\dots,a_{p-r}$

$$B(+,\dots,+,+,\dots,+) = A(a_1,\dots,a_{p-r},+,\dots,+,a_1,\dots,a_{p-r},+,\dots,+)$$

is skew-symmetric with respect to all arguments. For arbitrary $a_1,\dots,a_{p-r},b_{p-r+1},\dots,b_p,c_{p-r+1},\dots,c_p$ by means of the Bianchi identity we get

$$(-1)^{p-r+1}A(a_1,\dots,a_{p-r},b_{p-r+2},\dots,b_p,c_{p-r+1},b_{p-r+1},c_{p-r+2},\dots,c_p,a_1,\dots,a_{p-r}) + \dots + (-1)^{p}A(a_1,\dots,a_{p-r},b_{p-r+1},\dots,b_{p-1},c_{p-r+1},b_p,c_{p-r+2},\dots,c_p,a_1,\dots,a_{p-r}) + (-1)^{p+1}A(a_1,\dots,a_{p-r},b_{p-r+1},\dots,b_p,c_{p-r+1},\dots,c_p,a_1,\dots,a_{p-r}) = 0$$

hence

$$A(a_1,\dots,a_{p-r},b_{p-r+1},\dots,b_p,a_1,\dots,a_{p-r},c_{p-r+1},\dots,c_p) = 0$$

which contradicts E_r. This proves our theorem.

Theorem 8 cannot be enlarged to arbitrary $A \in \alpha^p(V)$ not fulfilling the Bianchi identity. Let us show this by means of an example where $m=4$ and $p=2$. Let $\{e_1,\dots,e_4\}$ be a basis of V and let us introduce a tensor $A \in \alpha^p(V)$ by defining its coordinates $A_{ijkl} = A(e_i,e_j,e_k,e_l)$ with respect to $\{e_1,\dots,e_4\}$ in the following way. Take

$$A_{1234} = A_{1423} = -A_{1324} = 1$$

and let $A_{ijkl} = 0$ if at least two of the indices i, j, k, l, coincide. It is easy to see that $A(a,b,a,b) = 0$ for arbitrary

$a,b \in V$. Hence for the non-zero tensor A, E_o is not fulfilled.

Theorem 9. For every non-zero tensor $A \in \mathfrak{S}^p(V)$ there exists a basis $\{e_1,\ldots,e_n\}$ of V such that every p-vector e_ι, $\iota \in I$, is non-A-isotropic.

Proof. From Theorem 8 we know that there exists a non-A-isotropic p-vector $e_1\wedge\ldots\wedge e_p$. Hence in the case $m=p$ the theorem is true.

We shall prove the theorem in the case $m > p$ by showing the following. Let n be a natural number with $p < n \le m$ and let V_{n-1} and V_n be linear subspaces of V with $\dim V_{n-1} = n-1$, $\dim V_n = n$ and $V_{n-1} \subseteq V_n$. Let $\{e_1,\ldots,e_{n-1}\}$ be a basis of V_{n-1} such that $e_{i_1}\wedge\ldots\wedge e_{i_p}$ for arbitrary distinct $i_1,\ldots,i_p \in \{1,\ldots,n-1\}$ is non-A-isotropic. Then $\{e_1,\ldots,e_{n-1}\}$ can be enlarged to a basis $\{e_1,\ldots,e_n\}$ of V_n with analogous properties.

For this let e be a point of $V_n \setminus V_{n-1}$. There exist reals $\varepsilon^1,\ldots,\varepsilon^{n-1}$ such that

$$X_{i_1\ldots i_{p-1}} = A(\varepsilon^i e_i, e_{i_1},\ldots,e_{i_{p-1}}, \varepsilon^i e_i, e_{i_1},\ldots,e_{i_{p-1}}) = 0$$

for arbitrary distinct $i_1,\ldots,i_{p-1} \in \{1,\ldots,n-1\}$. Moreover there exists a real $\varepsilon \neq 0$ such that

$$A(\varepsilon^i e_i+\varepsilon e, e_{i_1},\ldots,e_{i_{p-1}}, \varepsilon^i e_i+\varepsilon e, e_{i_1},\ldots,e_{i_{p-1}}) =$$

$$= X_{i_1\ldots i_{p-1}} + (\varepsilon)^2 A(e, e_{i_1},\ldots,e_{i_{p-1}}, e, e_{i_1},\ldots,e_{i_{p-1}}) +$$

$$+ 2\varepsilon\varepsilon^i A(e_i, e_{i_1},\ldots,e_{i_{p-1}}, e, e_{i_1},\ldots,e_{i_{p-1}})$$

for arbitrary distinct $i_1,\dots,i_{p-1}\in\{1,\dots,n-1\}$. Let $e_n=\varepsilon^i e_i+\varepsilon e$. Then $\{e_1,\dots,e_n\}$ is a basis of V_n and obviously we have $A(e_n,e_{i_1},\dots,e_{i_{p-1}},e_n,e_{i_1},\dots,e_{i_{p-1}})\neq 0$ for arbitrary distinct $i_1,\dots,i_{p-1}\in\{1,\dots,n-1\}$. This we wanted to prove.

For every $A\in\mathfrak{C}^p(V)$ by $g^{p,A}(V)$ we denote the set of all non-A-isotropic p-planes of V. $\{\mathrm{pr}^A_W;\ W\in g^{p,A}(V)\}$ is called the family of projections generated by A.

Theorem 10. Two tensors A, $A'\in\mathfrak{C}^p(V)$ generate the same family of projections if and only if there exists a real $\kappa\neq 0$ with $A=\kappa A$.

Proof. Evidently if $A=\kappa A'$ with $\kappa\neq 0$, then A and A′ generate the same family of projections.

Conversely assume that the families of projections generated by A and A′ are equal. Then $g^{p,A}(V)=g^{p,A'}(V)$. For every non-A-isotropic p-vector $d_1\wedge\dots\wedge d_p$ let

$$k(d_1,\dots,d_p)=\frac{A(d_1,\dots,d_p,d_1,\dots,d_p)}{A'(d_1,\dots,d_p,d_1,\dots,d_p)}\ .$$

Let $W=\mathrm{lin}\{a_1,\dots,a_p\}$ and $\mathrm{lin}\{b_1,\dots,b_p\}$ be arbitrary non-A-isotropic p-planes. For every sufficiently small $\varepsilon>o$, $W'=\mathrm{lin}\{a_1,\dots,a_{p-1},c_p\}$ with $c_p=\varepsilon a_p+b_p$ is non-A-isotropic p-plane and moreover we have $A(a_1,\dots,a_{p-1},c_p,a_1,\dots,a_p)\neq 0$ and $A'(a_1,\dots,a_{p-1},c_p,a_1,\dots,a_p)\neq 0$. From $\mathrm{pr}^A_W(c_p)=\mathrm{pr}^{A'}_W(c_p)$ and $\mathrm{pr}^A_{W'}(a_p)=\mathrm{pr}^{A'}_{W'}(a_p)$ we get

$$A(a_1,\dots,a_{p-1},c_p,a_1,\dots,a_p) =$$

$$k(a_1,\dots,a_p)A'(a_1,\dots,a_{p-1},c_p,a_1,\dots,a_p)$$

and

$$A(a_1,\dots,a_p,a_1,\dots,a_{p-1},c_p) =$$

$$k(a_1,\dots,a_{p-1},c_p)A'(a_1,\dots,a_p,a_1,\dots,a_{p-1},c_p)$$

hence

$$k(a_1,\dots,a_p) = k(a_1,\dots,a_{p-1},c_p).$$

By means of analogous considerations, taking $c_i = \varepsilon a_i + b_i$ for every $i \in \{1,\dots,p-1\}$, we see that for arbitrary sufficiently small $\varepsilon > 0$

$$k(a_1,\dots,a_{p-1},c_p) =$$

$$k(a_1,\dots,a_{p-2},c_{p-1},c_p) = \dots = k(c_1,\dots,c_p)$$

hence

$$k(a_1,\dots,a_p) = k(c_1,\dots,c_p).$$

Passing to the limit with $\varepsilon \longrightarrow 0$ we finally get

$$k(a_1,\dots,a_p) = k(b_1,\dots,b_p).$$

This shows that k is independent of the special choice of (the non-A-isotropic p-vector) $a_1\wedge\dots\wedge a_p$, i.e. is a constant κ. Thus we have

$$A(a_1,\dots,a_p,a_1,\dots,a_p) = \kappa A'(a_1,\dots,a_p,a_1,\dots,a_p)$$

for every non-A-isotropic $a_1\wedge\dots\wedge a_p$. By means of Theorem 8 it follows that $A = \kappa A'$, which we wanted to prove.

10. Let M be a smooth manifold of finite dimension $m>1$ and for every $x \in M$ let M_x be the tangent space of M at x. By F(M) we

denote the ring of all smooth real functions on M and by $\mathfrak{X}(M)$ the F(M)-module of all smooth vector fields tangent to M. Given a smooth submanifold N of M imbedded in M by $\iota : N \longrightarrow M$, we identify every $x \in N$ with $\iota(x) \in M$ and every $u \in N_x$ with $d_\iota(u) \in M_x$. Then evidently $\mathfrak{X}(N)$ is a submodule of F(N)-module $\mathfrak{X}(M)|_N$ of all smooth vector fields on M along N.

If (M,g) is a pseudo-Riemannian manifold, let R denote its curvature tensor field (given by the Levi-Civita connection on M) and let κ be the sectional curvature which is defined by

$$\kappa_x(\delta) = \frac{R_x(u,v,u,v)}{g_x(u,u)g_x(v,v) - g_x(u,v)^2}$$

for every $x \in M$, where u, v are linearly independent points of M_x and $\delta = \mathrm{lin}\{u,v\}$. A diffeomorphism $\varphi : M \longrightarrow M$ is said to be a conformal transformation of M if there exists a smooth positive function λ on M such that $\varphi^* g = \lambda g$ 3). If $\lambda = 1$, then φ is called an isometry of M.

11. M being a smooth manifold of finite dimension $m > 1$, by $\mathfrak{a}^p(M)$ we denote the set of all smooth tensor fields A on M of type (2p,0) such that A_x for every $x \in M$ satisfies (1) and (2). The most important cases of such tensor fields are the curvature structures on M which are such that additionally A_x for every $x \in M$ satisfies the Bianchi identity (4). The set of all curvature structures on M of type (2p,0) will be denoted by $\mathfrak{C}^p(M)$.

In the case of pseudo-Riemannian manifold (M,g) interesting

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3) $\varphi^* g_x(\ldots) = g_{\varphi(x)}(d\varphi(.), d\varphi(.))$.

elements of $\mathfrak{G}^p(M)$ are

i. the extended metric tensor field $\overset{p}{G}$, where $\overset{p}{G}_x$ for every $x \in M$ is simple and generated by g_x, and

ii. if $p=2$, the curvature tensor field R and if $p>2$, the extended curvature tensor field $\overset{p}{R}$, where $\overset{p}{R}_x$ for every $x \in M$ is defined by means of R_x as in section 5.

12. Let N be a smooth submanifold of M of dimension p and $A \in \mathfrak{G}^p(M)$. N is said to be non-A-isotropic at a point $x \in N$, if N_x is non-A_x-isotropic. If N is non-A-isotropic at every point $x \in N$, then N is said to be non-A-isotropic.

Let N be non-A-isotropic. For every $Y \in \mathfrak{X}(M)|_N$ and every $x \in N$ let

$$\mathrm{pr}_N^A(Y)(x) = \mathrm{pr}_{N_x}^{A_x}(Y(x))$$

with $\mathrm{pr}_{N_x}^{A_x}$ given by (3).

Theorem 11. pr_N^A is a smooth orthogonal projection of $\mathfrak{X}(M)|_N$ onto $\mathfrak{X}(N)$.

Proof. Because of Theorem 7 it suffices to show that $\mathrm{pr}_N^A(Y) \in \mathfrak{X}(N)$ for every $Y \in \mathfrak{X}(M)|_N$. Let $x_o \in N$. There exists an open neighbourhood U of x_o and smooth vector fields $X_1,\dots,X_p \in \mathfrak{X}(U)$ such that $N_x = \mathrm{lin}\{X_1(x),\dots,X_p(x)\}$ for every $x \in U$, hence because of (3)

$$\mathrm{pr}_N^A(Y)|_U = \sum_{i=1}^{p} \frac{A(X_1,\dots,X_{i-1},Y|_U,X_{i+1},\dots,X_p,X_1,\dots,X_p)}{A(X_1,\dots,X_p,X_1,\dots,X_p)}\, X_i$$

which already proves $\mathrm{pr}_N^A(Y) \in \mathfrak{X}(N)$.

For each $Y \in \mathfrak{X}(N)$ we have $pr_N^A(Y) = Y$. $\mathfrak{N}^A = \{X \in \mathfrak{X}(M)|_N;\ pr_N^A(X) = 0\}$ is an $F(N)$-submodule of $\mathfrak{X}(M)|_N$ which is called A-normal submodule of $\mathfrak{X}(M)|_N$. Obviously $\mathfrak{X}(M)|_N = \mathfrak{X}(N) \oplus \mathfrak{N}^A$. Thus for every $Y \in \mathfrak{X}(M)|_N$, $pr_N^A(Y)$ is the tangential and $Y - pr_N^A(Y)$ the A-normal component of Y with respect to N.

13. Let M be equipped with a linear connection ∇ and assume that N is non-A-isotropic.

Theorem 12. (cf. [6]). Let $X, Y \in \mathfrak{X}(N)$. For arbitrary extensions $\bar{X}, \bar{Y} \in \mathfrak{X}(M)$ of X respectively of Y, $\nabla_{\bar{X}}\bar{Y}|_N$ is independent of the special choice of $\bar{X}$ and $\bar{Y}$.

For arbitrary $X, Y \in \mathfrak{X}(N)$ and arbitrary $x \in N$ we define $pr_N^A(\nabla_X Y)(x)$ to be $pr_N^A(\nabla_{\bar{X}}\bar{Y})(x)$, where $\bar{X}$, $\bar{Y}$ are local extensions of X respectively of Y onto an open neighbourhood of x in M. Because of Theorem 12, $pr_N^A(\nabla_X Y)(x)$ is independent of the special choice of $\bar{X}$ and $\bar{Y}$

$$\nabla'_X Y = pr_N^A(\nabla_X Y),$$

$X, Y \in \mathfrak{X}(N)$, determines a linear connection ∇' on N which is said to be induced from ∇ by means of pr_N^A

$$h(X,Y) = \nabla_X Y - \nabla'_X Y,$$

$X, Y \in \mathfrak{X}(N)$, defines a (vector valued) tensor field h on N, called imbedding tensor field. For arbitrary $X, Y \in \mathfrak{X}(N)$ we have $h(X,Y) \in \mathfrak{N}^A$, i.e. $h(X,Y)$ is the A-normal component of $\nabla_X Y$.

Using well-known formulas, by means of ∇' on N a curvature tensor field and a so-called relative curvature tensor field are given.

14. Let (M,g) be a Riemannian manifold of dimension $m>1$.

Theorem 13. Consider the following assertions:

1. (M,g) is of constant sectional curvature $\neq 0$.

2. For every $x \in M$, R_x and $\overset{2}{G}_2$ generate the same family of projections.

Then $1 \Rightarrow 2$ and under the assumption that (M,g) is connected and $m>2$, conversely $2 \Rightarrow 1$.

Proof. Let 1 be true. Then

$$R_x(u,v,u,v) = \kappa(g_x(u,u)\, g_x(v,v) - g_x(u,v)^2) = \kappa\overset{2}{G}_x(u,v,u,v)$$

for every $x \in M$ and $u,\ v \in M_x$ From this by means of Theorem 8 for every $x \in M$ we get $R_x = \kappa\overset{2}{G}_x$, which implies the validity of 2.

Now let (M,g) be connected and $m>2$ and assume that 2. is true. Then Theorem 10 ensures that there exists a function κ on M with $\kappa(x) \neq 0$ for every $x \in M$ and with $R = \kappa G$. From a well-known theorem of Schur it follows that κ is constant. This proves the theorem.

Let $\varphi: M \longrightarrow M$ be a diffeomorphism and $\overset{2}{G}$ be the extended metric tensor field of type $(2,0)$ with respect to $\bar{g} = \varphi^{*}g$.

Theorem 14. φ is conformal if and only if $\overset{2}{G}_x$ and $\overset{\overset{2}{*}}{G}_x$ for every $x \in M$ generate the same family of projections.

Proof. φ being conformal, the mentioned condition is obvious. If this condition if fulfilled, then Theorem 10 ensures that there exists a function λ on M with $\lambda(x) \neq 0$ for every $x \in M$

and with $\overset{2}{G} = \lambda\overset{2}{\bar{G}}$. Obviously λ is smooth and positive.

Corollary. φ is an isometry if and only if $\overset{2}{G} = \overset{2}{\bar{G}}$.

Proof. If φ is an isometry, then $g = \varphi^{*}g = \bar{g}$, hence $\overset{2}{G} = \overset{2}{\bar{G}}$. If $\overset{2}{G} = \overset{2}{\bar{G}}$, then from Theorem 14 it follows that φ is conformal, thus there exists a positive smooth function λ on M such that $\bar{g} = \varphi^{*}g = \lambda g$. By means of $\overset{2}{G} = \overset{2}{\bar{G}}$ we get $\lambda = 1$

Let R and $\bar{R}$ as well as $\kappa_x(\sigma)$ and $\bar{\kappa}_x(\sigma)$ be the curvature tensor fields and the sectional curvatures, for σ at $x \in M$, generated by g and $\bar{g}$ respectively.

Theorem 15. If for every $x \in M$, R_x and $\bar{R}_x$ generate the same family of projections and $\kappa_x(\sigma) = \bar{\kappa}_x(\sigma) \neq 0$ for any plane σ and $x \in M$, then φ is a conformal transformation of M.

Proof. From the assumption and by the Theorem 10 we get $R_x = \lambda(x)\bar{R}_x$. Evidently λ is a smooth real function on M. Hence by the second part of the assumption we get the equality $\bar{R}_x(\lambda(x)\overset{2}{\bar{G}}_x - \overset{2}{G}_x) = 0$. Hence we get the thesis of our theorem.

REFERENCES

[1] S. Gähler, Z. Zekanowski : Tensors, 2-inner products and projections, Demonstratio Math. 19 (1986), 747-766.

[2] S. Gähler, Z. Zekanowski : Remarks on tensors

and families of projections, Math. Nachr. (submitted).

[3] F. R. Gantmacher : Applications of the theory of metrices Interscience Publishers, New York - London 1959.

[4] S. Kobayashi, K. Nomizu : Foundations of differential geometry, Interscience Publishers, New York - London 1963.

[5] A. U. Mal'cev : Foundations of linear algebra, Nauka, Moscow 1970 (in Russian).

[6] R. Sikorski : Wstęp do geometrii różniczkowej, PWN, Warszawa 1972.

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AN APPLICATION OF THE FINITE ELEMENT METHOD TO THE INITIAL VALUE PROBLEM FOR A SYSTEM OF ORDINARY DIFFERENTIAL EQUATIONS

Dedicated to the memory of Professor Edward Otto

In this work the method of finite elements is applied to initial value problems for a system of ordinary differential equations. The results of this paper are connected with some applications of the finite elements method, for example with [2], [3], and generalize the results obtained in [4], [5]. The generalizations lead in two directions: first, the results are carried from the equation to the system, and second, they are independent of concrete numeric integration formula.

1. Let $T > 0$ be a fixed constant. Let $f:\langle 0,T\rangle \times R^{2n} \longrightarrow R^n$ be a continuous vector function fulfilling the Lipschitz condition in the last two arguments, i.e. there exists some $L > 0$, such that for any $t \in \langle 0,T\rangle$ and for any $x_1, x_2, y_1, y_2 \in R^n$

$$(1) \qquad \|f(t,x_1,y_1)-f(t,x_2,y_2)\| \le L(\|x_1-x_2\|+\|y_1-y_2\|),$$

where $\|\cdot\|$ denotes the R^n norm.

For any x_0, $x_1 \in R^n$ let us consider the Cauchy problem (for $t \in \langle 0,T \rangle$):

$$\ddot{x} = f(t,x,\dot{x}), \tag{2}$$

$$x(0) = x_0 \;, \qquad \dot{x}(0) = x_1 . \tag{3}$$

By the assumptions made on the function f, problem (2),(3) has exactly one solution existing on $\langle 0,T \rangle$.

2. Now let us introduce the finite elements method by the standard procedure. Choose any natural integer N, let $h = \frac{T}{N}$. Let ϕ_p^h, $p = 0,1,2, \ldots ,N$, be the functions given by the formula

$$\phi_p^h(t) = \begin{cases} \frac{t}{h} - p + 1 & \text{for} \quad t \in (ph - h,\, ph\rangle \cap \langle 0,T\rangle, \\ -\frac{t}{h} + p + 1 & \text{for} \quad t \in (ph,\, ph + h) \cap \langle 0,T\rangle, \\ 0 & \text{for} \quad t \in \langle 0,T\rangle \backslash (ph - h,\, ph + h). \end{cases} \tag{4}$$

Notice that the functions ϕ_p^h are differentiable for all $t \in \langle 0,T \rangle$ except the points ph, $p = 1,2,\ldots,N-1$, where they have the left and right derivatives.

We will look for approximate solution x^h of the form

$$x^h(t) = \sum_{k=0}^{N} \phi_p^h(t)\alpha_k^h \;, \tag{5}$$

where $\alpha_k^h \in R^n$ are unknown vectors for $k= 0,1,2,\ldots,N$. Let us notice that for any $\alpha_k^h \in R^n$ the vector function x^h is differentiable for all $t \in \langle 0,T \rangle$ except perhaps some of the points kh, $k = 1,2,\ldots,N-1$, where it has left and right derivatives. From now on $\dot{x}^h(t)$ will denote the right derivative in t. Observe that

$$\alpha_h^k = x^h(kh). \tag{6}$$

The vector function x^h of the form (5) will be called an approximate solution of (2), (3) if it satisfies the initial condition

$$x^h(0) = x_0 \tag{7}$$

and the "Galerkin rule" of the form

$$\int_0^T [\dot\phi_l^h(t)\dot x^h(t)+\phi_l^h(t)f(t,x^h(t),\dot x^h(t))]dt = -\phi_l^h(0)x_1, \tag{8}$$

$$l = 0,\ 1,\ \dots\ ,N-1.$$

The condition (7) gives $\alpha_0^h = x_0$, from (8) for $l = 0$ we have (see (4))

$$\int_0^T [\dot\phi_0^h(t)\dot x^h(t)+\phi_0^h(t)f(t,x^h(t),\dot x^h(t))]dt= - x_1.$$

Inserting here x^h from (5) and integrating we obtain

$$- [(- \frac{1}{h})\alpha_0^h + \frac{1}{h}\alpha_1^h]+$$

$$+ \int_0^h \phi_0^h(t)f(t,\ \phi_0^h(t)\alpha_0^h + \phi_1^h(t)\alpha_1^h,\ \frac{1}{h}\ (\alpha_1^h - \alpha_0^h))dt = -x_1.$$

Then we have

$$\left\{\begin{aligned} &\alpha_0^h = x_0, \\ &\frac{1}{h}\ (\alpha_1^h - \alpha_0^h\ ) = \\ &\qquad = x_1+ \int_0^h \phi_0^h(t)f(t,\phi_0^h(t)\alpha_0^h + \phi_1^h(t)\alpha_1^h,\ \frac{1}{h}(\alpha_1^h - \alpha_0^h\ ))dt. \end{aligned}\right. \tag{9}$$

Now let us examine the system (8) for $l = 1,2,\dots,N-1$. The functions ϕ_l^h have support on $\langle lh - h,\ lh + h\rangle$, and thus

$$(10) \qquad \int_{lh-h}^{lh+h} [\dot{\phi}_l^h(t)\dot{x}^h(t) + \phi_l^h(t)\ f(t,x^h(t),\dot{x}^h(t))]dt = 0 \ ,$$

$$l = 1,\ 2,\ \ldots,\ N-1 \ .$$

Inserting x^h from (5) into (10) and taking into account that in the interval $\langle lh-h,\ lh+h\rangle$ only three functions ϕ_{l-1}^h, ϕ_l^h, ϕ_{l+1}^h are not identically equal to zero we obtain

$$(11) \qquad \int_{lh-h}^{lh+h} \sum_{k=l-1}^{l+1} \dot{\phi}_l^h(t)\dot{\phi}_k^h(t)\alpha_k^h + \phi_l^h(t)f(t,\ \sum_{k=l-1}^{l+1} \phi_k^h(t)\alpha_k^h,\ \sum_{k=l-1}^{l+1} \dot{\phi}_k^h(t)\alpha_k^h)\ dt = 0,$$

$$l = 1,\ 2,\ \ldots,\ N-1 \ .$$

Dividing the interval of integration into two parts $\langle lh-h, lh\rangle$, $(lh, lh+h\rangle$ and using (4) we get

$$(12) \qquad \frac{1}{h}\ (\alpha_{l+1}^h - 2\alpha_l^h + \alpha_{l-1}^h) =$$

$$= \int_{lh-h}^{lh} \phi_l^h(t)f(t,\phi_{l-1}^h(t)\alpha_{l-1}^h + \phi_l^h(t)\alpha_l^h,\frac{1}{h}(\alpha_l^h - \alpha_{l-1}^h))dt +$$

$$+ \int_{lh}^{lh+h} \phi_l^h(t)f(t,\phi_l^h(t)\alpha_l^h + \phi_{l+1}^h(t)\alpha_{l+1}^h,\frac{1}{h}(\alpha_{l+1}^h - \alpha_l^h))dt$$

$$l = 1,\ 2,\ \ldots,\ N-1 \ .$$

The resulting system (12) with conditions (9) will be treated as a difference scheme for the problem (2), (3). We will prove that for sufficiently small h all the vectors α_l^h, $l = 0,1,\ldots,N-1$, can be found from (9),(12) and that the corresponding approximate solutions (5) converge to the solution of (2), (3).

3. The proof of solvability of the system (9), (12) (for sufficiently small h) will be presented in a more general case.

Let $\varepsilon_k \in R^n$, $k = 0,1,\dots,N$; consider the system

$$(13)\quad \begin{cases} \beta_0 = x_0 + \varepsilon_0, \\ \frac{1}{h}(\beta_1 - \beta_0) = \\ = x_1 + \int\limits_0^h \phi_0^h(t) f(t, \phi_0^h(t)\beta_0 + \phi_1^h(t)\beta_1, \frac{1}{h}(\beta_1 - \beta_0))dt + \varepsilon_1, \\ \frac{1}{h^2}(\beta_{l+1} - 2\beta_l + \beta_{l-1}) = \\ = \frac{1}{h}\int\limits_{lh-h}^{lh} \phi_l^h(t)\, f(t, \phi_{l-1}^h(t)\beta_{l-1} + \phi_l^h(t)\beta_l, \frac{1}{h}(\beta_l - \beta_{l-1}))dt + \\ + \frac{1}{h}\int\limits_{lh}^{lh+h} \phi_l^h(t) f(t, \phi_l^h(t)\beta_l + \phi_{l+1}^h(t)\beta_{l+1}, \frac{1}{h}(\beta_{l+1} - \beta_l))dt + \varepsilon_{l+1}, \\ l = 1,2,\dots,N-1 . \end{cases}$$

The system (9), (12) can be obtained from (13) by taking $\varepsilon_k = 0$, $k = 0,1,\dots,N$.

For the proof of solvability we use the method of mathematical induction. Suppose that we have already found $\beta_0, \beta_1, \dots, \beta_l$. For the vector β_{l+1} we have

$$(14)\qquad \beta_{l+1} = 2\beta_l - \beta_{l-1} + h\int_{lh-h}^{lh} \phi_l^h(t)\, f\big(t, \phi_{l-1}^h(t)\beta_{l-1} + \phi_l^h(t)\beta_l, \tfrac{1}{h}(\beta_l - \beta_{l-1})\big)dt + h\int_{lh}^{lh+h} \phi_l^h(t) f\big(t, \phi_l^h(t)\beta_l + \phi_{l+1}^h(t)\beta_{l+1}, \tfrac{1}{h}(\beta_{l+1} - \beta_l)\big)dt + h^2\varepsilon_{l+1}.$$

Consider the operator $\mathcal{T}: R^n \longrightarrow R^n$ given by

$$\mathcal{T}\gamma = 2\beta_l - \beta_{l-1} + h\int_{lh-h}^{lh} \phi_l^h(t) f\big(t, \phi_{l-1}^h(t)\beta_{l-1} + \phi_l^h(t)\beta_l, \tfrac{1}{h}(\beta_l - \beta_{l-1})\big)dt + h\int_{lh}^{lh+h} \phi_l^h(t)\, f\big(t, \phi_l^h(t)\beta_l + \phi_{l+1}^h(t)\gamma, \tfrac{1}{h}(\gamma - \beta_l)\big)\, dt + h^2\varepsilon_{l+1}.$$

For two given vectors γ and $\tilde{\gamma}$, by (1) and (4), we have

$$\|\mathcal{T}\gamma - \mathcal{T}\tilde{\gamma}\| = \Big\|h\int_{lh}^{lh+h} \phi_l^h(t)\, \Big[f\big(t, \phi_l^h(t)\beta_l + \phi_{l+1}^h(t)\gamma, \tfrac{1}{h}(\gamma - \beta_l)\big) - f\big(t, \phi_l^h(t)\beta_l + \phi_{l+1}^h(t)\tilde{\gamma}, \tfrac{1}{h}(\tilde{\gamma} - \beta_l)\big)\Big]dt\Big\| \le$$

$$\le hL\int_{lh}^{lh+h} |\phi_l^h(t)|\, \Big[|\phi_{l+1}^h(t)|\, \|\gamma - \tilde{\gamma}\| + \tfrac{1}{h}\|\gamma - \tilde{\gamma}\|\Big]dt \le$$

$$\le hL\big(1 + \tfrac{1}{h}\big)\|\gamma - \tilde{\gamma}\|\, h = h\,(h + 1)\, L\, \|\gamma - \tilde{\gamma}\|\,.$$

If we assume that $h < \dfrac{-1 + \sqrt{1 + 4/L}}{2}$, then the operator $\mathcal{T}$ will be contractive, hence from the Banach principle there exists a unique solution of the equation $\mathcal{T}\gamma = \gamma$, i.e. a unique vector solution β_{l+1} of the (14).

4. To prove that the approximate solution x^h converges to the solution x of (2), (3), we shall show that the corresponding iterative scheme (see(6), (9), (12))

$$
(15)\left\{
\begin{aligned}
& x^h(0) - x_0 = 0 , \\
& \frac{1}{h}(x^h(h) - x^h(0)) - x_1 + \\
& -\int_0^h \phi_0^h(t) f(t, \phi_0^h(t)x^h(0) + \phi_1^h(t)x^h(h), \frac{1}{h}(x^h(h) - x^h(0)))dt = 0,
\end{aligned}
\right.
$$

$$
(16)\left\{
\begin{aligned}
& \frac{1}{h^2}(x^h(lh + h) - 2x^h(lh) + x^h(lh - h) - \\
& -\frac{1}{h}\int_{lh-h}^{lh} \phi_l^h(t) f(t, \phi_{l-1}^h(t)x^h(lh-h) + \phi_l^h(t)x^h(lh), \\
& \qquad \frac{1}{h}(x^h(lh) - x^h(lh-h)))dt + \\
& -\frac{1}{h}\int_{lh}^{lh+h} \phi_l^h(t) f(t, \phi_l^h(t)x^h(lh) + \phi_{l+1}^h(t)x^h(lh+h), \\
& \qquad \frac{1}{h}(x^h(lh+h) - x^h(lh)))dt = 0, \\
& l = 1, 2, \ldots, N-1,
\end{aligned}
\right.
$$

approximates the problem (2), (3) and is stable (see [1]).

5. Replacing the values of x^h in (15) by those of the solution x of the problem (2), (3) and then taking the norms we obtain

$\|x(0)-x_o\| = 0,$

$$\Big\|\frac{1}{h}(x(h)-x(0)) - x_1 +$$

$$-\int_0^h \phi_0^h(t)\, f\Big(t,\ \phi_0^h(t)x(0)+\phi_1^h(t)x(h),\ \frac{1}{h}(x(h)-x(0))\Big)\,dt\Big\| =$$

$$= \Big\|\int_0^h \phi_0^h(t)\ddot{x}(t)dt - \int_0^h \phi_0^h(t) f\Big(t,\ \phi_0^h(t)x(0)+\phi_1^h(t)x(h),$$

$$\frac{1}{h}(x(h)-x(0))\Big)dt\Big\| =$$

$$= \Big\|\int_0^h \phi_0^h(t)\Big[f(t,x(t),\dot{x}(t)) - f\Big(t,\phi_0^h(t)x(0)+\phi_1^h(t)x(h),$$

$$\frac{1}{h}(x(h)-x(0))\Big)\Big]dt\Big\| \le$$

$$\le L\int_0^h \Big[\|x(t)-\phi_0^h(t)x(0)-\phi_1^h(t)x(h)\| + \|\dot{x}(t)-\frac{1}{h}(x(h)-x(0))\|\Big]\,dt =$$

$$= L\int_0^h \Big[\|(x(t)-x(0))+\phi_1^h(t)(x(0)-x(h))\| + \|\dot{x}(t)-\dot{x}(\theta_1 h)\|\Big]\,dt \le$$

$$\le L\int_0^h \Big[\|t\dot{x}(\theta_2 t)\| + \phi_1^h(t)\ \|h\dot{x}(\theta_1 h)\| + h\ \|\ddot{x}(\theta_3 h)\|\Big]\,dt \le CLh^2\ ,$$

where $\theta_1,\ \theta_2,\ \theta_3 \in (0,1)$ and

$$\text{(17)} \qquad C := \sup_{t\in\langle 0,T\rangle} \big[2\ \|\dot{x}(t)\| + \|\ddot{x}(t)\|\ \big].$$

Thus, the initial conditions of (15) approximate (3) with the order 2. Similarly as above we can do with the formula (16).

Let A denote the left-hand side of (16) with x instead of x^h, then

$$\|A\| = \Big\|-\frac{1}{h}\int_{lh-h}^{lh} \phi_1^h(t)\ddot{x}(t)dt + \frac{1}{h}\int_{lh}^{lh+h} \phi_1^h(t)\ddot{x}(t)dt +$$

$$- \frac{1}{h}\int_{lh-h}^{lh} \phi_l^h(t)f(t,\ \phi_{l-1}^h(t)x(lh-h) + \phi_l^h(t)x(lh),$$

$$\cdot\frac{1}{h}(x(lh) - x(lh-h)))dt +$$

$$- \frac{1}{h}\int_{lh}^{lh+h} \phi_l^h(t)f(t,\ \phi_l^h(t)x(lh) + \phi_{l+1}^h(t)x(lh+h),$$

$$\frac{1}{h}(x(lh+h) - x(lh)))\ dt\ \| =$$

$$= \frac{1}{h}\ \| - \int_{lh-h}^{lh} \phi_l^h(t)\ [f(t,\ x(t),\ \dot{x}(t)) +$$

$$- f(t,\ \phi_{l-1}^h(t)x(lh-h) + \phi_l^h(t)x(lh),\ \frac{1}{h}(x(lh) - x(lh-h)))]\ dt +$$

$$+\int_{lh}^{lh+h} \phi_l^h(t)\ [f(t,\ x(t),\ \dot{x}(t)) +$$

$$- f(t,\ \phi_l^h(t)x(lh) + \phi_l^h(t)x(lh+h), \frac{1}{h}(x(lh+h) - x(lh)))]\ dt\ \| \leq$$

$$\leq\ \frac{L}{h}\ [\int_{lh-h}^{lh}\ \{\ \|x(t) - \phi_{l-1}^h(t)x(lh-h) - \phi_l^h(t)x(lh)\| +$$

$$+\ \|\dot{x}(t) - \frac{1}{h}(x(lh-h) - x(lh))\|\ \}\ dt +$$

$$+ \int_{lh}^{lh+h}\ \{\ \|x(t) - \phi_l^h(t)x(lh) - \phi_{l+1}^h(t)x(lh+h)\| +$$

$$+\ \|\dot{x}(t) - \frac{1}{h}\ (x(lh+h) - x(lh))\|\ \}\ dt\ ] \leq$$

$$\leq \frac{L}{h}\ [\ \int_{lh-h}^{lh}\ \{\ \|x(t) - x(lh-h)\| + \phi_l^h(t)\ \|x(lh) - x(lh-h)\| +$$

$$+ \|\dot{x}(t)-\dot{x}(lh-\theta_1 h)\| \} \, dt + \int_{lh}^{lh+h} \{ \|x(t) - x(lh)\| +$$

$$+ \phi^h_{l+1}(t) \|x(lh+h) - x(lh)\| + \|\dot{x}(t) - \dot{x}(lh+\theta_2 h)\|\} \, dt \,] \le$$

$$\le \frac{L}{h} [\int_{lh-h}^{lh} \{ 2h \|\dot{x}(t_1)\| + h \|\ddot{x}(t_2)\| \} \, dt +$$

$$+ \int_{lh}^{lh+h} \{ 2h \|\dot{x}(t_3)\| + h\|\ddot{x}(t_4)\| \} \, dt \,] \le \frac{LC}{h} h^2 = CLh,$$

where $t_1, t_2 \in (lh-h, lh)$, $t_3, t_4 \in (lh, lh+h)$ and the constant C is defined by (17).

Hence, we have proved that (16) approximates (2) with the order 1, thus the iterative scheme (15), (16) approximates (2), (3) with the order 1.

It is easy to understand, that using linear elements (4) irrespective of the regularity of function f we can not receive greater order of approximation than 1 .

6.Now we want to show that the scheme (15), (16) is stable (see [1]). Together with the scheme (15), (16) we shall examine the perturbed scheme of the form

$$(18)\quad \begin{cases} \tilde{x}^h(0) - x_0 = \varepsilon_0, \\ \frac{1}{h}(\tilde{x}^h(h) - \tilde{x}^h(0)) - x_1 + \\ - \int_0^h \phi_0^h(t)f(t,\ \phi_0^h(t)\tilde{x}^h(0) + \phi_1^h(t)\tilde{x}^h(h), \\ \qquad\qquad \frac{1}{h}(\tilde{x}^h(h) - \tilde{x}^h(0)))dt = \varepsilon_1, \\ \frac{1}{h^2}(\tilde{x}^h(lh+h) - 2\tilde{x}^h(lh) + \tilde{x}^h(lh-h)) + \\ -\frac{1}{h}\int_{lh-h}^{lh} \phi_l^h(t)f(t,\ \phi_{l-1}^h(t)\tilde{x}^h(lh-h) + \phi_l^h(t)\tilde{x}^h(lh), \\ \qquad\qquad \frac{1}{h}(\tilde{x}^h(lh) - \tilde{x}^h(lh-h)))]dt + \\ -\frac{1}{h}\int_{lh}^{lh+h} \phi_l^h(t)f(t,\ \phi_l^h(t)\tilde{x}^h(lh) + \phi_{l+1}^h(t)\tilde{x}^h(lh+h), \\ \qquad\qquad \frac{1}{h}(\tilde{x}^h(lh+h) - \tilde{x}^h(lh)))dt = \varepsilon_{l+1}, \\ l = 1,\ 2,\ \ldots,\ N-1. \end{cases}$$

From the section 3 we know that from the system (18) it is possible to find all vectors $\tilde{x}^h(ph)$, $p = 0,1,\ldots,N$, for $h < \frac{-1 + \sqrt{1 + 4/L}}{2}$.

Let $c = 2eLT(2 + 2T + h)(1 - Lh - Lh^2)^{-1}$. In the sequel we restrict ourselves to integers $N \geq c$.

In R^{N+1} we introduce a special norm: if

$$y = (\ \|y_0\|,\ \|y_1\|,\ \ldots,\ \|y_N\|\ ) \in R^{N+1},$$

then let

$$(19)\qquad |||y||| := \max_{k=0,1,\ldots,N} (\|y_k\|u\Psi(k)),$$

where

$$\Psi(k) = \exp\left(-\frac{k}{N}\,c\right) \tag{19'}$$

and $\|\cdot\|$ denotes the norm in R^n.

To show that the scheme (15), (16) is stable, we must prove that there exists $K \in R$ such that for all $N \geq c$, for the corresponding vectors

$$x^h = (x^h(0),\ x^h(h),\ \ldots,\ x^h((N-1)h),\ x^h(T))$$

$$\tilde{x}^h = (\tilde{x}^h(0),\ \tilde{x}^h(h),\ \ldots,\ \tilde{x}^h((N-1)h),\ \tilde{x}^h(T))$$

(where $h = \frac{T}{N}$) and for $\varepsilon = (\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_N)$ (where $\varepsilon_i \in R^n$ for $i = 0,1,\ldots,N$), from (15), (16), (18) we have

$$|||x^h - \tilde{x}^h||| \leq K|||\varepsilon||| .$$

For simplicity we introduce new vectors $\gamma_l, \tilde{\gamma}_l \in R^n$ for $l = 0,1,\ldots,N$ taking $\gamma_0 = \tilde{\gamma}_0 = 0$ and

$$\left\{\begin{aligned} \gamma_l &:= \frac{1}{h}(x^h(lh) - x^h(lh-h)), \\ \tilde{\gamma}_l &:= \frac{1}{h}(\tilde{x}^h(lh) - \tilde{x}^h(lh-h)), \quad l=1,2,\ldots,N. \end{aligned}\right. \tag{20}$$

Then

$$x^h(kh) = x^h(0) + h\sum_{p=1}^{k} \gamma_p \;, \quad \tilde{x}^h(kh) = \tilde{x}^h(0) + h\sum_{p=1}^{k} \tilde{\gamma}_p \,. \tag{21}$$

For these new vectors the schemes (15), (16), (18) take the forms

$$
(22)\left\{
\begin{aligned}
&\gamma_0 = 0,\\
&\gamma_1 = x_1 + \int_0^h \phi_0^h(t) f(t,\ x_0 + \phi_1^h(t) h\gamma_1,\ \gamma_1)\,dt,\\
&\gamma_{k+1} = \gamma_k + \\
&+\int_{kh-h}^{kh} \phi_k^h(t) f\Big(t, \phi_{k-1}^h(t)\big(x_0 + h\sum_{l=1}^{k-1}\gamma_l\big) + \phi_k^h(t)\big(x_0 + h\sum_{l=1}^{k}\gamma_l\big), \gamma_k\Big)\,dt +\\
&+\int_{kh}^{kh+h} \phi_k^h(t) f\Big(t, \phi_k^h(t)\big(x_0 + h\sum_{l=1}^{k}\gamma_l\big) + \phi_{k+1}^h(t)\big(x_0 + h\sum_{l=1}^{k+1}\gamma_l\big), \gamma_{k+1}\Big)\,dt,
\end{aligned}
\right.
$$

$$
(23)\left\{
\begin{aligned}
&\tilde{\gamma}_0 = 0,\\
&\tilde{\gamma}_1 = x_1 + \varepsilon_1 + \int_0^h \phi_0^h(t) f(t,\ x_0 + \varepsilon_0 + \phi_1^h(t) h\tilde{\gamma}_1,\ \tilde{\gamma}_1)\,dt,\\
&\tilde{\gamma}_{k+1} = \tilde{\gamma}_k + \int_{kh-h}^{kh} \phi_k^h(t) f\Big(t,\ \phi_{k-1}^h(t)\big(x_0 + \varepsilon_0 + h\sum_{l=1}^{k-1}\tilde{\gamma}_l\big) +\\
&\qquad\qquad\qquad + \phi_k^h(t)\big(x_0 + \varepsilon_0 + h\sum_{l=1}^{k}\tilde{\gamma}_l\big), \tilde{\gamma}_k\Big)\,dt +\\
&+\int_{kh}^{kh+h} \phi_k^h(t) f\Big(t,\ \phi_k^h(t)\big(x_0 + \varepsilon_0 + h\sum_{l=1}^{k}\tilde{\gamma}_l\big) +\\
&\qquad\qquad + \phi_{k+1}^h(t)\big(x_0 + \varepsilon_0 + h\sum_{l=1}^{k+1}\tilde{\gamma}_l\big), \tilde{\gamma}_{k+1}\Big)\,dt + h\varepsilon_{k+1}.
\end{aligned}
\right.
$$

By adding the last equations in (22) and (23) in k from 1 to p-1 we receive, respectively,

$$\begin{cases} \gamma_0 = 0, \\ \gamma_1 = x_1 + \displaystyle\int_0^h \phi_0^h(t) f(t,\ x_0 + \phi_1^h(t) h\gamma_1,\ \gamma_1) dt, \\ \gamma_p = \gamma_1 + \displaystyle\sum_{k=1}^{p-1} \int_{kh-h}^{kh} \phi_k^h(t) f(t,\ x_0 + h\sum_{l=1}^{k-1} \gamma_l + \phi_k^h(t) h\gamma_k,\ \gamma_k) dt + \\ \quad + \displaystyle\sum_{k=1}^{p-1} \int_{kh}^{kh+h} \phi_k^h(t) f(t, x_0 + h\sum_{l=1}^{k} \gamma_l + \phi_{k+1}^h(t) h\gamma_{k+1}, \gamma_{k+1}) dt \\ , \end{cases}$$

and

$$\begin{cases} \tilde{\gamma}_0 = 0, \\ \tilde{\gamma}_1 = x_1 + \varepsilon_1 + \displaystyle\int_0^h \phi_0^h(t) f(t,\ x_0 + \varepsilon_0 + \phi_1^h(t) h\tilde{\gamma}_1,\ \tilde{\gamma}_1) dt, \\ \tilde{\gamma}_p = \tilde{\gamma}_1 + \displaystyle\sum_{k=1}^{p-1} \int_{kh-h}^{kh} \phi_k^h(t) f(t,\ x_0 + \varepsilon_0 + h\sum_{l=1}^{k-1} \tilde{\gamma}_l + \phi_k^h(t) h\tilde{\gamma}_k, \tilde{\gamma}_k) dt + \\ \quad + \displaystyle\sum_{k=1}^{p-1} \int_{kh}^{kh+h} \phi_k^h(t) f(t,\ x_0 + \varepsilon_0 + h\sum_{l=1}^{k} \tilde{\gamma}_l + h\phi_{k+1}^h(t) \tilde{\gamma}_{k+1}, \tilde{\gamma}_{k+1}) dt + \\ \quad + h\displaystyle\sum_{k=1}^{p-1} \varepsilon_{k+1} \quad . \end{cases}$$

For p = 2, 3, ..., N from the Lipschitz condition we obtain

$$(24)\quad \|\tilde{\gamma}_p-\gamma_p\| \le \|\tilde{\gamma}_1-\gamma_1\| +$$

$$+\sum_{k=1}^{p-1}\int_{kh-h}^{kh} L\phi_k^h(t)[\|\varepsilon_0\|\cdot h\sum_{l=1}^{k-1}\|\tilde{\gamma}_l-\gamma_l\|+\|\tilde{\gamma}_k-\gamma_k\|(h\phi_k^h(t)+1)]dt\ +$$

$$+\sum_{k=1}^{p-1}\int_{kh}^{kh+h} L\phi_k^h(t)[\|\varepsilon_0\|+h\sum_{l=1}^{k}\|\tilde{\gamma}_l-\gamma_l\|+\|\tilde{\gamma}_{k+1}-\gamma_{k+1}\|(h\phi_k^h(t)+1)]dt\ +$$

$$+\ h\sum_{k=1}^{p-1}\|\varepsilon_{k+1}\|.$$

But

$$(25)\quad \|\tilde{\gamma}_1-\gamma_1\| \le \|\varepsilon_1\|+L\int_0^h \phi_0^h(t)[\|\varepsilon_0\|+(h\phi_1^h(t)+1)\|\tilde{\gamma}_1-\gamma_1\|]dt \le$$

$$\le \|\varepsilon_1\|+Lh[\|\varepsilon_0\|+(h+1)\|\tilde{\gamma}_1-\gamma_1\|].$$

Taking into account that $h < \dfrac{-1+\sqrt{1+4/L}}{2}$ we have $(h^2+h)L < 1$ and $(1-(h^2+h)L)\|\tilde{\gamma}_1-\gamma_1\| \le \|\varepsilon_1\| + Lh\|\varepsilon_0\|$, hence

$$\|\tilde{\gamma}_1-\gamma_1\| \le (1-(h^2-h)L)^{-1}[\,Lh\,\|\varepsilon_0\| + \|\varepsilon_1\|\,]\,.$$

Putting $B_p:=\|\tilde{\gamma}_p-\gamma_p\|$ for $p = 1, 2, \ldots, N$ from (24) we have

$$B_p \le B_1+\sum_{k=1}^{p-1} L\int_{kh-h}^{kh} [\|\varepsilon_0\|+h\sum_{l=1}^{k-1} B_l+B_k(h+1)]\,dt\ +$$

$$+\sum_{k=1}^{p-1} L\int_{kh}^{kh+h} [\|\varepsilon_0\|+h\sum_{l=1}^{k} B_l+B_{k+1}(h+1)]dt+h\sum_{k=1}^{p-1}\|\varepsilon_{k+1}\| =$$

$$= B_1+Lh[2(p-1)\|\varepsilon_0\|+2h\sum_{k=1}^{p-1}\sum_{l=1}^{k}B_l+\sum_{k=1}^{p-1}(B_k+(h+1)B_{k+1})]+$$

$$+\ h\sum_{k=1}^{p-1}\|\varepsilon_{k+1}\|.$$

But

$$\sum_{k=1}^{p-1}\sum_{l=1}^{k}B_l=\sum_{l=1}^{p-1}(p-1-l)\ B_l\ \le\ (p-1)\sum_{k=1}^{p-1}B_k$$

and

$$\sum_{k=1}^{p-1}(B_k+(h+1)B_{k+1})\ =\ B_1+(h+2)\sum_{k=2}^{p-1}B_k+(h+1)B_p\ \le$$

$$\le\ (h+2)\sum_{k=1}^{p-1}B_k+(h+1)B_p\ ,$$

hence

$$B_p\ \le\ 2Lh(p-1)\|\varepsilon_0\|\ +\ (2Lh^2(p-1)+Lh(h+2))\sum_{k=2}^{p-1}B_k+$$

$$+\ (1+2Lh^2(p-1)+Lh)B_1+Lh(h+1)B_p+h\sum_{k=1}^{p-1}\|\ \varepsilon_{k+1}\|.$$

Having $p-1 \le N$; $hN=T$ and $L(h^2+h)\le 1$, we get

$$B_p\ \le\ (1-L(h^2+h))^{-1}[2Lh(p-1)\|\varepsilon_0\|\ +\ h\sum_{k=1}^{p-1}\|\varepsilon_{k+1}\|+(2Lh^2(p-1)+$$

$$+\ Lh(h+2))\sum_{k=2}^{p-1}B_k+(1+2LTh+Lh)B_1]$$

and then, using (25),

$$B_p\ \le\ (1-L(h^2+h))^{-1}[2LT\|\varepsilon_0\|+h\sum_{k=1}^{p-1}\|\varepsilon_{k+1}\|+(2LTh+Lh^2+2Lh)\sum_{k=2}^{p-1}B_k+$$

$$+\ (1-Lh^2-Lh)^{-1}(1+2TLh+Lh)(Lh\|\varepsilon_0\|\ +\ \|\varepsilon_1\|)]\ =$$

$$=\ (1-Lh^2-Lh)^{-1}[(1-Lh^2-Lh)^{-1}((2LT-2L^2Th+Lh+L^2h^2)\|\varepsilon_0\|\ +$$

$$+\ (1+2TLh+Lh)\|\varepsilon_1\|)\ +\ h\sum_{k=2}^{p-1}\|\varepsilon_k\|\ +\ (2LTh+2Lh+Lh^2)\sum_{k=2}^{p-1}B_k],$$

$$p = 2, 3, \ldots, N.$$

Multiplying the received inequalities by $\Psi(p)$ and using the norm defined by (19) we observe that

$$B_p\Psi(p) \le$$

$$\le (1-Lh^2-Lh)^{-1}[(1-Lh^2-Lh)^{-1}((2LT-2L^2Th+Lh+L^2h^2)\|\varepsilon_0\|\Psi(p)+$$

$$+ (1+2TLh+Lh)\|\varepsilon_1\|\Psi(p))+h\sum_{k=2}^{p-1}\frac{\Psi(p)}{\Psi(k)}\|\varepsilon_k\| \Psi(k)+$$

$$+ (2LTh+2Lh+Lh^2)\sum_{k=2}^{p-1}\frac{\Psi(p)}{\Psi(k)} B_k\Psi(k)].$$

Hence ($B_p = \|\tilde{\gamma}_p - \gamma_p\|$) we have

$$\|\tilde{\gamma}_p-\gamma_p\|\Psi(p) \le$$

$$\le (1-Lh^2-Lh)^{-1}[(1-Lh^2-Lh)^{-1}((2LT-2L^2Th+Lh+L^2h^2)\Psi(p)|||\varepsilon||| +$$

$$+(1+2TLh+Lh)|||\varepsilon|||\frac{\Psi(p)}{\Psi(1)})+ h\sum_{k=2}^{p-1}\frac{\Psi(p)}{\Psi(k)}|||\varepsilon||| +$$

$$+ (2LTh+2Lh+Lh^2)\sum_{k=2}^{p-1}\frac{\Psi(p)}{\Psi(k)}|||\tilde{\gamma} - \gamma|||\ ] .$$

Observing that $\Psi(p) \le \Psi(1) \le 1$ and that, by (19'),

$$\sum_{k=2}^{p}\frac{\Psi(p)}{\Psi(k)} = \sum_{k=2}^{p}\exp(-(p-k)c/N) =$$

$$\sum_{k=2}^{p}\exp(-(k-2)c/N) \le \frac{1}{1-\exp(-c/N)} ,$$

we can write

$$\|\tilde{\gamma}_p-\gamma_p\|\psi(p) \le$$

$$\le(1-Lh^2-Lh)^{-1}\Big[(1-Lh^2-Lh)^{-1}(1+2TLh+2LT-2L^2Th+2Lh+L^2h^2)\,|\!|\!|\varepsilon|\!|\!| +$$

$$+\frac{h}{1-\exp(-c/N)}\,|\!|\!|\varepsilon|\!|\!| + h(2LT+2L+Lh)\frac{1}{1-\exp(-c/N)}\,|\!|\!|\tilde{\gamma}-\gamma|\!|\!|\Big],$$

for $p = 2,3,\ \ldots,N$. It is easy to prove that this equality holds also for $p = 0$ and $p = 1$.

Because $h = \frac{T}{N}$, we have

$$|\!|\!|\tilde{\gamma}-\gamma|\!|\!| \le$$

$$\le (1-Lh^2-Lh)^{-1}\ \Big\{\ \Big[(1-Lh^2-Lh)^{-1}(1+2TLh+2LT-2L^2Th+2Lh+L^2h^2) +$$

$$+\frac{T}{N(1-\exp(-c/N))}\Big]|\!|\!|\varepsilon|\!|\!| + \frac{(2LT+2L+Lh)T}{N(1-\exp(-c/N))}\,|\!|\!|\tilde{\gamma}-\gamma|\!|\!|\ \Big\}.$$

Since $1-\exp(-x) \ge \frac{1}{e}\,x$ for $x\in(0,1\rangle$, and we have assumed that $N \ge c$, thus $\frac{c}{N} \le 1$, so we have $1-\exp(-c/N) \ge \frac{c}{eN}$ and $\frac{1}{N(1-\exp(-c/N))} \le \frac{e}{c}$. Putting

$$c_1= 2\left[\ \frac{1+2LT+2LTh-2TL^2h+2Lh+Lh^2}{(1-Lh-Lh^2)^2} + \frac{eT}{c(1-Lh-Lh^2)}\right] ,$$

we have

$$|\!|\!|\tilde{\gamma}-\gamma|\!|\!| \le \frac{1}{2}c_1|\!|\!|\varepsilon|\!|\!| + \frac{1}{2}\,|\!|\!|\tilde{\gamma}-\gamma|\!|\!|$$

which implies that

$$|\!|\!|\tilde{\gamma}-\gamma|\!|\!| \le c_1|\!|\!|\varepsilon|\!|\!| .$$

But

$$\tilde{x}^h(lh)-x^h(lh)=\tilde{x}^h(0)-x^h(0)+h\sum_{p=1}^{l}(\tilde{\gamma}_p-\gamma_p) = \varepsilon_0+h\sum_{p=1}^{l}(\tilde{\gamma}_p-\gamma_p) ,$$

so we can write

$$\| \tilde{x}^h(lh)-x^h(lh) \| \Psi(l) \le \| \varepsilon_0 \| \Psi(l)+h\sum_{p=1}^{l}\|\tilde{\gamma}_p-\gamma_p\| \Psi(l) \le$$

$$\le |||\varepsilon||| + h\sum_{p=1}^{l}\frac{\Psi(l)}{\Psi(p)}|||\tilde{\gamma}-\gamma||| \le |||\varepsilon||| + \frac{h}{1-\exp(-c/N)}|||\tilde{\gamma}-\gamma||| \le$$

$$\le |||\varepsilon||| + \frac{eT}{c}|||\tilde{\gamma}-\gamma||| \le (1+\frac{eTc_1}{c})|||\varepsilon|||$$

for $l = 0, 1, \ldots, N$. Thus, for

$$c_2 = 1+\frac{eTc_1}{c} =$$

$$= 1+\frac{L}{2+2T+h}\left(\frac{1+2LT+2LTh-2TL^2h+2Lh+Lh^2}{(1-Lh-Lh^2)^2}+\frac{1-Lh-Lh^2}{2L(2+2T+h)}\right)$$

we have

$$|||\tilde{x}^h - x^h||| \le c_2|||\varepsilon|||,$$

which was to be proved.

REFERENCES

[1] С. К. Годунов, В. С. Рябенький : Разностные схемы, Москва 1987.

[2] Z. Kączkowski : The method of finite space-time elements in dynamics of structures, J. Tech. Phys. 16 (1985) 69-84.

[3] Z. Kączkowski, M. Zyszko : Drgania giętne

pręta metodą czasoprzestrzennych elementów skończonych, Arch. Inż. Ląd., 24 (1) (1978) 59-78.

[4] K. Litewska, J. Muszyński : Сходимость метода конечных элементов для некоторой начальной задачи, Demonstratio Math. 15 (1982) 935-945.

[5] K. Litewska, J. Muszyński : The application of the finite elements method to the initial problems of nonlinear ordinary differential equations, Demonstratio Math. 18 (1985) 279-293.

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THE STRUCTURE OF DESCRIPTIVE GEOMETRY OF THE FOUR-DIMENSIONAL LOBACHEVSKI SPACE

Dedicated to the memory
of Professor Edward Otto

1. Introduction

In the following article we shall present the method of developing geometry of four-dimensional space of Lobachevski (L^4) realized in the three-dimensional Euclidean space complemented with an improper point.

In order to attain this result the space L^4 should be mapped by means of rectilinear projection onto the surface of the hyperhorysphere embedded in the space L^4.

Then the fact that on the hyperhorysphere three-dimensional Euclidean geometry is realized enables us to obtain the solution of the problems concerning space L^4 in the three-dimensional Moebius space (M^3). The examples of the constructions of common elements, belonging of the elements, and mensural constructions will be presented below.

D e f i n i t i o n 1. The hyperhorysphere is defined as a surface spread over the hyperpencil of parallel lines with the improper vertex for which the line linking it to points is equioblique to the lines of the hyperpencil passing through these points.

Let us consider an arbitrary hyperhorysphere $\Sigma \subset L^4$ spread over the hyperpencil of lines hw with improper center R^∞.

Definition 2. Parallel transformation is defined as a rectilinear transformation Φ of space L^4 onto the hyperhorysphere Σ realized by means of a hyperpencil hw of parallel lines with the improper vertex R^∞ and such that

$$\Phi(X) = X^\Sigma \quad X^\Sigma = \Sigma \cap r(X, R^\infty)^{1)}.$$

It results from the definition 1 and 2 that

$$\forall X \neq R^\infty \; \exists X^\Sigma : \Phi(X) = X^\Sigma.$$

The following theorem is true:

Theorem. On the surface of hyperhorysphere in the set of points and horycycles and horyspheres there are fulfilled all axioms of the Euclidean space geometry.

Proof. Let us have a hyperhorysphere Σ . It is possible to prove directly that in the set of points of the hyperhorysphere Σ all axioms of the absolute geometry are fulfilled where the horycycles are taken for lines and the horyspheres for planes.

---

[1] The elements of the space L^4 are denoted in the following way: points - A, B, C..., lines-a, b, c.., planes-α, β, γ...., hyperplanes-$\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$..., improper with index "∞". The elements of the space M^3 are denoted in the same way but with a mark "M" above. The elements of the hyperhorysphere are denoted by a mark "Σ". The elements determining a given element are given in brackets e.g. $r(X, R^\infty)$-the function of the points X and R^∞.

Thus it is enough to prove that the Euclidean axiom is fulfilled in the set of elements of the hyperhorysphere as well. In order to attain this, let us take an arbitrary horycycle h_1 and the point A ($A \notin h_1$). The horycycle h_1 and the point A determine the hyperhorysphere $H \subset \Sigma$. Let us consider the hyperhorysphere H and denote by γ_1 the plane in which the horycycle h_1 is included. All the axes of the hyperhorysphere Σ (it implies that those of the horysphere H as well) which are not uniplanar with the plane γ_1 are parallel to this plane.

One and only one plane parallel to γ_1 passes through each axis of the horysphere H. Such a plane γ_2 passing through the axis of H with the origin in the point A and parallel to the plane γ_1 intersects horysphere H in the horycycle h_2 disjoint with h_1.

The horycycles h_1 and h_2 are defined as parallel ones. An arbitrary plane γ_3 ($\gamma_3 \neq \gamma_1$) passing through axis of H the origin in A intersects the plane γ_1 in the axis of the horysphere H, too. The horycycle $h_3 = H \cap \gamma_3$ intersects horycycle h_1. Thus on the hyperhorysphere Σ through a point not lying on a given horycycle passes one and only one horycycle parallel (q.e.d.).

2. Mapping of the space L^4 onto three-dimensional Moebius space

Let us assume that $R^{\infty} \in \Sigma$ so $R^{\infty} = R^{\Sigma\infty}$.

Definition 3. We define the mapping Γ as the composition of parallel projection Φ and isomorphic mapping ψ [1] of the hyperhorysphere surface Σ onto three-dimensional Moebius space.

Thus we have $\Gamma(X) = \Psi(\Phi(X)) = X^M$. Of course we have $\forall X \; \exists X^M : \Gamma(X) = X^M$.

The definition of the transformation Γ implies that in the set of improper points of the space L^4 this transformation is invertible.

Let us consider an arbitrary line $l(L_1^\infty, L_2^\infty)$.

Definition 4. The points $\Gamma(L_1^\infty) = L_1^M$, $\Gamma(L_2^\infty) = L_2^M$ are called traces of the line l.

$\Gamma(l) = [L_1^M, L_2^M]$ - the closed segment of the ends L_1^M and L_2^M. The proper points of the line l are transformed onto the points of the interior (L_1^M, L_2^M) of the segment $[L_1^M, L_2^M]$.

Let us denote by α^∞ the set of points which are improper in the arbitrary plane α.

Definition 5. $\Gamma(\alpha^\infty) = O_\alpha^M$, where O_α^M is a circle and we call it the trace of the plane α

$$\Gamma(\alpha) = [\, O_\alpha^M \,].$$

The proper points of the plane α are transformed onto the points of the interior (O_α^M) of the circle $[\, O_\alpha^M \,]$.

Let $\mathscr{A}^\infty$ denote the set of improper points of an arbitrary hyperplane $\mathscr{A}$.

Definition 6. $\Gamma(\mathscr{A}^\infty) = S_{\mathscr{A}}^M$, where $S_{\mathscr{A}}^M$ is the sphere called the trace of the hyperplane $\mathscr{A}$

$$\Gamma(\mathscr{A}) = [\, S_{\mathscr{A}}^M \,].$$

The proper points of the hyperplane $\mathscr{A}$ are transformed onto the points of the interior $(S_{\mathscr{A}}^M)$ of the sphere $[\, S_{\mathscr{A}}^M \,]$.

The invertibility of the transformation Γ for proper points of the space L^4 follows from the invertibility of the transformation Γ for improper points and the rule of belonging as well.

A detailed description of the transformation Γ has been presented in thesis [2].

3. Some examples

Example 1. The points $A \in k[K_1^\infty, K_2^\infty]$ and $B \in l[L_1^\infty, L_2^\infty]$ are given. Determine traces of the line $m(A,B)$.

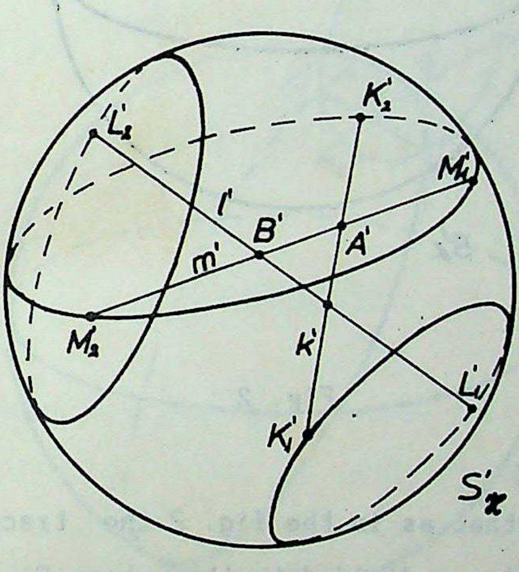

Fig. 1

Let lines k and l determine the hyperplane $\mathcal{X}$ (fig. 1). The points K'_1, K'_2, L'_1, L'_2 determine the sphere $S'_{\mathcal{X}}$ - the trace of the hyperplane $\mathcal{X}$. The traces M_1' and M_2' are situated on the sphere $S'_{\mathcal{X}}$.

Example 2. The hyperplane $\mathcal{L}$ and the line $p(P_1^\infty, P_2^\infty)$ are given. Determine the point of the intersection of the hyperplane $\mathcal{L}$ by the line p.

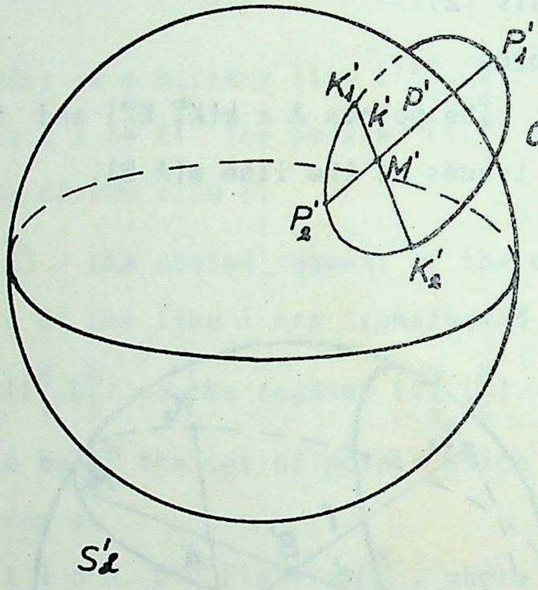

Fig. 2

Let us assume that as in the fig. 2 the traces P_1' and P_2' are given by the line p divided by the sphere $S'_{\mathcal{L}}$ - a trace of the hyperplane $\mathcal{L}$ (then the line gets through the hyperplane). An arbitrary plane β is drawn through the line p. The trace O_β' of the plane β containing points P_1' and P_2', goes through the sphere $S'_{\mathcal{L}}$ in points K_1' and K_2'. The line $k(K_1^\infty, K_2^\infty)$ whose traces are points K_1' and K_2' is the edge of the plane β and the hyperplane

$\mathcal{L}$. The traces K_1', K_2' and P_1', P_2' of the lines k and p are divided on the circle O_β' so these lines intersect. The point M of intersection of lines k and p is the point of puncture of the hyperplane $\mathcal{L}$ by the line p.

Example 3. We are given the hyperplane $\mathcal{L}$ and the line $a(A_1^\infty, A_2^\infty)$ intersecting this hyperplane. Determine the angle formed by the line a and hyperplane $\mathcal{L}$.

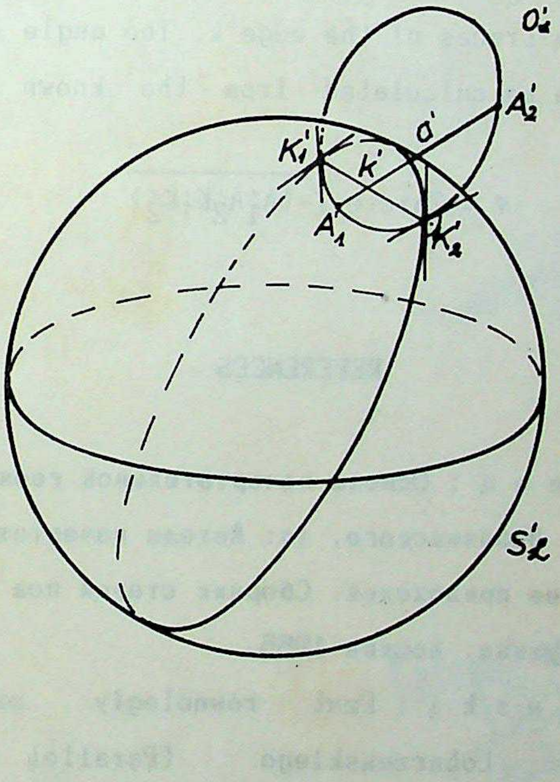

Fig. 3

The sphere S'_α - the trace of the hyperplane $\mathcal{L}$ separates points A'_1 and A'_2- the traces of the line a intersecting the hyperplane $\mathcal{L}$ (fig. 3). Through this line a we draw the plane α perpendicular to the hyperplane $\mathcal{L}$. The edge $k = \alpha\cap\mathcal{L}$ is orthogonal projection of the line a onto hyperplane $\mathcal{L}$, and the angle in question is the angle between the lines a and k.

Accordingly through the points A'_1 and A'_2 we draw the circle O'_α- the trace of the plane α perpendicular to a sphere S_α (a condition of perpendicularity of a plane to a hyperhorysphere). The points K'_1 and K'_2 of the intersection of the sphere S'_α by a circle O'_α are the traces of the edge k. The angle formed by the lines a and k can be calculated from the known formula (see [3]):

$$\varphi = 2\operatorname{arctg}\sqrt{-(A'_1A'_2K'_1K'_2)}$$

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REFERENCES

[1] З. А. Скопец : Основы начертательной геометрии пространства Лобачевского, in: Методы начертательной геометрии и ее приложения. Сборник статей под редакцией М. Ф. Четверухина. Москва 1955.

[2] A. Bieliński : Rzut równoległy czterowymiarowej przestrzeni Lobaczewskiego (Parallel projection four-dimensional space of Lobachevski). Doctoral thesis. Technical University of Warsaw, 1977.

[3] B. Grochowski : Rzut środkowy przestrzeni

Łobaczewskiego (Central projection space of Lobachevski). Zeszyty Nauk. Politech. Warszaw. Geometria, No. 1 1959 39-76.

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